Homework 3

Graph Theory (Fall 2011)
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1. Let G be a d-regular n-vertex graph and let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of its adjacency matrix.
   (a) What can you say about G if $\lambda_n = -d$?
   (b) Show that the size any independent set in G is at most:
   $$\frac{-\lambda_n}{d - \lambda_n} \cdot n.$$
   This is called the Hoffman bound.
   (c) How do parts (a) and (b) relate to each other?

2. Let G be a connected d-regular n-vertex graph. Let λ_2 be the second eigenvalue of the adjacency matrix of G. Let $f : V_G \to \mathbb{R}$ be a function with $\sum_v f(v) = 0$ and $\sum_v f(v)^2 = 1$.
   (a) Show that $\max_{u,v \in V_G}(f(u) - f(v)) \geq \frac{C}{\sqrt{n}}$, for some universal constant C.
   (b) Let u and v be vertices witnessing the above inequality. Suppose ℓ is the distance between u and v. Show that
   $$\langle f, Lf \rangle \geq \Omega(\frac{1}{n\ell}),$$
   where L is the Laplacian of G.
   (c) Thus show that in any graph $\lambda_2 \leq d - \frac{1}{n^2}$.
   (d) Define a lazy random walk on a graph to be the following process. v_0 is a fixed vertex; for each $k \geq 0$, v_{k+1} is chosen as follows: with probability $1/2$, $v_{k+1} = v_k$, and with probability $1/2$, v_{k+1} is a uniformly random neighbor of v_k.
   Use the bound on λ_2 to show that for any vertex w, the lazy random walk visits w within $O(n^2 \log n)$ steps with probability $1 - o(1)$.

3. Read the proof of Cheeger's inequality from the notes.

4. Let G be an n vertex graph. Let $\alpha(G)$ denote the size of the largest independent set in G. Let $\chi(G)$ denote the chromatic number of G. Recall that for every graph G, we have $\chi(G) \geq \frac{n}{\alpha(G)}$.
   Show that if G is a Cayley graph $\mathsf{Cay}(\Gamma, S)$ (for some n-element group Γ and some $S \subseteq \Gamma$), then:
   $$\chi(G) \leq O\left(\frac{n}{\alpha(G)} \cdot \log n\right).$$

5. Recall the result of Ajtai-Komlos-Szemeredi stating that if G is an n-vertex triangle-free graph with maximum degree at most d, then G has an independent set of size $\Omega(n \cdot \frac{\log d}{d})$.

   Let $\epsilon > 0$ be a constant. Now suppose G is a d-regular graph which has $nd^{2-\epsilon}$ triangles. Use the above result to show that G has an independent set of size $\Omega_\epsilon(n \cdot \frac{\log d}{d})$.