Ramsey Theory

Graph Theory (Fall 2011)
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1 The Finite Ramsey Theorem

Theorem 1. *For every c, s, there exists n_0 such that for all $n \geq n_0$, for every c-coloring of the edges of K_n, there is a monochromatic K_s.*

Let us make some preliminary remarks about this theorem.

A c-coloring of the edges of K_n gives us c different n-vertex graphs (those given by edges of a given color). This theorem guarantees the existence of a K_s in one of these graphs. We know that one of these graphs has density at least $\frac{1}{c}$; however, this density is not enough to guarantee the existence of a K_s. The truth of this theorem fundamentally relies on the hypothesis that the edges of K_n are *partitioned* into c classes.

To highlight the contrast, consider the following theorem, which follows immediately from our study of Turan numbers.

Theorem 2. *For every c, s, there exists n_0 such that for all $n \geq n_0$, for every c-coloring of the edges of K_n, there is a monochromatic P_s.*

Proof. Some color has at least $\frac{1}{c}\binom{n}{2}$ edges. We also know that $\mathsf{ex}(n, P_s) = \frac{(s-1)}{2} \cdot n$. Thus for $n - 1 \geq c \cdot (s-1)$, we conclude that some color contains a P_s. $\square$

The proof of the Ramsey Theorem must use fundamentally different ideas.

The proof will be by induction. Actually, we will prove the following stronger statement, which is more amenable to induction.

Theorem 3. *For every c, for every nonnegative integers $s_1, \ldots, s_c$, there exists n_0 such that for all $n \geq n_0$, for every c-coloring of the edges of K_n, there is some $i \in [c]$ for which there is a monochromatic K_{s_i} of color i.*

Proof. We will prove this by induction. Let $R(s_1, \ldots, s_c)$ denote the smallest n_0 which has this property; we will show that it is finite.

Clearly $R(2, 2, \ldots, 2) = 2$. We now show that

$$R(s_1, \ldots, s_c) \leq R(s_1 - 1, s_2, \ldots, s_c) + R(s_1, s_2 - 1, s_3, \ldots, s_c) + \ldots \tag{1}$$

$$+ R(s_1, s_2 \ldots, s_{i-1}, s_i - 1, s_{i+1}, \ldots, s_c) + R(s_1, s_2, \ldots, s_{c-1}, s_c - 1). \tag{2}$$

Let n_0 denote the right hand side of the above inequality.

Let $n \geq n_0$, and consider any c-coloring of K_n. Let v be any vertex in K_n. Partition the remaining $n-1$ vertices of K_n according to the color of the edge from that vertex to v. By the pigeon hole principle, there must be some color $i \in [c]$ such that the set U of vertices u joined to v by an edge of color i has cardinality at least $R(s_1, s_2 \ldots, s_{i-1}, s_i - 1, s_{i+1}, \ldots, s_c)$.

By the induction hypothesis, U contains either a monochromatic s_j-clique of color j for some $j \neq i$, or else it contains an $(s_i - 1)$-clique. In the first case, we have found a monochromatic s_j-clique of color j in K_n, and in the other case, we have a monochromatic s_i clique of color i (which consists of the $(s_i - 1)$-clique just found by the induction hypothesis along with v). □

The Ramsey theorem for 2-colors can be stated just in terms of graphs (without mentioning coloring): For every s_1, s_2, every sufficiently large graph either contains a clique of size s_1 or an independent set of size s_2.

2 The Infinite Ramsey Theorem

Next we discuss the Ramsey theorem for infinite sets.

Theorem 4. *Let K_ω denote the complete graph on the vertex set $\mathbb{N}$. For every $c > 0$, if we color the edges of K_ω with c colors, then there must be an infinite monochromatic clique.*

The finite Ramsey theorem readily implies that there exist arbitrarily large monochromatic cliques. It is not clear how to make the jump from this to an infinite monochromatic clique. Even the method of proof of the finite Ramsey theorem, induction on the size of the clique, does not give us any hints for the infinite Ramsey theorem.

The subject of "proof theory" can formalize the statement that the infinite Ramsey theorem does not follow from the finite Ramsey theorem.

Proof. Start with any vertex v_0. There must be some color c_0 such that there are infinitely many v such that the edge (v_0, v) is colored c_0. Let S_0 be the set of those vertices v. By choice of c_0, S_0 is infinite.

All the action now shifts into S_0. Pick any vertex $v_1 \in S_0$. There must be some color c_1 such that there are infinitely many $v \in S_0$ such that (v_1, v) is colored c_1. Let S_1 be the set of those vertices v. By choice of c_1, S_1 is infinite.

Keep repeating this process. (Pick $v_i \in S_{i-1}$. Get color c_i and an infinite set S_i such that for each $v \in S_i$, (v_i, v) is colored c_i).

We get a sequence $(v_i, c_i, S_i)_{i=1}^{\infty}$ such that for every $i < j$, (v_i, v_j) is colored c_i.

Now there must be some color c^* such that $I = \{i \mid c_i = c^*\}$ is infinite. Then the desired monochromatic clique is $\{v_i \mid i \in I\}$, all of whose edges are colored c^*. □

3 Quantitative Aspects

One can use equation (1) to derive quantitative upper bounds for for $R(s_1, \ldots, s_c)$.

We do this in three cases of particular interest:

- Diagonal Ramsey numbers for 2-coloring $R(s,s)$: By induction, one can show that $R(s,t) \leq \binom{s+t-2}{s-1}$. Thus $R(s,s) \leq \binom{2s-1}{s} \leq O\left(\frac{1}{\sqrt{s}}4^s\right)$.

  Equivalently, this says that every graph on $\Omega(\frac{1}{\sqrt{s}}4^s)$ vertices contains either a clique or an independent set on s vertices.

  Coming up next, we will see Erdos's beautiful probabilistic argument showing that $R(s,s) \geq \sqrt{2}^s$.

- $R(3,k)$: The above bound shows that $R(3,k) \leq k^2$. We will see in a later class the stronger bound (due to Ajtai, Komlos, Szemeredi) $R(3,k) \leq (1+o(1)) \cdot \frac{k^2}{\log k}$. A delicate probabilistic construction of Kim shows that this is the correct behavior: $R(3,k) = (1+o(1))\frac{k^2}{\log k}$.

  Equivalently, this says that a triangle-free graph on $(1+o(1))\frac{k^2}{\log k}$ vertices contains an independent set of size k.

- Multicolor Ramsey numbers for K_3: By induction, one can show that $R(3,3,\ldots,3) \leq 3c!$.

  In words, this says that if $n \geq 3c!$, any c-coloring of the edges of K_n contains a monochromatic triangle. In the homework, you will show that $R(3,3,\ldots,3) \geq \exp(c)$.

3.1 Lower bound for $R(s,s)$

Theorem 5. *For each s, $R(s,s) \geq \Omega(s \cdot 2^{s/2})$.*

Proof. Let $n = \frac{1}{10}s \cdot 2^{(s-1)/2}$. Let $G = (V,E)$ be the random graph $G(n,1/2)$. This is the random graph with n vertices, such that each pair of vertices $\{u,v\}$ is an edge with probability $1/2$ (independently).

We will show that with positive probability, G does not contain an s-clique or an s-independent set.

For a fixed set S of s vertices, let B_S denote the event that S is either a clique or an independent set. Clearly $\Pr[B_S] = 2 \cdot 2^{-\binom{s}{2}}$.

Thus,

$$\Pr[\bigvee_{S \in \binom{V}{s}} B_S] \leq \binom{n}{s} \cdot 2 \cdot 2^{-\binom{s}{2}}.$$

By choice of n, this probability is at most 1. Therefore there is a graph G such that none of the B_S occur. This is the desired graph with no clique or independent set of size s. □

Probabilistic arguments can also be used to give lower bounds on $R(s_1,\ldots,s_c)$. You should try to work out these bounds. Sometimes these bounds are the best known, but not always. For example, the probabilistic argument only gives a weak lower bound of $R(3,3,\ldots,3) \geq c^{O(1)}$.