Homework 1

Combinatorics I (Fall 2012)
Rutgers University
Swastik Kopparty

Due Date: September 24, 2012.

1. Let $(a_n)_{n=0}^{\infty}$ be a sequence of complex numbers. Show that a_n is a linear-recurrent sequence if and only if there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and polynomials $P_1(X), \ldots, P_k(X) \in \mathbb{C}[X]$ such that for every n, $a_n = \sum_{i=1}^{k} P_i(n)\lambda_i^n$.

2. Let $S(n, k)$ denote Stirling numbers of the second kind.

   (a) Give a bijection to show that $S(n, k) = kS(n-1, k) + S(n-1, k-1)$.

   (b) Consider the following generating function: $F_k(X) = \sum_{n \geq k} S(n, k)X^n$. Show that $F_k(X) = \frac{X^k}{\prod_{j=1}^{k}(1-jX)}$.

   (c) Show that
   $$S(n, k) = \sum_{\substack{a_1,\ldots,a_k \geq 0 \\ \sum a_i = n-k}} \prod_{j=1}^{k} j^{a_j}.$$

   (d) Give a direct bijective proof of the above equation.

3. Let $F(X) = (1-X)^k$. Use $F(X)$ to show that:

   - For $n < k$,
   $$\sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n = 0.$$
   - For $n \geq k$,
   $$\sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n$$
   is an integer divisible by $k!$.

   Observe that these two facts are obvious when the LHS is interpreted as counting something.

4. Define a sequence a_n by the rules:

   - $a_0 = 1$
   - for each $n \geq 1$,
   $$a_n = \sum_{k \geq 1} \sum_{\substack{i_1+\ldots+i_k = n-k \\ i_1,\ldots,i_k \geq 0}} \prod_{j=1}^{k} a_{i_j}.$$

   Using generating functions, find a formula for a_n.

   Having found this formula, identify the counting problem from class which clearly satisfies this recurrence.