Local Structure: Forbidden Subgraphs III

Graph Theory (Fall 2011)
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We will now see the powerful Erdos-Stone-Simonovits theorems, which determine the asymptotic behavior of a Turan numbers in many situations.

Theorem 1 (Erdos-Stone-Simonovits). *Let F be any graph. Let r be the chromatic number of F. Then*

$$\mathsf{ex}(n, F) = (1 - \frac{1}{r-1})\binom{n}{2} + o(n^2).$$

If $r = 2$ (i.e., if F is bipartite), then this theorem does not tell us much. But for $r > 2$, this theorem not only tells us that $\mathsf{ex}(n, F)$ is $\Theta(n^2)$, but also what the leading constant is.

Even better, we have:

Theorem 2 (Erdos-Stone-Simonovits). *Let $\mathcal{F}$ be any finite set of graphs. Let r be the minimum chromatic number of $F \in \mathcal{F}$. Then*

$$\mathsf{ex}(n, \mathcal{F}) = (1 - \frac{1}{r-1})\binom{n}{2} + o(n^2).$$

We will derive both these theorems from the following theorem, that guarantees the existence of a fairly big complete r-partite graph in every dense enough graph. Let $K_r(t)$ denote the complete r-partite graph with t vertices in every part.

Theorem 3 (Erdos-Stone). *For every integer $r \geq 2$, every $\epsilon > 0$ and every sufficiently large n, if G is a graph on n vertices with $(1 - 1/r - 1 + \epsilon) \cdot \binom{n}{2}$ edges, then G contains a copy of $K_r(t)$, for $t = \Omega_{r,\epsilon}(\log n)$.*

Given the Erdos-Stone theorem, the Erdos-Stone-Simonovits theorems follow easily (in fact, all we need quantiatively about t in the Erdos-Stone theorem is that $t = \omega(1)$).

Let $\mathcal{F}$ be a finite collection of graphs, and let r be the minimum of the chromatic numbers of $F \in \mathcal{F}$. If G is a graph with $(1 - \frac{1}{r-1} + \epsilon)\binom{n}{2}$ edges, then the Erdos-Stone theorem implies that it contains a copy of $K_r(t)$. Now $K_r(t)$ contains a copy of every r-chromatic graph with at most t vertices. Thus, for sufficiently large n, we have $t > |V_F|$ for every $F \in \mathcal{F}$, and so for some $F \in \mathcal{F}$ we have a copy of F in G.

The other direction follows easily by considering the complete $r-1$-partite graph, with parts of size $\frac{n}{r-1}$.

We now prove the Erdos-Stone theorem.

Proof. Let $G = (V, E)$. We start with the special case when every vertex has degree at least $(1 - \frac{1}{r-1} + \epsilon) \cdot n$.

The proof in this case is by induction on r.

For the base case $r = 2$, we basically saw this argument in the Kovary-Sos-Turan bound for $\mathsf{ex}(n, K_{t,t})$. G will contain a $K_{t,t}$ if there are t vertices which have t common neighbors. Let us count the number of pairs $(u, \{v_1, \ldots, v_t\})$ where $v_1, \ldots, v_t$ are all neighbors of u. Clearly this number equals $\sum_u \binom{d(u)}{t}$. If this value exceeds $t \cdot \binom{n}{t}$, then there would be some $\{v_1, \ldots, v_t\}$ such that $v_1, \ldots, v_t$ have at least t common neighbors, and this would give us the desired $K_{t,t}$.

Now since $d(u) \geq \epsilon n$ for each u, this condition will be satisfied if

$$n \cdot \binom{\epsilon n}{t} \geq t \cdot \binom{n}{t},$$

which will hold if $n \geq \frac{t}{\epsilon^t}$, and this is true for $t = \Omega_\epsilon(\log n)$. Thus G must contain a $K_{t,t}$.

For general r, the induction hypothesis guarantees the existence of a copy of $K_{r-1}(T)$ in G, for some $T = \Omega_{r,\epsilon}(\log n)$. Let $U_1, \ldots, U_{r-1} \subseteq V$ be the $r - 1$ parts of a copy of $K_{r-1}(T)$ in G (thus $|U_i| = T$ and every vertex in U_i is adjacent to every vertex in U_j, for each $i \neq j$). We will now use $U_1, \ldots, U_{r-1}$ to find a copy of $K_r(t)$ in G.

Our goal is to find $W_1, \ldots, W_{r-1}$ such that $W_i \subseteq U_i$, $|W_i| = t$ for each i, such that there are at least t vertices of G adjacent to all the vertices in $\bigcup_{i=1}^{r-1} W_i$. Then $W_1, \ldots, W_{r-1}$ along with t vertices adjacent to all of $\bigcup_{i=1}^{r-1} W_i$ will form the $K_r(t)$.

Let N denote the cardinality of the set $A = \{(v, W_1, \ldots, W_{r-1})$ such that for each i, (i) $|W_i| = t$, (ii) $W_i \subseteq U_i$, (iii) v is adjacent to every vertex of W_i $\}$. We will show that $N \geq \binom{T}{t}^{r-1} \cdot t$; this will imply the existence of $(W_1, \ldots, W_{r-1})$ for which there are at least t such v's, and this gives the desired $K_r(t)$.

Let $f(v)$ denote the number of neighbors that v has in $\bigcup_{i=1}^{r-1} U_i$. We have

$$\sum_{v \in V} f(v) \geq T \cdot (r-1) \cdot \left(\left(1 - \frac{1}{r-1} + \epsilon \right) n - T(r-1) \right).$$

Thus,

$$\mathbb{E}_{v \in V}[f(v)] \geq T \cdot (r-1) \cdot \left(1 - \frac{1}{r-1} + \frac{\epsilon}{2} \right).$$

Since for every $v \in V$, $f(v) \leq T \cdot (r-1)$, we have that

$$\Pr_{v \in V} \left[f(v) > T \cdot (r-1) \cdot \left(1 - \frac{1}{r-1} + \frac{\epsilon}{4} \right) \right] \geq \frac{\epsilon}{4}.$$

Let S be the set of such v, namely $S = \{v \mid f(v) > T \cdot (r-1) \cdot \left(1 - \frac{1}{r-1} + \frac{\epsilon}{4}\right)\}$. We thus have $|S| \geq \frac{\epsilon}{4} \cdot n$. Each $v \in S$ must have at least $\frac{\epsilon}{4} \cdot T$ neighbors in every U_i (this is the crucial step where the "$1 - \frac{1}{r-1}$" gets used). Thus for each $v \in S$, the the number of $(W_1, \ldots, W_{r-1})$ for which $(v, W_1, \ldots, W_{r-1}) \in A$ is at least $\binom{\frac{\epsilon}{4}T}{t}^{r-1}$. Thus

$$N \geq \frac{\epsilon}{4} \cdot n \cdot \binom{\frac{\epsilon}{4}T}{t}^{r-1}.$$

If $t = \frac{\epsilon}{8}T$, then

$$N \geq \frac{\epsilon}{4} 2^{c_\epsilon T \cdot (r-1)} \cdot n \geq n \geq \binom{T}{t}^{r-1} \cdot t,$$

(where we need $T < \frac{1}{2}\frac{1}{r-1} \cdot \log n$). This completes the proof of the Erdos-Stone theorem in the special case where all vertices have high degree.

Finally, we show how to reduce the general case to the above special case.

We start with G with n vertices and $> (1 - \frac{1}{r-1} + \epsilon) \cdot \binom{n}{2}$ edges. We will show that G has a subgraph G' with $n' \geq \frac{1}{2}\sqrt{\epsilon} n$ vertices and with each vertex having degree at least $(1 - \frac{1}{r-1} + \epsilon/2) \cdot n'$. By the above special case, we find that G', and hence G, has $K_r(t)$ as a subgraph for some $t = \Omega(\log n)$.

To find such a subgraph G', we do the following: Start with G and keep removing vertices as follows. Suppose we currently have a subgraph on m vertices. If there is a vertex of degree at most $(1 - \frac{1}{r-1} + \epsilon/2)m$, delete it.

Suppose this process continues until we are left with s vertices. Then the total number of edges remaining is at least

$$\left(1 - \frac{1}{r-1} + \epsilon\right)\binom{n}{2} - \left(1 - \frac{1}{r-1} + \frac{\epsilon}{2}\right) \cdot (n + (n-1) + \ldots + (s+1)),$$

which is at least $\frac{\epsilon}{2}\frac{n^2 - 3n}{2}$. But this cannot be larger than $\binom{s}{2}$, and so $s \geq \frac{1}{2}\sqrt{\epsilon} n$.

This completes the proof of the Erdos-Stone theorem. □