Expander Graphs

Graph Theory (Fall 2011)
Rutgers University
Swastik Kopparty

Throughout these notes G is a d-regular graph.

1 The Spectrum

Let A_G be the adjacency matrix of G. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of A_G.

Sometimes we will also be interested in the Laplacian matrix of G. This is defined to be $L_G = D - A_G$, where D is the diagonal matrix where D_{vv} equals the degree of the vertex v. For d-regular graphs, $L_G = dI - A_G$, and hence the eigenvalues of L_G are $d - \lambda_1$, $d - \lambda_2, \ldots, d - \lambda_n$.

Lemma 1.

- $\lambda_1 = d$.
- $\lambda_2 = \lambda_3 = \ldots = \lambda_k = d$ *if and only if* G *has at least* k *connected components.*

Proof. For the first part, verify that the vector $\mathbf{1}$ is an eigenvector with eigenvalue d.

For the second part, let $C_1, \ldots, C_r$ be all the connected components of G. Then $1_{C_1}, \ldots, 1_{C_r}$ are all mutually orthogonal eigenvectors with eigenvalue d. Thus if G has at least k connected components, then $\lambda_2 = \ldots = \lambda_r = d$.

Suppose there is an eigenvector f not in the span of $1_{C_1}, \ldots, 1_{C_r}$ with eigenvalue d. Let C_i be a component on which that eigenvector is not constant. Let $v \in C_i$ be a vertex such that $f(v)$ is maximum. Since f has eigenvalue d, $df(v) = \sum_{u \in \Gamma(v)} f(u)$, and by maximality this implies that $f(u) = f(v)$ for each $u \in \Gamma(v)$. Repeating this, we conclude that f must be constant on the component C_i, a contradiction. □

In particular, connected graphs have $\lambda_2 < d$. We now study the notion of expansion, where λ_2 is less than d by a significant amount; this can be thought of as a strong form of connectedness.

2 Expansion

Definition 2 (Eigenvalue Expansion)**.** *We say a* d*-regular graph is a* λ *eigenvalue expander if* $\lambda_2 \leq \lambda$.

We say a d*-regular graph is a* λ *absolute eigenvalue expander if* $|\lambda_2|, |\lambda_n| \leq \lambda$.

If we have a family of graphs with n tending to ∞ and with d constant, we informally call these graphs expander graphs if they are all λ eigenvalue expanders for some constant $\lambda < d$. Similarly we call these graphs absolute expander graphs if they are all λ absolute eigenvalue expanders for some constant $\lambda < d$.

Having made this definition, we now show that families of expander graphs (and absolute expander graphs) exist.

Theorem 3. *For every $d \geq 3$, there exists $\lambda < d$ such that for all sufficiently large n, there exists an n-vertex d-regular λ absolute eigenvalue expander graph.*

The proof is deferred to the end of these notes. We first see why this notion is useful.

3 Properties of Expanders

Many combinatorial properties of a graph can be expressed in terms of the eigenvalues of the adjacency matrix. In the case of eigenvalue expanders, this connection becomes very clean and powerful.

Let $G = (V_G, E_G)$ be a d-regular n-vertex graph with eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Let $v_1 = \frac{1}{\sqrt{n}}\mathbf{1}, v_2, \ldots, v_n$ be an orthonormal basis consisting of eigenvectors for $\lambda_1, \ldots, \lambda_n$ respectively.

Let $v : V_G \to \mathbb{R}$. Then v can be written as $a_1 v_1 + \ldots + a_n v_n$, where $a_i = \langle v, v_i \rangle$. We have $\|v\|^2 = \sum_{i=1}^n a_i^2$.

Note that

$$\langle v, A_G v \rangle = \sum_{x \in V_G} \sum_{y \in \Gamma(x)} v(x)v(y),$$

$$\langle v, L_G v \rangle = d \sum_{x \in V_G} v(x)^2 - \sum_{x \in V_G} \sum_{y \in \Gamma(x)} v(x)v(y) = \frac{1}{2} \sum_{x \in V_G} \sum_{y \in \Gamma(x)} (v(x) - v(y))^2.$$

We can also express these quantities in terms of the eigenvalues. We have $A_G v = \sum a_i \lambda_i v_i$ and $L_G v = \sum a_i (d - \lambda_i) v_i$. Therefore $\langle v, A_G v \rangle = \sum a_i^2 \lambda_i$ and $\langle v, L_G v \rangle = \sum a_i^2 (d - \lambda_i)$.

Utilizing such expressions, we can derive a large number of combinatorial properties of expander graphs.

Theorem 4 (Edge Expansion)**.** *Suppose G is a λ eigenvalue expander. Then for every $S \subseteq V_G$ with $|S| \leq n/2$ we have*

$$e(S, S^c) \geq \frac{d - \lambda}{2} |S|.$$

Proof. We first want to express the quantity $e(S, S^c)$ in terms of the adjacency matrix. Let $\mathbf{1}_S : V_G \to \mathbb{R}$ be the indicator function of the set S. Then

$$e(S, S^c) = \langle \mathbf{1}_S, A_G(\mathbf{1} - \mathbf{1}_S) \rangle = d|S| - \langle 1_S, A_G 1_S \rangle.$$

Let $\not\vDash_S = \sum_{i=1}^n a_i v_i$.

We have $a_1 = \langle \mathbf{1}_S, \frac{1}{\sqrt{n}}\mathbf{1} \rangle = \frac{|S|}{\sqrt{n}}$. Further, we have $\sum a_i^2 = |\mathbf{1}_S|^2 = |S|$.

So

$$\begin{aligned}
\langle 1_S, A_G 1_S\rangle &= \langle \sum a_i v_i, \sum a_i \lambda_i v_i\rangle \\
&= \sum a_i^2 \lambda_i \\
&= \frac{|S|^2}{n} d + \sum_{i=2}^{n} a_i^2 \lambda_i \\
&\leq \frac{|S|^2}{n} d + \lambda(|S| - \frac{|S|^2}{n}) \\
&= \frac{|S|^2}{n}(d-\lambda) + \lambda |S|.
\end{aligned}$$

Thus

$$e(S, S^c) = (d-\lambda)\left(|S| - \frac{|S|^2}{n}\right) \geq \frac{d-\lambda}{2}|S|.$$

□

Theorem 5 (Expander Mixing Lemma)**.** *Suppose G is a λ absolute eigenvalue expander. Then for every $S, T \subseteq V_G$, we have*

$$|e(S,T) - |S||T|\frac{d}{n}| \leq \lambda\sqrt{|S||T|}.$$

Proof. Let $1_S = \sum a_k v_i$ and $1_T = \sum b_i v_i$. We have $a_1 = \frac{|S|}{\sqrt{n}}$ and $\sum a_i^2 = |S|$, and similarly $b_1 = \frac{|T|}{\sqrt{n}}$ and $\sum b_i^2 = |T|$.

The quantity of interest $e(S,T)$ can be expressed spectrally as follows:

$$\begin{aligned}
e(S,T) &= \langle 1_S, A_G 1_T\rangle \\
&= \langle \sum_i a_i v_i, \sum_j b_j \lambda_j v_j\rangle \\
&= \sum_{i,j} a_i b_j \lambda_j \langle v_i, v_j\rangle \\
&= \sum_i a_i b_i \lambda_i \\
&= a_1 b_1 d + \sum_{i=2}^{n} a_i b_i \lambda_i \\
&= \frac{d|S||T|}{n} + \sum_{i=2}^{n} a_i b_i \lambda_i.
\end{aligned}$$

The absolute value of second term in this expression can be bounded (using the Cauchy-Schwarz inequality) by:

$$\lambda \cdot \left(\sum_{i=2}^{n} a_i^2\right)^{1/2} \left(\sum_{i=2}^{n} b_i^2\right)^{1/2} \leq \lambda\sqrt{|S||T|}.$$

This completes the proof. □

Theorem 6 (Vertex Expansion)**.** *Suppose G is a λ eigenvalue expander. Let $S \subseteq V_G$. Then the neighborhood of S is large:*

$$|\Gamma(S)| \geq |S| \cdot \frac{1}{\frac{\lambda^2}{d^2} + \left(1 - \frac{\lambda^2}{d^2}\right)\frac{|S|}{n}}.$$

In this theorem, if $\lambda < (1-\Omega(1))d$, and $|S| < (1-\Omega(1))n$, then $|\Gamma(S)| = |S| \cdot (1+\Omega(1))$, which is called "vertex expansion".

Proof. Let $1_S = \sum_i a_i v_i$. Then $a_1 = \frac{|S|}{\sqrt{n}}$ and $\sum a_i^2 = |S|$.

$|\Gamma(S)|$ is not a quantity that can be captured precisely in terms of the spectrum of G (unlike $e(S, S^c)$). However it can be lower bounded using the following observation: If f is a function, then $|\mathsf{supp}(f)| \geq \frac{\|f\|_1^2}{\|f\|_2^2}$.

We apply this to $f = A \cdot 1_S$. Then $\mathsf{supp}(f)$ clearly equals $|\Gamma(S)|$.

We now compute $\|f\|_1$ and $\|f\|_2$. $\|f\|_1$ equals $d \cdot |S|$.

$$\|f\|_2 = \|A 1_S\|_2 = \|\sum_i a_i \lambda_i v_i\|_2 \leq \left(|S|^2 \frac{d^2}{n} + \lambda^2 |S| \left(1 - \frac{|S|}{n}\right)\right)^{1/2}.$$

Thus

$$\begin{aligned}|\Gamma(S)| &\geq \frac{d^2|S|^2}{\frac{|S|^2 d^2}{n} + \lambda^2 |S| \frac{n-|S|}{n}} \\ &= \frac{n|S|}{|S| + \frac{\lambda^2}{d^2}(n - |S|)} \\ &= |S| \cdot \frac{1}{\frac{\lambda^2}{d^2} + \left(1 - \frac{\lambda^2}{d^2}\right)\frac{|S|}{n}}.\end{aligned}$$

This completes the proof. □

Theorem 7 (Rapid Mixing of Random Walks)**.** *Suppose G is a λ absolute eigenvalue expander. Let $v_0 \in V_G$ be any vertex.*

Let μ be the probability distribution of the k-th step of the random walk on G starting at v_0. Then the statistical distance of μ from uniform is bounded as follows:

$$\|\mu - U\|_1 \leq \sqrt{n}\left(\frac{\lambda}{d}\right)^k,$$

where U is the uniform distribution on V_G.

Proof. Let P be the matrix $\frac{1}{d} \cdot A_G$. If $\pi \in \mathbb{R}^{V_G}$ is a probability distribution on the vertices of G, then $P\pi$ is the probability distribution of the vertex v, obtained by picking a vertex u according to π and then letting v be a random neighbor of that vertex.

Let π_0 be the probability distribution supported on v_0. Then probability distribution μ of the k-th step of the random walk equals $P^k \pi_0$.

Let $\pi_0 = \sum_i a_i v_i$. Note that $a_1 = \frac{1}{\sqrt{n}}$, and $\sum_i a_i^2 = \sum_{x \in V_G} \pi_0(x)^2 = 1$.

Then

$$\mu = P^k \pi_0 = \sum_i a_i \left(\frac{\lambda_i}{d}\right)^k v_i = U + \sum_{i=2}^{n} a_i \left(\frac{\lambda_i}{d}\right)^k v_i.$$

Therefore

$$\begin{aligned} \|\mu - U\|_2 &= \left(\sum_{i=2}^{n} a_i^2 \left(\frac{\lambda_i}{d}\right)^{2k}\right)^{1/2} \\ &\le \left(\frac{\lambda}{d}\right)^k. \end{aligned}$$

Thus

$$\|\mu - U\|_1 \le \sqrt{n} \left(\frac{\lambda}{d}\right)^k.$$

□

The next theorem says that absolute eigenvalue expanders have no large independent sets and hence they have large chromatic number.

Theorem 8 (Independent Sets/Chromatic Number)**.** *The largest independent set in G has size at most $\frac{-\lambda_n}{d-\lambda_n} \cdot n$. Thus the chromatic number of G is at least $\frac{d-\lambda_n}{-\lambda_n}$.*

You will prove this in your homework.

4 Limits on eigenvalue expansion

How good an eigenvalue expander can a d-regular graph be? This question is answered by a theorem of Alon and Boppana.

Theorem 9 (Alon-Boppana, simpler version)**.** *Let G be a d-regular graph which is a λ absolute eigenvalue expander. Then $\lambda \ge 2\sqrt{d-1} - o(1)$ (where the $o(1)$ term tends to 0 as the number of vertices n tends to infinity).*

$2\sqrt{d-1}$ is not just any old number, it has very deep origins. The infinite adjacency matrix of the infinite d-regular tree (this is the ultimate expander graph) has its spectral radius equal to $2\sqrt{d-1}$.

Proof. The main idea is to study the eigenvalues through the trace of a high even power of A_G.

We have:

$$\mathsf{Tr}(A_G^{2k}) = \sum_{i=1}^{n} \lambda_i^{2k} \le d^{2k} + (n-1)\lambda^{2k}.$$

On the other hand, $\mathsf{Tr}(A_G^k)$ counts the number of closed walks of length $2k$ in G.

It is easy to see that the number of such walks is at least as large as n times the number of such walks in the infinite d-regular tree. The latter can be counted by elements σ of $[d] \times ([d-1] \cup \beta)^{2k-1}$

with exactly k β's, having the property that any prefix of σ of length t has at most $t/2$ $\beta's$. This can be counted easily in terms of Catalan numbers; we get that the total number of closed walks in G of length $2k$ is at least:

$$n \cdot \left(\frac{1}{k+1} \binom{2k}{k} d \cdot (d-1)^{k-1} \right).$$

Thus

$$d^{2k} + (n-1)\lambda^{2k} \geq n \cdot \left(\frac{1}{k+1} \binom{2k}{k} d \cdot (d-1)^{k-1} \right).$$

Taking $k = \omega(1)$ and simplifying, we get:

$$\lambda \geq 2\sqrt{d-1} - o(1).$$

□

The full theorem of Alon-Boppana shows that if G is a λ eigenvalue expander (not necessarily an absolute eigenvalue expander), even then $\lambda \geq 2\sqrt{d-1} - o(1)$. This is a more delicate fact and the proof is more serious.

5 Combinatorial versions of eigenvalue expansion

Define the edge expansion of the graph G, denoted $h(G)$, by:

$$h(G) = \min_{S \subseteq V_G, |S| \leq n/2} \frac{e(S, S^c)}{|S|}.$$

$h(G)$ being large means that most edges incident on vertices of S are between S and S^c. By Theorem 4, we have that if G is a λ eigenvalue expander, then $h(G) \geq \frac{d-\lambda}{2}$. In particular, if $\lambda = d - \Omega(1)$, then $h(G) \geq \Omega(1)$.

We now see the converse of this statement: if $h(G) \geq \Omega(1)$, then G is a λ eigenvalue expander for $\lambda \geq d - \Omega(1)$.

Theorem 10 (Cheeger-Alon-Milman)**.** *Suppose $\lambda \in \mathbb{R}$ satisfies*

$$h(G) \leq \sqrt{2d \cdot (d-\lambda)}.$$

Then G is a λ eigenvalue expander.

Proof. Let $f : V_G \to \mathbb{R}$ be a unit norm eigenvector of the eigenvalue λ_2. Thus $\sum_{\{x,y\} \in E_G} (f(x) - f(y))^2 = \langle f, Lf \rangle = d - \lambda_2$.

We will use f to find a set $S \subseteq V_G$, with $|S| \leq n/2$, such that $e(S, S^c) \leq \sqrt{2d \cdot (d-\lambda_2)}|S|$ (this will show that $h(G) \leq \sqrt{2d \cdot (d-\lambda_2)}$, as desired).

First write $f = f^+ - f^-$, where f^+ and f^- are nonnegative valued. Let V^+ and V^- be their supports. Note that for each $x \in V^+$,

$$Lf^+(x) \leq Lf(x) = (d-\lambda_2)f(x) = \lambda_2 f^+(x),$$

and for each $x \in V^-$,

$$Lf^-(x) \geq -Lf(x) = -(d-\lambda_2)f(x) = -(d-\lambda_2)f^-(x).$$

Thus $\langle f^+, Lf^+\rangle \leq (d-\lambda_2)\|f^+\|_2^2$ and $\langle f^-, Lf^-\rangle \leq (d-\lambda_2)\|f^-\|_2^2$.

Now at least one of the functions f^+ and f^- has support of size at most $n/2$. Let us call that function g. We just showed that $\langle g, Lg\rangle \leq (d-\lambda_2)\|g\|_2^2$. We will now find the desired S as a subset of the support of g. It will be through a delicate probabilistic argument.

Pick $a \in [0, \max_{x\in V_G} g(x)]$ uniformly at random. Let $T = \{x \in V_G \mid g(x)^2 \geq a\}$ (note that T is a random set, with $0 < |T| \leq n/2$).

Then for an edge $\{x,y\} \in E_G$, the probability that it gets counted in $e(T,T^c)$ equals the probability that a lies in between $g(x)^2$ and $g(y)^2$, which equals $|g(x)^2 - g(y)^2|$. Thus:

$$\begin{aligned}
\mathbb{E}[e(T,T^c)] &= \sum_{\{x,y\}\in E_G} |g(x)^2 - g(y)^2| \\
&= \sum_{\{x,y\}\in E_G} |(g(x)-g(y))| \cdot |(g(x)+g(y))| \\
&\leq \left(\sum_{\{x,y\}\in E_G} |(g(x)-g(y))|^2\right)^{1/2} \cdot \left(\sum_{\{x,y\}\in E_G} |(g(x)+g(y))|^2\right)^{1/2} \\
&\leq \langle g, Lg\rangle^{1/2} \cdot \left(2d\|g\|_2^2\right)^{1/2} \\
&\leq \sqrt{2d(d-\lambda_2)}\|g\|_2^2.
\end{aligned}$$

On the other hand, $\mathbb{E}[|T|] = \sum_{x\in V_G} g^2(x) = \|g\|_2^2$.

Thus

$$\frac{\mathbb{E}[e(T,T^c)]}{\mathbb{E}[|T|]} \leq \sqrt{2d(d-\lambda_2)}.$$

This implies (verify this!) that with positive probability, $\frac{e(T,T^c)}{|T|} \leq \sqrt{2d(d-\lambda_2)}$, as desired. $\square$

Absolute eigenvalue expansion can also be characterized in terms of combinatorial properties of a graph. A fundamental theorem of Bilu and Linial roughly says that a graph is an absolute eigenvalue expander if and only if it satisfies the conclusion of the expander mixing lemma.