Partitions of an integer.

Note Title 9/23/2012

Basic problem: Given n, how many ways can we write n as a sum of other positive integers?

Many versions.

(1) $S_k(n) = \left\{ (a_1, \dots a_k) \;:\; \begin{array}{c} a_1 + \dots + a_k = n \\ a_i > 0 \end{array} \right\}$

we know $|S_k(n)| = \binom{n-1}{k-1}$

(2) $S(n) = \bigcup_{k \geq 1} S_k = \left\{ (a_1, \dots, a_k) : \begin{array}{c} \sum a_i = n \\ a_i > 0 \\ k > 0 \end{array} \right\}$

$|S(n)| = 2^{n-1}$

(3) $P_k(n) = \left\{ (a_1, \dots a_k) \;:\; \begin{array}{c} \sum a_i = n \\ a_1 \geq a_2 \geq \dots a_k > 0 \end{array} \right\}$

$\boxed{\text{Partitions of } n \text{ into } k \text{ parts.}}$

$|P_k(n)|$ is denoted $p_k(n)$.

(4) $P(n) = \bigcup_{k \geq 1} P_k(n)$

$\boxed{\text{Partitions of } n}$

$|P(n)|$ is denoted $p(n)$.

eg

$$p(5) = \{ (5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1) \}.$$

$p(5) = 7.$

Generating function for $p(n)$.

Every partition $n = a_1 + \cdots + a_k$ gives a unique sequence

$$(b_1, b_2, \cdots \quad)$$

s.t. $\sum_i i \cdot b_i = n.$

(b_i counts the # of occurrences of i in $(a_1, \cdots, a_k)$)

and vice versa.

So

$$F(x) = \sum_{n=0}^{\infty} p(n)\, x^n = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots) \cdots (1 + x^i + x^{2i} + \cdots) \cdots$$

So $\quad F(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$.

Very fundamental formula.

Leads to asymptotic formula for $p(n)$.

$$p(n) \sim \frac{1}{4\sqrt{3}} \cdot \frac{1}{n} \cdot e^{\pi \cdot \sqrt{\frac{2}{3}} \cdot \sqrt{n}}$$

via complex analysis.

(Later we will get $p(n) \ll e^{\pi\sqrt{\frac{2}{3}}\sqrt{n}}$

easily using real variable methods).

Generating function for $p_k(n)$? (k fixed).

Not so clear ...

Try it.

Fun fact

\# partitions of n into $\le k$ parts

= \# partitions of n where each part is $\le k$

Prove it. $**$

\# partitions of n into exactly k parts = ?

Find the generating fn of $p_k(n)$.
(k fixed)

$***$

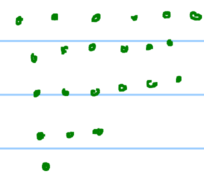
Hint for $**$

Ferrer's diagram of a partition.

Each partition

$n = a_1 + a_2 + \cdots a_k \qquad a_1 \geq \cdots \geq a_k > 0$

corresponds to a unique diagram with n dots, arranged in rows, with non-increasing rowlength.



$\leftarrow a_1$ dots

$\leftarrow a_2$ dots

$\vdots$

$***$ Answer: $\dfrac{x^k}{\prod_{i=1}^{k} (1-x^i)}$

Asymptotics of $p_k(n)$.

For k fixed :

$$p_k(n) \sim \frac{n^{k-1}}{(k-1)!\, k!}$$

Rough idea: most representations of n as $a_1 + \cdots + a_k$ have distinct a_i

Proof :

$$p_k(n) \geq \frac{|S_k(n)|}{k!} = \frac{\binom{n-1}{k-1}}{k!}$$

$$\sim \frac{n^{k-1}}{(k-1)!\, k!}$$

$$p_k(n) \leq \frac{\left|S_k\left(n + \frac{k(k-1)}{2}\right)\right|}{k!} \sim \frac{n^{k-1}}{(k-1)!\, k!}$$

Each partition $n = a_1 + \cdots a_k$ gives a partition into distinct parts

$$n + \frac{k(k-1)}{2} = (a_1 + k-1) + (a_2 + k-2) + \cdots + (a_k + 0)$$

Complete the details ... $\square$

Fun fact

\# partitions of n into distinct parts

$=$

\# partitions of n into odd numbers

Proof Equivalent to showing:

$$(1+x)(1+x^2)(1+x^3)\cdots = \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdots$$

Observe:

$$(1-x)\cdot\prod_{i=0}^{\infty}\left(1+x^{2^i}\right) = 1$$

$$(1-x^3)\cdot\prod_{i=0}^{\infty}\left(1+x^{3\cdot 2^i}\right) = 1$$

$\vdots$

So done!

Wow!

Bijective proof?

Bounds for $p(n)$.

$$F(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$$

Plan: $\sum_{n=0}^{\infty} p(n) x^n = F(x)$.

$p(n) \geq 0 \quad \forall n$

So $\quad p(n) \leq \frac{F(x)}{x^n} \quad$ for every $x \geq 0$

Let us bound $F(x)$

$$\log F(x) = -\sum_{i=1}^{\infty} \log(1-x^i)$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{x^{ij}}{j} \quad (\text{if } x<1)$$

$$= \sum_{j=1}^{\infty} \frac{1}{j} \sum_{i=1}^{\infty} x^{ij}$$

$$= \sum_{j=1}^{\infty} \frac{1}{j} \frac{x^j}{1-x^j}$$

$$\log p(n) \leq \log F(x) - n \log x$$

$$= \sum_{j=1}^{\infty} \frac{1}{j} \frac{x^j}{1-x^j} - n \log x$$

If $x = 1-\epsilon$, then

$$\begin{cases} x^j \approx 1-\epsilon j & \text{if } j \ll \frac{1}{\epsilon} \\ x^j \approx e^{-\epsilon j} & \text{if } j \gg \frac{1}{\epsilon} \\ \log x \approx -\epsilon & \end{cases}$$

all these approximations should be justified in the following steps.

$$\sum_{j=1}^{\infty} \frac{1}{j} \frac{x^j}{1-x^j} - n \log x$$

$$\approx \sum_{j=1}^{1/\epsilon} \frac{1}{j} \frac{1-\epsilon j}{\epsilon j} + \sum_{1/\epsilon}^{\infty} \frac{e^{-\epsilon j}}{j \cdot (1-e^{-\epsilon j})}$$

$$+ n\epsilon$$

$$\leq \frac{1}{\epsilon} \cdot \left(\sum_{j=1}^{\infty} \frac{1}{j^2} \right) + O(1) + n\epsilon$$

$$\leq \frac{1}{\epsilon} \cdot \frac{\pi^2}{6} + O(1) + n\epsilon.$$

Optimize ϵ to get $\epsilon = \sqrt{\frac{\pi^2}{6n}}$

So $\log p(n) \leq \pi \sqrt{\frac{2n}{3}}$

$$p(n) \leq \exp\left(\pi \sqrt{\frac{2n}{3}} \right)$$ Almost the true asymptotic!

Next Class

Set Systems

Matchings

⋮

Might see more applications of partitions
later in the course.

See you on Wednesday.