Homework 4

Combinatorics I (Fall 2012)
Rutgers University
Swastik Kopparty

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1. Let Σ be a finite set of cardinality q. Formulate and prove a version of the Sauer-Shelah lemma for subsets of Σ^n (The original Sauer-Shelah lemma deals with the case $q = 2$).

2. (a) Let S be a set of n points in $\mathbb{R}^m$. Let $\mathcal{F}$ be the family of subsets of S which are of the form $S \cap \{x \in \mathbb{R}^d \mid Q(x) = 0\}$ for some degree $\leq d$ polynomial $Q(X_1, \ldots, X_m) \in \mathbb{R}[X_1, \ldots, X_m]$.
   Show that the VC dimension of $\mathcal{F}$ is at most $\binom{m+d}{d}$. Thus deduce an upper bound on $|\mathcal{F}|$. Bonus: Show that this bound is tight.

   (b) Let $Q_1, \ldots, Q_k$ be polynomials in $\mathbb{R}[X_1, \ldots, X_m]$ of degree at most d. Let $\mathcal{G}$ be the family of all subsets of $[k]$ which are of the form $\{i \in [k] \mid Q_i(x) = 0\}$ for some $x \in \mathbb{R}^m$.
   For each $G \in \mathcal{G}$, let $Q_G(X_1, \ldots, X_m)$ be the polynomial $\prod_{i \notin G} Q_i(X_1, \ldots, X_m)$. Show that the Q_G, as G varies in $\mathcal{G}$, are linearly independent. Thus deduce an upper bound on $|\mathcal{G}|$.

3. Let L be a set of s nonnegative integers. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be such that $|A \cap B| \in L$ for distinct $A, B \in \mathcal{F}$. We will prove the uniform Ray-Chaudhuri Wilson inequality:

$$|\mathcal{F}| \leq \binom{n}{s}.$$

   (In class we showed the weaker inequality $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \ldots + \binom{n}{0}$).

   For $A \in \mathcal{F}$, let $f_A(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$ be the polynomial

$$f_A(X_1, \ldots, X_n) = \prod_{\ell \in L} \left(\sum_{i \in A} X_i - \ell \right).$$

   For $I \subseteq [n]$, let $v_I \in \{0,1\}^n$ be the indicator vector of I. For $J \subseteq [n]$, let $x_J : \{0,1\}^n \to \mathbb{R}$ be the function $x_J(v) = \prod_{j \in J} v_j$.

   (a) Suppose $f : \{0,1\}^n \to \mathbb{R}$ is such that $f(v_I) \neq 0$ for each $I \subseteq [n]$ with $|I| \leq t$. Show that the functions $(f \cdot x_J)_{|J| \leq t}$ are linearly independent.

   (b) Show that the $|\mathcal{F}| + \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{s-1}$ functions:
   - f_A with $A \in \mathcal{F}$,
   - $x_J \cdot (\sum_{i=1}^{n} x_i - k)$ with $|J| \leq s - 1$,

   are all linearly independent.

   (c) Deduce that $|\mathcal{F}| \leq \binom{n}{s}$.

4. We saw that with high probability, a random graph $G(N, 1/2)$ has no clique or independent set of size $\geq 2 \log N$. We will study a version of this on groups.

   Let Γ be a group, and let S be a subset of $\Gamma \setminus \{0\}$. Then the *Cayley graph* $Cay(\Gamma, S)$ is the (possibly directed) graph whose vertex set is Γ, where x is adjacent to y if $x - y \in S$.

   Now we take Γ to be the group $\mathbb{Z}_2^n$, and S to be a uniformly random subset of $\Gamma \setminus \{0\}$. Let $G = Cay(\Gamma, S)$. G is an undirected graph with $N = 2^n$ vertices. We will see that G has a clique of size $\geq \Omega(n \log n) = \Omega(\log N \log \log N)$.

   (a) How many k dimensional subspaces does $\mathbb{Z}_2^n$ have?

   (b) How many pairs (A, B) of k-dimensional subspaces of $\mathbb{Z}_2^n$ are there which intersect in a subspace of dimension exactly ℓ?

   (c) Let X be the number of k dimensional subspaces of $\mathbb{Z}_2^n$ that $S \cup \{0\}$ contains. Find the expectation and variance of X.

   (d) Prove that with high probability G has a clique of size $\Theta(n \log n) = \Theta(\log N \log \log N)$.

   The following is also true, but quite a bit deeper: With high probability, the largest clique and largest independent set of G are of size $\Theta(\log N \log \log N)$.

   I believe it is open whether there exists a Cayley graph of $\mathbb{Z}_2^n$ which has no clique or independent set of size $\geq \Omega(\log N)$.

Hints 2.(a). Associate to each $x = (x_1, \ldots, x_m) \in S$ the high dimensional vector consisting of all monomials of degree at most d evaluated at x. Bonus: Probabilistic method.