Trees

Graph Theory (Fall 2011)
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1 Some basic definitions

Let $G = (V, E)$ be a graph.

Definition 1 (Degree)**.** *The* degree *of a vertex $v \in V$, denoted $d(v)$, is the number of $e \in E$ that are incident on v.*

Lemma 2.

$$\sum_{v \in V} d(v) = 2|E|.$$

Proof. Count the number of $(v, e) \in V \times E$ such that e is incident on v. $\square$

Definition 3 (Walk)**.** *A* walk *in G is a sequence of vertices $v_0, \ldots, v_k$ of G such that for each i, $\{v_i, v_{i+1}\}$ is an edge of G. k is called the length of the walk. This walk is said to be "from v_0 to v_k".*

Definition 4 (Path)**.** *A* path *in G is a walk $v_0, \ldots, v_k$ such that $v_i \neq v_j$ for all $i \neq j$.*

Definition 5 (Cycle)**.** *A* cycle *in G is a walk $v_0, \ldots, v_k$ such that $k \geq 3$, $v_0 = v_k$, and for all $i \neq j$ with $\{i, j\} \neq \{0, k\}$, we have $v_i \neq v_j$.*

A cycle in G of length k is called a k-cycle of G. Note the condition $k \geq 3$; there is no such thing as a cycle of length 2.

Definition 6 (Acyclic Graph)**.** *A graph is called* acyclic *if there are no cycles in G.*

Definition 7 (Connected)**.** *G is* connected *if for every $u, v \in V$, there is a path in G from u to v.*

Definition 8 (Subgraph and Induced Subgraph)**.** *A graph $G' = (V', E')$ is called a* **subgraph** *of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E \cap \binom{V'}{2}$.*

In case $E' = E \cap \binom{V'}{2}$, then G' is called the **induced subgraph** *on V' of G, denoted $G[V']$ or $G|_{V'}$.*

The subgraph relation forms a partial order on all graphs. Under this partial order, we may speak of a maximal element of a collection of graphs. This enables the following definition.

Definition 9 (Connected Component)**.** *A connected component of G is a maximal connected subgraph of G.*

2 Trees

Definition 10 (Tree)**.** *A* tree *is a* **connected**, **acyclic** *graph.*

Theorem 11. *Let $G = (V, E)$ be a tree with $|V| = n$. Then $|E| = n - 1$.*

Proof. By induction on n. The theorem is trivial for $n = 1$.

Pick any vertex $u \in V$. Let $e_1, e_2, \ldots, e_k$ be the neighbors of u in G. For $i \in [k]$, let V_i be the set of vertices $v \in V$ such that there is a path from u to v passing through edge e_i.

Observe that the V_i are pairwise disjoint; otherwise if $v \in V_i \cap S_j$, we could use the two paths from u to v (one via e_i and another via e_j) to construct a cycle in G (Make sure you can see this argument through!).

Observe also that $G|_{V_i}$ is a tree.

Let $n_i = |V_i|$ and let $m_i = |E \cap \binom{V_i}{2}|$ be the number of edges in $G|_{V_i}$. By the induction hypothesis, we have $m_i = n_i - 1$.

Then $V = \bigcup V_i \cup \{u\} = V$, and so $\sum n_i = n - 1$.

Also, $E = \bigcup \left(E \cap \binom{V_i}{2}\right) \cup \{e_1, \ldots, e_k\}$, and so $|E| = \sum m_i + k = \sum (n_i - 1) + k = \sum n_i = n - 1$. □

Now two quick corollaries, showing how delicately balanced the properties of connectedness and acyclicity are with respect to the number of edges in a tree.

Lemma 12. *Let G be a graph with n vertices and m edges. If $m > n - 1$, then G has a cycle.*

Proof. Suppose not. Let $G_1, G_2, \ldots, G_k$ be the connected components of G. Let n_i be the number of vertices of G_i, and let m_i be the number of edges of G_i. We have $\sum n_i = n$ and $\sum m_i = m \geq n$. Thus there must be some i such that $m_i \geq n_i$. This G_i is a connected graph, and since $m_i > n_i - 1$, it cannot be a tree (By Theorem 11). □

Lemma 13. *Let G be a graph with n vertices and m edges. If $m < n - 1$, then G is disconnected.*

Proof. Suppose G was connected.

Suppose G contains a cycle C. Pick any edge e of C and remove it from the graph G. Then G is still connected (because any path in the original G that uses e can be modified to use other edges of C and avoid e). We may keep doing this as long as G has a cycle. We end up with G being acyclic and connected, and thus a tree.

Now by Theorem 11, this resulting graph must have $n-1$ edges. But since we started with $m < n-1$ edges, and our operations only deleted edges, this is a contradiction. □

Using Theorem 11, we can also calculate the average degree of vertices in a tree. Let G be a tree with $|V| = n$. Then the average degree of G is given by: $\frac{1}{n} \sum_{v \in V} d(v) = \frac{1}{n} \cdot (2|E|) = 2\left(1 - \frac{1}{n}\right)$.

3 Spanning Trees

Definition 14 (Spanning Tree)**.** *Let G be a graph. A* spanning tree *of G is a subgraph $T = (V, E')$ (T has the same vertex set as G) which is a tree.*

Lemma 15. *Every connected graph G has a spanning tree.*

Proof. Take a minimal connected subgraph of G on the same vertex set as G. Observe that this subgraph is acyclic, and is thus a spanning tree of G.

Alternatively, take a maximal acyclic subgraph of G. Observe that this graph is connected, and is thus a spanning tree of G. □

This motivates a simple algorithm to find a spanning tree of a graph.

Algorithm to find a Spanning Tree of $G = (V, E)$

1. Start with $E' = \emptyset$.
2. While (V, E') is not connected:
   - Let $e \in E$ be any edge such that $(V, E' \cup \{e\})$ is acyclic.
     If such an e does not exist, then FAIL (G is disconnected, and thus does not have a spanning tree).
   - Set $E' := E' \cup \{e\}$.
3. $T = (V, E')$ is the desired spanning tree of G.

One could also design an algorithm which starts from E and keeps deleting edges, maintaining the property that the graph is connected. When this algorithm cannot proceed, what remains is a spanning tree of G.