Homework 3

Combinatorics I (Fall 2012)
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Due Date: November 19, 2012.

1. Generalize the Erdos-Rado argument from class to show the following relationship between 2-color k-uniform hypergraph Ramsey numbers and 2-color $k-1$-uniform hypergraph Ramsey numbers.
$$R(s,s;k) \leq 2^{\binom{R(s,s;k-1)}{k}}.$$

2. Let $\delta \in (0, 1/2)$ be a constant. We want to find a subset C of $\{0,1\}^n$ of size as large as possible such that no two elements have Hamming distance $\leq \delta n$. This is a discrete "sphere packing" problem. Express all your answers in terms of the binary entropy function H.
   (a) Pick C to be a random set of size K. How large can you take K such that C has the desired property with probability at least 1/2?
   (b) Use the method of alterations to find a larger set C with the desired property. How large can you make C this way?
   (c) By a volume packing argument, show that no such C can have $|C| \geq 2^{(1-H(\delta/2)+o(1))n}$.
   (d) Now we consider a dual covering problem. We want to find a subset C of $\{0,1\}^n$ of size as small as possible such that every element of $\{0,1\}^n$ is within Hamming distance ϵn of some element of C.
   (e) How small a C can you find with this property?
   (f) Show that no such C can have $|C| \leq 2^{(1-H(\epsilon)-o(1))n}$.

3. A collection of sets $S_1, \ldots, S_m \subseteq [n]$ with $|S_i| = k$ for each i is called an (ϵ, δ)-sampler if for every $T \subseteq [n]$:
$$\Pr_{i \in [m]}\left[\left|\frac{|S_i \cap T|}{k} - \frac{|T|}{n}\right| > \epsilon\right] < \delta.$$
Show that there exists an (ϵ, δ)-sampler with $k = O(\frac{1+\log\frac{1}{\delta}}{\epsilon^2})$ and $m = O(\frac{n}{\delta^2})$.

4. Let S be a set of $\omega(q)$ lines in the projective plane over $\mathbb{F}_q$. We want to show that the union of all lines in S has size $(1-o(1)) \cdot q^2$.
   (a) Using inclusion-exclusion (more precisely, the Bonferroni inequality), show that that the union of all lines in S has size $\geq \frac{q^2}{2}$.
   (b) Let A denote the incidence matrix of the projective plane over $\mathbb{F}_q$ (ie. the 0/1 matrix with rows indexed by points and columns indexed by lines, and a 1 in entry (p, ℓ) denotes that line ℓ passes through point p). Compute $A^T A$.
   (c) Let 1_S be the indicator vector of S. Compute the norms $\|A1_S\|_1$ and $\|A1_S\|_2$, and use this to show that the union of all lines in S has size $(1-o(1)) \cdot q^2$.

Hints 3. Choose the S_i at random. First fix a T, and then take a union bound over all possible T. 4. Use the computation of $A^T A$ to help you compute $\|A 1_S\|_2$.