Local Structure: Forbidden Subgraphs II

Graph Theory (Fall 2011)
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We continue considering Turan numbers. As we will see, Turan numbers can have very different behaviors.

1 Turan numbers for trees

We begin with some more graphs for which we know the exact Turan numbers.

Lemma 1. *Let F be the $(k+1)$-vertex star (i.e., one vertex of degree k and all other vertices with degree 1). We have*

$$\mathsf{ex}(n, F) = \frac{(k-1)}{2} \cdot n.$$

Proof. Obvious. The extremal graph is the disjoint union of k-cliques. □

Lemma 2. *Let P_k be the path of length k (P_k has $k+1$ vertices). We have*

$$\mathsf{ex}(n, P_k) = \frac{(k-1)}{2} \cdot n.$$

Proof. Again, the extremal graph is the disjoint union of k-cliques.

Now consider a graph G with n vertices and $> \frac{k-1}{2} \cdot n$ edges. By successively removing any vertex with degree $< \frac{k-1}{2}$, we are left with a nonempty graph G' with n' vertices and $> \frac{k-1}{2} \cdot n'$ edges, where every vertex has degree $\geq (k-1)/2$.

We will find a path of length k in G'. Without loss of generality, assume that G' is connected (otherwise we can focus on any connected component of G' with the largest ratio of edges to vertices).

Now consider a maximum length path $v_0, \ldots, v_t$ in G'. If $t \geq k$, then G' and hence G contains a P_k.

Otherwise, $t \leq k-1$. By maximality of the path, all the neighbors of v_0 are in the path, and all the neighbors of v_t are in the path.

If v_0 is adjacent to v_t, then $v_0, \ldots, v_t, v_0$ is a cycle, and hence $v_i, v_{i+1}, \ldots, v_t, v_0, \ldots, v_{i-1}$ is also a path of maximum length, and therefore we deduce that all the neighbors of v_i are also in the path. Thus $v_0, \ldots, v_t$ are all the vertices of G', and hence the number of edges of $G' = \binom{t+1}{2} \leq (k-1)/2 \cdot n'$, a contradiction.

Thus assume that v_0 is not adjacent to v_t. Since v_0 and v_t both have degree $\geq (k-1)/2$ and all their neighbors are amongst $v_1, \ldots, v_{t-1}$, there must exist some v_i, v_{i+1} such that v_i is adjacent to v_t and v_{i+1} is adjacent to v_0. This gives us a cycle $v_0, v_1, \ldots, v_i, v_t, v_{t-1}, \ldots, v_{i+1}, v_0$. We thus conclude, as before, that $v_0, \ldots, v_t$ are all the vertices of G', and get a contradiction.

Therefore t must be at least k. □

For general trees, we have the following.

Lemma 3. *Let T be a tree with $k+1$ vertices. Then*

$$\frac{k-1}{2} \cdot n \leq \mathsf{ex}(n, T) \leq (k-1) \cdot n.$$

Proof. The lower bound comes from the disjoint union of k-cliques.

The upper bound comes by first reducing to the case where every vertex has degree at least $k-1$, and then showing that T can be embedded in in any such graph greedily. □

The Erdos-Sos conjecture states that for every tree T on $k+1$ vertices, $\mathsf{ex}(n, T) = \frac{k-1}{2} \cdot n$. Recently Ajtai-Komlos-Simonovits-Szemeredi showed that the conjecture holds for all sufficiently large k.

2 Complete Bipartite Graphs

We saw some examples of graphs with Turan number $\Theta(n^2)$ and other examples of graphs with Turan number $\Theta(n)$.

Theorem 4 (Erdos-Renyi-Sos)**.**

$$\mathsf{ex}(n, K_{2,2}) \geq \frac{n + n\sqrt{4n-3}}{4}$$

Furthermore, for infinitely many n,

$$\mathsf{ex}(n, K_{2,2}) = \frac{n + n\sqrt{4n-3}}{4}$$

Proof. Notice that a graph is $K_{2,2}$ free if and only if for every two vertices u, v, their neighborhoods $N(u), N(v)$ have no more than one vertex in their intersection.

Let $G = (V, E)$ be an n-vertex graph with no $K_{2,2}$. We know that for every $u \neq v$, $|N(u) \cap N(v)| \leq 1$. Thus if we let:

$$S = \{(z, \{u, v\}) \mid z, u, v \in V, \{z, u\}, \{z, v\} \in E\},$$

we have $|S| \leq \binom{n}{2}$.

On the other hand, we know that $|S| = \sum_z \binom{d(z)}{2}$ (where $d(z)$ denotes the degree of vertex z). Thus

$$\sum_z \binom{d(z)}{2} \leq \binom{n}{2}.$$

$$\sum_z \frac{d(z)^2 - d(z)}{2} \leq \binom{n}{2}.$$

We want to bound the number of edges, $m = \frac{1}{2}\sum d(z)$. We can lower bound the right hand side in terms of m (using Cauchy-Schwarz) by $2m^2/n - m$.

Simplifying, we get

$$m \leq \frac{n + n\sqrt{4n-3}}{4}. \tag{1}$$

This theorem is completely tight! One can construct graphs which have $\Theta(n^{3/2})$ edges and no $K_{2,2}$ as follows. We are looking for a graph where the neighborhoods of vertices intersect in at most one point. A familiar example of a collection of sets which intersect in at most one point is the collection of lines. Motivated by this, we can construct a graph based on lines in the projective plane over a finite field $\mathbb{F}_p$. The vertices are equivalence classes of $\mathbb{F}_p^3 \setminus \{(0,0,0)\}$, where (a,b,c) is considered equivalent to (x,y,z) if there is a nonzero λ with $(a,b,c) = \lambda(x,y,z)$. The edges are given by: (a,b,c) is adjacent to (x,y,z) if $ax + by + cz = 0$ (this respects the equivalence classes). This graph has $(p^3-1)/(p-1) = p^2+p+1$ vertices, each vertex has degree $(p^2-1)/(p-1) = p+1$, and there is no $K_{2,2}$. One can check that that this gives equality in Equation (1). □

In fact, this argument generalizes to $K_{r,s}$ quite easily to yield:

Theorem 5 (Kovary-Sos-Turan)**.** *For every r, s, we have*

$$\mathsf{ex}(n, K_{r,s}) \leq (\frac{s-1}{2})^{1/r} \cdot n^{2-1/r}.$$

In particular,

$$\mathsf{ex}(n, K_{r,s}) \leq O(n^{2-1/\min(r,s)}).$$

Proof. As in the earlier proof, we use the fact that a graph is $K_{r,s}$-free if and only if for every r vertices $u_1, \ldots, u_r$, their neighborhoods $N(u_1), \ldots, N(u_r)$ have at most $s-1$ vertices in their intersection.

Start with a graph G with n vertices, m edges, and This leads to the following inequality:

$$\sum_z \binom{d(z)}{r} \leq (s-1) \cdot \binom{n}{r}.$$

Simplifying in terms of $m = \frac{1}{2}\sum d(z)$,

$$n \cdot \binom{2m/n}{r} \leq (s-1) \cdot \binom{n}{r}.$$

This gives $m \leq (\frac{s-1}{2})^{1/r} \cdot n^{2-1/r}$. □

It is not known whether this theorem is tight for general r, s. For $r = 2$ this is tight (Furedi). For $r = s = 3$ this is tight (Brown). For $s > (r-1)!+1$ the exponent of n is tight (Kollar-Ronyai-Szabo, Alon-Ronyai-Szabo). The graphs that witness these results come from algebraic phenomena (and are very elegant).

3 Cycles

We now talk about $\mathsf{ex}(n, C_k)$. Notice that $C_3 = K_3$ and $C_4 = K_{2,2}$, and so their Turan numbers are $\frac{1}{2}\binom{n}{2}$ and $\Theta(n^{3/2})$ respectively. Because of the example $K_{n/2,n/2}$, we see that for k odd $\mathsf{ex}(n, C_k) \geq \frac{1}{2}\binom{n}{2}$.

In the next section we will see that for k odd, $\mathsf{ex}(n, C_k) \leq \frac{1}{2}\binom{n}{2} + o(n^2)$ (and thus $K_{n/2,n/2}$ is essentially the extremal example).

On the other hand, for k even, a result of Bondy and Simonovits shows that $\mathsf{ex}(n, C_k) \leq O(n^{1+2/k})$. This is known to be tight for $k = 4, 6, 10$. For other k, it is known that $\mathsf{ex}(n, C_k) \geq \Omega(n^{1+c/k})$ for some $c < 2$.