Homework 2

Graph Theory (Fall 2011)
Rutgers University
Swastik Kopparty

Due Date: November 10, 2011.

1. Let $f(c)$ equal the c-color Ramsey number $R(3, 3, \ldots, 3)$ (namely, the smallest integer n such that any c-coloring of the edges of K_n contains a monochromatic triangle).

   Show that $(f(c_1 + c_2) - 1) \geq (f(c_1) - 1) \cdot (f(c_2) - 1)$. Thus show that $f(c) \geq 5^{\lfloor c/2 \rfloor}$.

   Recall that the proof of the Ramsey theorem shows that $f(c) \leq O(c!) = \exp(c \log c)$. Today it is unknown what the correct behavior for $f(c)$ is.

2. Let $p \in (0, 1)$ be fixed. We will study the chromatic number of $G(n, p)$.

   Recall that $G(n, p)$ has clique number $(2 + o(1)) \cdot \log_{\frac{1}{p}} n$. Thus $G(n, p)$ has independence number $(2 + o(1)) \cdot \log_{\frac{1}{1-p}} n$, and so $G(n, p)$ has chromatic number at least

   $$(1 + o(1)) \cdot \frac{n}{2 \log_{\frac{1}{1-p}} n}.$$

   Show that the greedy coloring algorithm colors $G(n, p)$ with $O\left(\frac{n}{\log n}\right)$ colors with high probability.

3. Let s, t be fixed. For which $\alpha \in (0, 2)$ is $G(n, n^{-\alpha})$ almost sure to have no $K_{s,t}$? For which $\alpha \in (0, 2)$ is $G(n, n^{-\alpha})$ almost sure to have fewer than $n/2$ copies of $K_{s,t}$?

   Use this to prove a lower bound on the Turan number $\mathsf{ex}(n, K_{s,t})$. For which s, t does this match the upper bound we saw in class?

4. (a) Let $H = (L, R, E)$ be any bipartite graph with $|L| = |R| = n$ and $|E| = e$. Show that there is a subgraph $H' = (L, R', E')$ of H with at least $e/2$ edges such that every vertex in R' has degree at least $e/2n$ in H'.

   (Aside: Must there be a subgraph $H'' = (L'', R'', E'')$ with at least $e/100$ edges such that every vertex in L'' and every vertex in R'' has degree at least $e/100n$?)

   (b) Let T be a tree with k edges. Let v be a vertex of T.

   Prove, by induction on T, that in every bipartite graph $H = (L, R, E)$ with $|L| = |R| = n$ and $|E| = e$,

   $$|\{\varphi \in \hom(T, H) \mid \varphi(v) \in R\}| \geq c_k \left(\frac{e}{n}\right)^k \cdot n,$$

   where $c_0 = 1$, and $c_i = c_{i-1} \cdot 2^{-i}$ for each $i > 0$.

   Hint: Use H' from part (a).

   (c) Show that any graph G with n vertices and edge density β, we have:

   $$\frac{\hom(T, G)}{n^{k+1}} \geq c_k \beta^k.$$

(d) **Sidorenko's conjecture for trees:** Use the previous statement to show that in any graph G with n vertices and edge density β, we have:

$$\frac{\hom(T, G)}{n^{k+1}} \geq \beta^k.$$

5. **(Fun with Homomorphisms)**

(a) Write $F \to G$ if $\hom(F, G) > 0$. Give examples of graphs F, G such that $F \to G$ and $G \to F$.

(b) A *core* of a graph G is a graph F such that $F \to G$, $G \to F$, and every homomorphism from F to F is an isomorphism. Show that every graph G has a core.

(c) Suppose F_1 and F_2 are graphs such that for all graphs G, $F_1 \to G$ if and only $F_2 \to G$. What can you say about F_1 and F_2?

(d) Suppose G_1 and G_2 are graphs such that for all graphs F, $F \to G_1$ if and only if $F \to G_2$. What can you say about G_1 and G_2?

(e) Suppose $\hom(F_1, G) = \hom(F_2, G)$ for all graphs G. What can you say about F_1 and F_2?

(f) Suppose $\hom(F, G_1) = \hom(F, G_2)$ for all graphs F. What can you say about G_1 and G_2?