More Linear Algebra in Graph Theory

Graph Theory (Fall 2011)
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1 The number of trees

How many trees are there on the vertex set $\{1, \ldots, n\}$? The answer is given by Cayley's formula: n^{n-2}.

We will now consider a question that is vastly more general than this, and come up with a surprisingly elegant answer to it.

Let G be a graph. How many spanning trees does it have? If $G = K_n$, then this is precisely the question we asked earlier. Surely this question cannot have a simple answer; it depends so intricately on the possibly very complex edge structure of G.

Nevertheless, the amazing matrix-tree theorem of Kirchoff does give a very succinct and elegant answer to this question.

The answer is in terms of the Laplacian matrix of the graph G. For d-regular graphs we already encounted the Laplacian of G, defined to be $dI - A_G$. For general graphs the Laplacian of G is defined to be $D - A_G$, where D is the $V \times V$ diagonal matrix whose (v, v) entry equals the degree of the vertex v. It is easy to check that the Laplacian matrix is positive semidefinite, and has the $\mathbf{1}$ vector as an eigenvector[1] with eigenvalue 0.

Theorem 1. *Let L be the Laplacian of G. Let $0 = \lambda_1 \leq \lambda_2 \leq \ldots \lambda_n$ be its eigenvalues. Then the number of spanning trees of G equals:*

$$\frac{1}{n} \prod_{i=2}^{n} \lambda_i.$$

Proof. Notice that $\prod_{i=2}^{n} \lambda_i$ equals the $(n-1)$-th elementary symmetric polynomial of $\lambda_1, \ldots, \lambda_n$.

Thus $\prod_{i=2}^{n} \lambda_i$ equals $\sum_{v \in V} \det(L_v)$, where L_v is the $(n-1) \times (n-1)$ matrix obtained by removing the v-th row and v-th column from L.

Fix any $v_0 \in V$. We now focus on computing $\det(L_{v_0})$. We will show that it equals the number of spanning trees of G.

Let D be the $V \times E$ matrix with the following entries: for $e = \{u, v\} \in E$, the column of D corresponding to e has exactly two nonzero entries; $D_{u,e} = +1$ and $D_{v,e} = -1$ (here it will not matter which of the two vertices of e is taken to be u and which is taken to be v).

[1] There is no such characterization of the top eigenvalue of the adjacency matrix of a not-necessarily-regular graph. This is one of the reasons the Laplacian is a more natural linear-algebraic object associated with a graph than the adjacency matrix.

We have the following relationship between D and L:

$$DD^T = L.$$

Let D' be the matrix obtained by removing the v_0 row from D. Then

$$D'(D')^T = L_{v_0}.$$

Given this equality, we now compute $\det(L_{v_0})$. By the Cauchy-Binet formula[2],

$$\det(L_{v_0}) = \sum_{S \subseteq E, |S|=n-1} \det(D'_S)\det((D')^T_S) = \sum_{S \subseteq E, |S|=n-1} \det(D'_S)^2,$$

(where for $S \subseteq E$, we let D'_S denotes the submatrix of D' given by the columns of S).

The amazing fact is that $\det(D'_S)^2$ equals 1 precisely when the edges in S form a spanning tree of G, and otherwise $\det(D'_S)^2$ equals 0. You should try to prove this yourself; the main point is that in the expansion of the determinant $\det(D'_S)$, the nonzero terms correspond to matchings between $V \setminus \{v_0\}$ and S, and this has something to do with S being a spanning tree of G.

The amazing fact above immediately tells us that $\det(L_{v_0})$ equals the number of spanning trees of G, and this completes the proof. □

2 Unit-distance graphs and positive semidefinite matrices

The beautiful unit-distance problem of Erdos asks: For n distinct points $x_1, \dots, x_n \in \mathbb{R}^2$, what is the largest number of $(i, j) \in [n]^2$ such that x_i and x_j are at a distance of exactly 1 from each other?

By considering a $\sqrt{n} \times \sqrt{n}$ grid with spacing $1/c$, where c is an integer such that c^2 can be written as $a^2 + b^2$ for integer a, b in many different ways, we see that the number of unit distances can be slightly superlinear in n. Erdos conjectured that the number of unit distances is at most $O(n^{1+\epsilon})$ for every $\epsilon > 0$.

Call a graph G a unit distance graph if it can be realized via the following embedding: every vertex of G is drawn as a point in $\mathbb{R}^2$, and two vertices are adjacent if and only if the corresponding points are exactly a unit distance apart. In this language, the Erdos conjecture bounds the number of edges in a unit distance graph.

Is every graph a unit distance graph? One immediately realizes that K_4 is not a unit distance graph. In fact, a unit distance graph cannot contain K_4 as a subgraph. Thus every unit distance

[2] The Cauchy-Binet formula says that if $AB = C$, where A, B, C are $m \times n$, $n \times m$, $m \times m$ matrices, then

$$\det(C) = \sum_{S \subseteq [n], |S|=m} \det(A_S)\det(B_S),$$

where A_S, respectively B_S are the submatrices of A, respectively B, corresponding to the columns, respectively rows, indexed by S.

The $m = 1$ and $m = n$ cases of this equality are familiar; this provides an elegant interpolation between them.

graph is K_4-free, and by Turan's theorem the number of edges cannot be more than $\frac{2}{3}\binom{n}{2}$. This is nice to know, but still terribly weak.

To really get some mileage, we would like to use our bounds on Turan numbers for bipartite graphs. It is easy to see that $K_{2,3}$ is not a unit distance graph, and that every unit distance graph is $K_{2,3}$-free. Thus we conclude that the number of unit distances is at most $O(n^{3/2})$.

It is known that the number of unit distances cannot be more that $O(n^{4/3})$. This has a short proof using the crossing number inequality. Let $x_1, \ldots, x_n$ be points in $\mathbb{R}^2$. Draw unit circles around each x_i. Consider the graph that is induced by this drawing: the vertices are the n points $x_1, \ldots, x_n$, and there is an edge between x_i and x_j if there is an arc of a circle in this drawing joining x_i to x_j (with no other point x_k on this arc). Then the number of edges m in this graph is at least as large as twice the number of unit distances. Finally, the number of crossings in this drawing of the graph is at most $O(n^2)$, since there are n circles and two circles intersect in at most 2 points. By the crossing number inequality:

$$O(n^2) \geq \frac{m^3}{n^2},$$

and so $m \leq O(n^{4/3})$.

A natural variant of the above problem is to consider unit distance graphs in higher dimension: how many unit distances can be there between n points in $\mathbb{R}^d$? Already in 4 dimensions there can be $\Omega(n^2)$ unit distances! Yet K_6 is not a unit distance graph in 4 dimensions.

Linear algebraic tools come in when we study unit distance graphs in high dimension. Some very simple considerations involving positive semidefinite matrices shows that *every* graph is a unit distance graph in n dimensions. Furthermore, if G is regular, then G is a unit distance graph in $n-1$ dimensions.

Theorem 2. *Every graph G is a unit distance graph in n dimensions. If G is regular, then G is a unit distance graph in $n-1$ dimensions.*

Proof. Since I is a positive definite matrix, there is a sufficiently small ϵ such that $I - \epsilon A$ is positive definite. Thus there exists an $n \times n$ real matrix C such that $I - \epsilon A = CC^T$. Let the rows of C be $(x_v)_{v \in V}$. Then

$$\|x_v\|^2 = (I - \epsilon A)_{v,v} = 1,$$

and

$$\|x_v - x_w\|^2 = \|x_v\|^2 + \|x_w\|^2 - 2\langle x_v, x_w \rangle = 1 + 1 - 2(I - \epsilon A)_{v,w},$$

which equals $2 - 2\epsilon$ if v, w are adjacent, and equals 2 otherwise. Scaling the vectors x_v by a constant gives us the desired unit distance embedding.

Now let G be regular. Let L be the Laplacian of G. Since L is positive semidefinite and singular, there is an $n \times (n-1)$ real matrix C such that $L = CC^T$. The rows of C, scaled appropriately, give the desired unit distance embedding. □

The above embeddings in fact have the (much much stronger) property that the distance between any pair of points not adjacent in the graph is the same. Thus there are only 2 possible distances between any two points of the embedding!

A nice result of Maehara and Rodl states that if G has maximum degree d, then G can be represented as a unit distance graph in $2d$ dimensions.