1.25 Let G be a graph with $\geqslant 5$ vertices.
Suppose G is bipartite. We will show that $\overline{G}$ is not bipartite.

Let L, R be the two parts of the bipartite graph G. Since $|L|+|R| = \#\text{vertices}(G) \geq 5$, either $|L|$ or $|R|$ is at least 3.

Thus there are 3 vertices in G with no edges between any 2 of them.

These 3 vertices form a Δ in $\overline{G}$.

Thus $\overline{G}$ is not bipartite (since graphs with odd cycles are not bipartite).

1.26. We will show that G is bipartite.

Let L be the vertices that are even integers.
Let R be - - - - - - - - - - odd - - - .

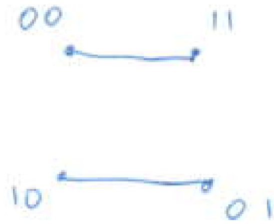
For any $x, y \in L$, $x+y$ is even, so $x-y$ is not an edge.

For any $x, y \in R$, $x+y$ is even, so $x-y$ is not an edge.

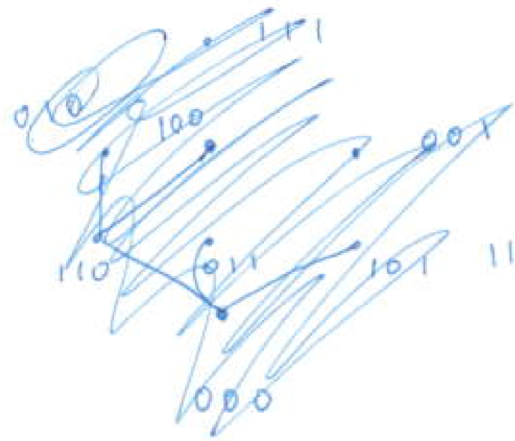
Thus L, R is a bipartition of the vertex set with no edges within L or within R.
Thus G is bipartite.

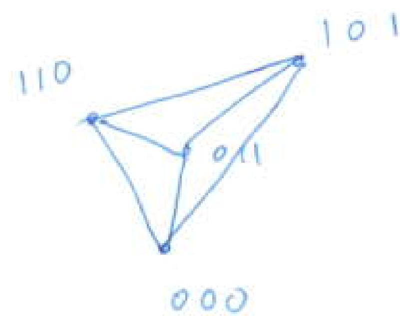
1.28

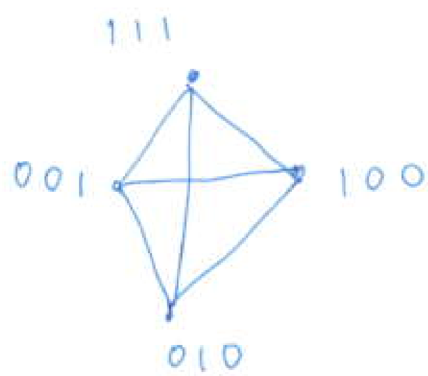
(a) R_2



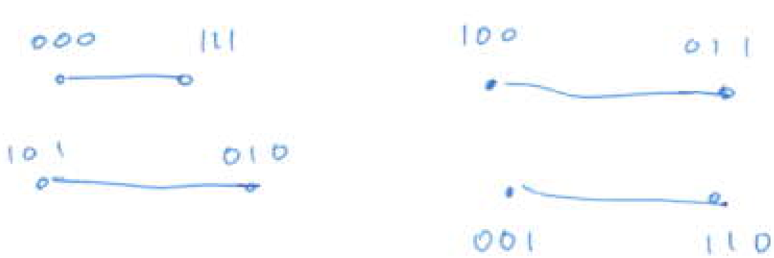
R_3



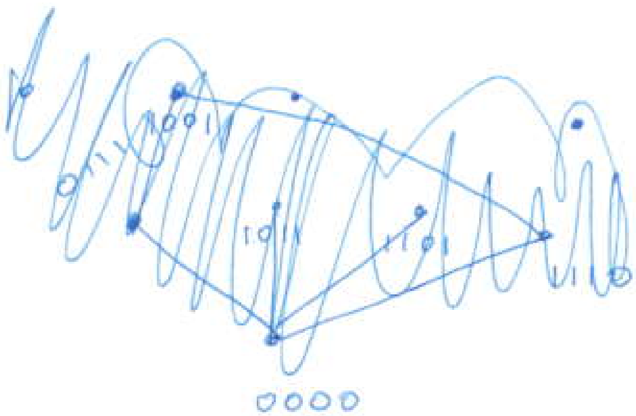


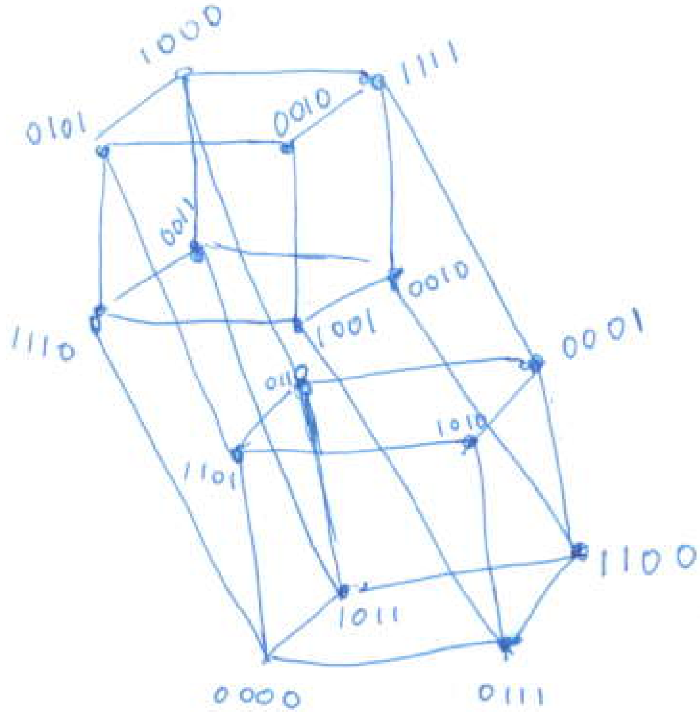


(b) S_3



S_4.

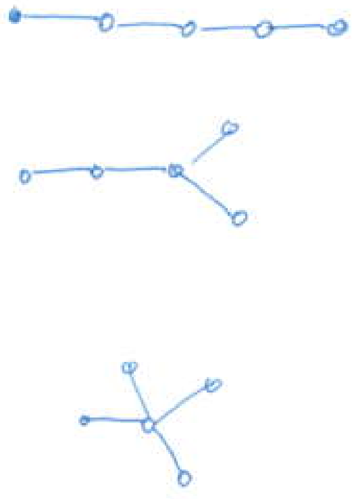




R_2, S_3, S_4 are all Bipartite.

4.7.

(a)



Sum of degrees $= 8$

≥ 2 degrees are 1

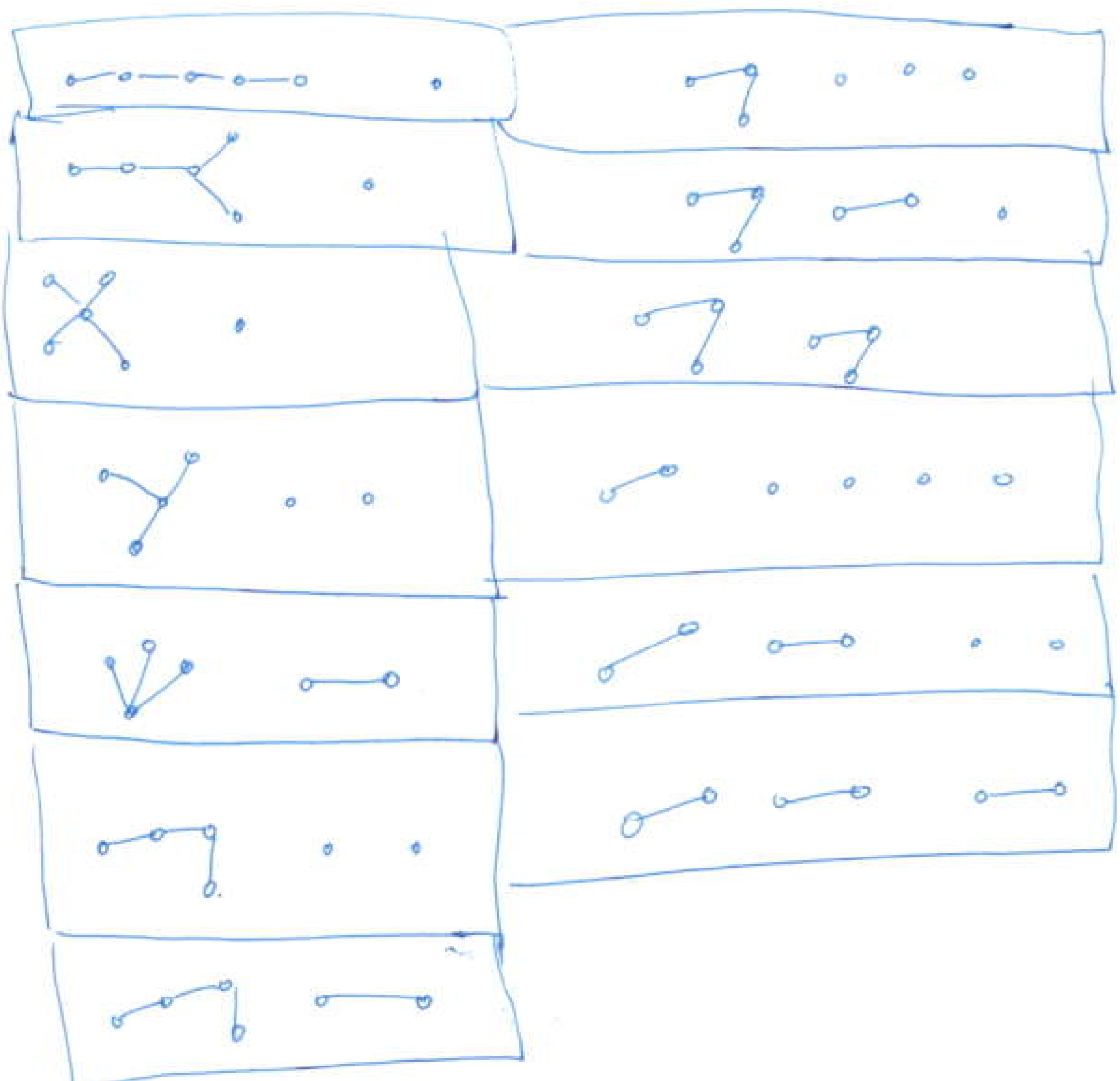
$1+1+\_+\_+\_=8$

so either $1,2,3$

$1,1,4$

$2,2,2.$

(b)



4.8

Let G_1 be a connected component of G. Let n_1 be the # vertices of G_1

Then all the degrees of G_1 are ≥ 2.

So sum of degrees of $G_1 \geq 2 \cdot n_1$

By Handshake lemma, # edges $(G_1) \geq \frac{2n_1}{2} \geq n_1$.

So G_1 is not a tree. Since G_1 is connected, G_1 is not acyclic.

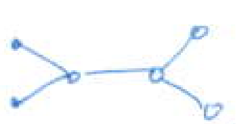
(There are other proofs too, eg by considering the longest path in G_1)

4.9.

$n = 4$
3 edges

4.16.

(a)



(b). Let n be the number of vertices.

Sum of degrees $= \frac{2}{3} \cdot n + 3 \cdot \frac{n}{3} = \frac{5n}{3}$

This should equal (by handshake lemma + fact that trees have $n-1$ edges) $2(n-1)$. $\frac{5n}{3} = 2(n-1)$
$\Rightarrow n = 6$.

$G =$

4.17

(c) Let the fixed degree be c.

$$2(n-1) = \text{Sum of degrees} = \frac{3}{4} \cdot n + \frac{1}{4} \cdot n \cdot c$$

$$8n - 8 = (3+c) \cdot n$$

$$(5-c)\, n = 8$$

n, c are integers.

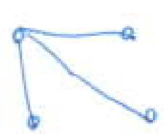
Only solutions are with $n \mid 8$

So $n = 4,\ c = 3$

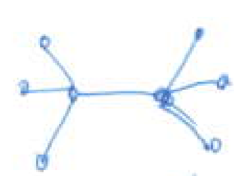
$n = 8,\ c = 4$.

$n = 2,\ c = 1$ → doesn't count, since the fixed degree shouldn't be 1.

Ans 1



Ans 2



6.1 Yes, there are exactly 2 vertices with odd degree.

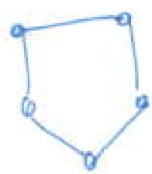
R6 − R9 − R6 − R5 − R4 − R7 − R8 − R5 − R2 − R1 − R2 − R3

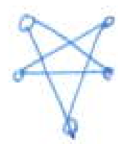
6.2. G has an Eulerian trail.

Start at v_1, do an Eulerian tour of G_1 (ending at v_1), cross $v_1 - v_2$, do an Eulerian tour of G_2 ending at v_2.

6.4.

(a)

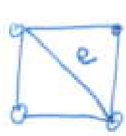
G. 

$\bar{G}$ 

(b)

G. 

(c)

G. 

6.8

(a). If we introduce double edges between every two ~~v~~ vertices that had an edge in G, we get a (multi-) graph where every vertex has even degree, so the new graph is Eulerian. This gives the desired walk on the original graph.

(b) Introducing triple edges, the degrees get tripled, so the desired walk exists if and only if the original Graph is Eulerian.