Bipartite Graphs and Matchings

Graph Theory (Fall 2011)
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Definition 1. *A graph $G = (V, E)$ is called* bipartite *if there is a partition of V into two disjoint subsets: $V = L \cup R$, such every edge $e \in E$ joins some vertex in L to some vertex in R.*

When the bipartition $V = L \cup R$ is specified, we sometimes denote this bipartite graph as $G = (L, R, E)$.

Theorem 2. *$G = (V, E)$ is bipartite if and only if G has no cycles of odd length.*

Proof. We first prove the easier 'only if' direction. Suppose $G = (L, R, E)$ is bipartite and let $v_0, \ldots, v_{k-1}, v_k = v_0$ be a cycle in G. Suppose $v_0 \in L$. Then $v_1 \in R$, since $\{v_0, v_1\} \in E$. Then $v_2 \in L$, since $\{v_1, v_2\} \in E$. Continuing this way, we see that if i is odd, then $v_i \in R$, and if i is even then $v_i \in L$. Thus, since $v_k = v_0 \in L$, this implies that k is even, and thus the cycle is of even length.

We now prove the 'if' direction. Suppose G has no cycles of odd length. We may assume that G is connected (otherwise we consider the connected components of G).

Pick a vertex $u_0 \in V$. For every vertex $v \in V$, let p_v be any path from u_0 to v, and let d_v be its length. Set $L = \{v \in V \mid d_v \text{ is even}\}$ and let $R = \{v \in V \mid d_v \text{ is odd}\}$. Clearly $V = L \cup R$ is a partition of V. We now show that (L, R, E) is bipartite.

If not, then there is some $\{u, v\} \in E$ such that both $u, v \in L$ or both $u, v \in R$. In either case, there is a closed walk in G given by $p_u, \{u, v\}, p_v$ (from u_0 to u, then u to v, then v to u_0), whose total length is $d_u + d_v + 1$, which is odd. Since G has a closed walk of odd length, then G also has a cycle of odd length (Why?). This is a contradiction.

Thus $G = (L, R, E)$ is bipartite. □

Definition 3 (Matchings and Perfect Matchings)**.** *Let $G = (V, E)$ be a graph. A* matching *in G is a set of edges $M \subseteq E$ such that for every $e, e' \in M$, there is no vertex v such that e and e' are both incident on v.*

The matching M is called perfect *if for every $v \in V$, there is some $e \in M$ which is incident on v.*

If a graph has a perfect matching, then clearly it must have an even number of vertices. Furthermore, if a bipartite graph $G = (L, R, E)$ has a perfect matching, then it must have $|L| = |R|$.

For a set of vertices $S \subseteq V$, we define its set of neighbors $\Gamma(S)$ by:

$$\Gamma(S) = \{v \in V \mid \exists u \in S \text{ s.t. } \{u, v\} \in E\}.$$

Our goal now is to get a characterization of when a bipartite graph has a perfect matching.

Suppose $G = (L, R, E)$ has a perfect matching M. Then for every set $S \subseteq L$, we have that $|\Gamma(S)| \geq |\Gamma(S) \cap M| \geq |S|$ (since every vertex of S is matched in M).

It turns out that the converse of this is also true. This gives us the nice consequence that whenever a bipartite graph does not have a perfect matching, there is a short proof that demonstrates this.

Theorem 4 (Hall's Marriage Theorem). *Let $G = (L, R, E)$ be a bipartite graph with $|L| = |R|$. Suppose that for every $S \subseteq L$, we have $|\Gamma(S)| \geq |S|$. Then G has a perfect matching.*

Proof. By induction on $|E|$. Let $|E| = m$. Suppose we know the theorem for all bipartite graphs with $< m$ edges.

We take cases depending on whether there is slack in the hypothesis or not.

- **Case 1: For every $S \subseteq L$ with $0 < |S| < |L|$, we have $|\Gamma(S)| \geq |S| + 1$.** In this case, pick any edge $e = \{u, v\} \in E$ and include it in the matching M. Apply the induction hypothesis to the induced bipartite graph on $L \setminus \{u\}$ and $R \setminus \{v\}$: this gives us a matching M' between $L \setminus \{u\}$ and $R \setminus \{v\}$. The desired matching between L and R is $M' \cup \{e\}$.
- **Case 2: For some $S \subseteq L$ with $0 < |S| < |L|$, we have $|\Gamma(S)| = |S|$.** In this case, first apply the induction hypothesis to the the induced bipartite graph on S and $\Gamma(S)$. This gives us a matching M' between S and $\Gamma(S)$.

  Now let $T = L \setminus S$ and $U = R \setminus \Gamma(S)$. Applying the induction hypothesis again, we get that the induced bipartite graph on T and U has a perfect matching M'' (Why? Suppose there was some $S' \subseteq T$ such that $|\Gamma(S') \cap U| < |S'|$. Then $\Gamma(S \cup S') = |\Gamma(S') \cap U| + |\Gamma(S)| < |S'| + |S| = |S' \cup S|$, a contradiction).

  The desired matching between L and R is $M' \cup M''$.

□

How does one go about finding a perfect matching in a bipartite graph G (assuming one exists)? The following greedy algorithm does not achieve this (Why not?): Maintain a matching, starting with the empty matching, and keep adding edges from E to it as long as possible.

In the next section, we see another proof of the marriage theorem. This proof has the advantage of also giving an efficent algorithm to find a perfect matching in G whenever it exists. In case a perfect matching does not exist, the algorithm finds a set $S \subseteq L$ such that $|\Gamma(S)| < |S|$.