Homework 1

Graph Theory (Fall 2011)
Rutgers University
Swastik Kopparty

Due Date: October 6, 2011.

Answer any 5 questions.

1. Let $G = (V, E)$ be an undirected graph. Let M be the $V \times E$ matrix (the rows correspond to the vertices and the columns correspond to the edges), where the column corresponding to the edge $\{u, v\}$ has a $+1$ in row u and a -1 in row v (the order here does not matter). Let $D : \mathbb{R}^V \to \mathbb{R}^E$ be the linear transformation given by $x \mapsto x \cdot M$ (here x is a row vector).

   Using the linear transformation D, give new proofs for all the theorems about trees and spanning trees from the notes.

   Also prove that if G is an acyclic graph with n vertices and m edges, then G has $n - m$ connected components.

2. Let $G = (V, E)$ be a graph. A c-coloring of G is a map $C : G \to \{1, 2, \ldots, c\}$ such that for every edge $\{u, v\} \in E$, $C(u) \neq C(v)$.

   Suppose Δ is the maximum vertex degree in G. Prove that G has a $(\Delta + 1)$-coloring.

   For every integer Δ, give an example of a connected graph which has maximum vertex degree Δ and which has no Δ-coloring[1].

3. Suppose $G = (L, R, E)$ is a bipartite graph where every vertex has degree d. Prove that G has a perfect matching.

4. Let T be a tree on n vertices. Show that T contains a vertex v such that each connected component of the graph $T \setminus \{v\}$ has at most $n/2$ vertices.

5. A regular polyhedron is a polyhedron where all faces are bounded by the same number of edges, and all vertices have the same number of incident edges.

   Every polyhedron naturally gives rise to a planar graph (with the same vertex, edge and face structure).

   Using Euler's formula on this planar graph, show that there are only 5 regular polyhedra.

   This theorem is at least 3000 years old!

6. In this problem, we will see a more illuminating proof of Euler's formula.

   Let $G = (V, E)$ be a connected planar graph (here $E \subseteq \binom{V}{2}$). Let F be the set of faces of a planar drawing of G.

   Let $\tilde{E} \subseteq V^2$ be the set of ordered pairs arise from edges in E (for every $\{u, v\}$ in E, we have both (u, v) and (v, u) in $\tilde{E}$). Thus $|\tilde{E}| = 2|E|$.

[1] A fundamental theorem of Brooks says that there are no other examples!

Let $A = \mathbb{R}^V$. Let $B = \{\mathbf{b} \in \mathbb{R}^{\tilde{E}} \mid b_{(u,v)} = -b_{(v,u)}\}$. Let $C = \mathbb{R}^F$. Note that $\dim(A) = |V|$, $\dim(B) = |E|$ and $\dim(C) = |F|$.

Now consider the following linear transformations.

- $$D_1 : A \to B,$$

  given by: $D_1(\mathbf{a}) = \mathbf{b}$, where for every $(u, v) \in \tilde{E}$, we have $b_{(u,v)} = a_v - a_u$.

- $$D_2 : B \to C,$$

  given by $D_2(\mathbf{b}) = \mathbf{c}$, where for every face f, if $v_0, v_1, \ldots, v_k = v_0$ is the closed walk which traverses f in the clock-wise direction,

  $$c_f = \sum_{i=0}^{k-1} b_{(v_i, v_{i+1})}.$$

(a) What is the kernel of D_1?

(b) Show that the image of D_1 equals the kernel of D_2. This has two parts: first show that $D_2 \circ D_1 = 0$; then show that for every $\mathbf{b}$ with $D_2(\mathbf{b}) = 0$, there exists some $\mathbf{a}$ such that $\mathbf{b} = D_1(\mathbf{a})$. *This second part is where the planarity is used crucially! It fails if we considered graphs drawn on other surfaces.*

(c) Show that the image of D_2 equals $\{\mathbf{c} \in C \mid \sum_{f \in F} c_f = 0\}$.

(d) Conclude, using the rank-nullity theorem for D_1 and D_2, that $\dim(A) - \dim(B) + \dim(C) = 2$, and thus $|V| - |E| + |F| = 2$.