Homework 2

Combinatorics I (Fall 2012)
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1. Let Σ be a finite set, with $|\Sigma| = q$. For a sequence s, let $|s|$ denote its length.

   Let $s_1, s_2, \ldots, s_k$ be (nonempty) sequences composed of elements from Σ.

   (a) We say that $s_1, \ldots, s_k$ is *prefix-free* if there are no $i \neq j$ such that s_i is a prefix of s_j. If $s_1, \ldots, s_k$ is prefix-free, show that $\sum_{i=1}^{k} q^{-|s_i|} \leq 1$.

   (b) We say that $s_1, \ldots, s_k$ is *uniquely decodable* if the following holds: for every m, n, and every $(i_1, \ldots, i_m) \in [k]^m$ and $(j_1, \ldots, j_n) \in [k]^n$, if

   $$s_{i_1} \cdot s_{i_2} \cdot \cdots \cdot s_{i_m} = s_{j_1} \cdot s_{j_2} \cdot \cdots \cdot s_{j_n},$$

   then we must have $m = n$ and $i_\ell = j_\ell$ for each ℓ. (For two sequences a, b, we use $a \cdot b$ to denote the concatenation of the sequences a and b).

   Notice that if $s_1, \ldots, s_k$ is prefix-free, then it is uniquely decodable.

   Show that if $s_1, \ldots, s_k$ is uniquely decodable, then $\sum_{i=1}^{k} q^{-|s_i|} \leq 1$.

2. Look up Stirling's formula and its proof (Not to be submitted).

   For the rest of this problem, submit only the answers and do not submit your calculations.

   Use Stirling's formula to find an asymptotic formula for $\binom{n}{\alpha n}$, where $\alpha \in [0, 1]$ is constant, and $n \to \infty$.

   Express your answer in terms of the "binary entropy function" $H : [0, 1] \to [0, 1]$ defined by

   $$H(\alpha) = \alpha \log_2 \frac{1}{\alpha} + (1 - \alpha) \log_2 \frac{1}{1 - \alpha},$$

   and $H(0) = H(1) = 0$. Draw of graph of H. Note the special case of $\alpha = 1/2$ and think about why that seems reasonable in terms of tossing n independent coins.

   How large should c be for $\sum_{i=0}^{c} \binom{n}{i}$ to be $\Omega(2^n)$? How large should c be for $\sum_{i=0}^{c} \binom{n}{i}$ to be at least $\Omega(2^n/n^k)$ for a given $k > 0$?

3. **Tight cases of the LYM inequality**

   (a) Let $\mathcal{A} \subseteq \binom{[n]}{k}$. Let

   $$\partial_{k,r}\mathcal{A} = \{B \in \binom{[n]}{r} \mid \exists A \in \mathcal{A} \text{ with } B \subseteq A\}.$$

   Show that $\frac{|\partial_{k,r}\mathcal{A}|}{\binom{n}{r}} \geq \frac{|\mathcal{A}|}{\binom{n}{k}}$, with equality iff $\mathcal{A} = \binom{[n]}{k}$.

(b) Suppose $\mathcal{F} \subseteq 2^{[n]}$ is an antichain. Let $\mathcal{F}_k = \mathcal{F} \cap \binom{[n]}{k}$.
What can you say about the relationship between $\partial_{k,r}\mathcal{F}_k$ and $\partial_{k',r}\mathcal{F}_{k'}$?

(c) Use this to give another proof of the LYM inequality:

$$\sum_{k=1}^{n} \frac{|\mathcal{F}_k|}{\binom{[n]}{k}} \leq 1,$$

and show that equality holds iff $\mathcal{F}_k = \binom{[n]}{k}$ for some k.

4. Suppose $x_1, \ldots, x_n \in \mathbb{R}$ are such that $|x_i| \geq 1$ for all i. Erdos showed that:

$$|\{A \subseteq [n] \mid |\sum_{i \in A} x_i| < 1/2\}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

(This is the original Littlewood-Offord problem, which is slightly different from the version we saw in class. Make sure that you can adapt the proof from class to this setting also.)

In this problem, we will see a generalization of this to vectors in $\mathbb{R}^d$.

A chain $A_1 \subseteq A_2 \ldots \subseteq A_k$ in $2^{[n]}$ is called a symmetric chain if $|A_{i+1} = |A_i| + 1$ for each i, and $|A_k| = n - |A_1|$. A symmetric partition of $2^{[n]}$ is a partition consisting entirely of symmetric chains.

(a) Show that every symmetric partition of $2^{[n]}$ has exactly $\binom{n}{\lfloor n/2 \rfloor}$ parts. (In fact, a symmetric partition has 1 part of cardinality $n+1$, and for each $1 \leq i \leq n/2$ it has exactly $\binom{n}{i} - \binom{n}{i-1}$ parts cardinality $n + 1 - 2i$).

(b) Show how to construct a symmetric partition of $2^{[n]}$ given a symmetric partition of $2^{[n-1]}$. (Note that this now gives us another proof of Sperner's theorem.)

(c) Let $v_1, \ldots, v_n \in \mathbb{R}^d$ with $|v_i| \geq 1$ for each i (here $|\cdot|$ denotes the ℓ_2 norm). For $A \subseteq [n]$, let v_A denote $\sum_{i \in A} v_i$.

A family $\mathcal{A}$ of subsets of $2^{[n]}$ is called sparse if for all $A, B \in \mathcal{A}$, we have $|v_A - v_B| \geq 1$. A partition of $2^{[n]}$ is called pseudo-symmetric if it has exactly 1 part of cardinality $n+1$, and for each $1 \leq i \leq n/2$ it has exactly $\binom{n}{i} - \binom{n}{i-1}$ parts cardinality $n + 1 - 2i$.

Show (by induction on n) that $2^{[n]}$ has a pseudo-symmetric partition where each part is sparse.

(d) Deduce that

$$|\{A \subseteq [n] \mid |v_A| < 1/2| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

5. Let π be a random permutation from S_n. Let $C_k(\pi)$ be the number of cycles of π of length k. Show that

$$\mathbb{E}\left[\binom{C_k(\pi)}{i}\right] = \frac{1}{k^i \cdot i!}.$$

6. Let $A_1, \ldots, A_m \subseteq [n]$ and $B_1, \ldots, B_m \subseteq [n]$ be such that:

- $A_i \cap B_i = \emptyset$ for each i.
- $A_i \cap B_j \neq \emptyset$ for each $i \neq j$.

Show that

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

Hints

Problem 1: Consider the generating function

$$F(X) = \sum_{i=1}^{k} X^{|s_i|}.$$

What does the unique decodability property say about $F(X)$?

Problem 4(c) Without loss of generality, we may take v_n to be the vector $(\alpha, 0, 0, \ldots, 0)$ where $|\alpha| \geq 1$. For each part $\mathcal{A}$ of a given sparse pseudo-symmetric partition of $2^{[n-1]}$, produce up to two parts of a sparse pseudo-symmetric partition of $2^{[n]}$; these produced parts will depend on the first coordinates of the vectors $\{v_A \mid A \in \mathcal{A}\}$.

Problem 5: Use the exponential formula for permutations. Try to think of manipulations and substitutions into that formula (analogous to the $i = 1$ case that we saw in class).

Problem 6: This generalizes one of the inequalities we saw in class. Try to adapt the proof of that inequality.