HARPER'S THEOREM

Note Title 10/10/2012

Working in $\{0,1\}^n$

$wt(x) =$ # 1's in x

$\Delta(x,y) =$ # coordinates where x, y differ

<u>Simplicial order on $\{0,1\}^n$</u>

$x < y$ if $wt(x) < wt(y)$

OR $\left(\begin{array}{l} wt(x) = wt(y) \\ \text{and } \{i : x_i = 1\} < \{j : y_j = 1\} \end{array}\right.$

in lex order

For $S \subseteq \{0,1\}^n$

define $N(S) = \{ y : \exists x \in S \text{ with } \Delta(x,y) \le 1 \}$

<u>Harper's Theorem</u> $S, T \subseteq \{0,1\}^n$

<u>If</u> $|T| = |S|$ and T is a prefix of the simplex order, then

$$|N(T)| \le |N(S)|$$

<u>Proof</u>

Define C_i operation on subsets of $\{0,1\}^n$ as follows.

Fix i.

Let $H_0 = \{ x \in \{0,1\}^n : x_i = 0 \}$

$H_1 = \{ x \in \{0,1\}^n : x_i = 1 \}$

Given $S \subseteq \{0,1\}^n$

Let $S_0 = S \cap H_0$

$S_1 = S \cap H_1$

Define $S_0', S_1' \subseteq \{0,1\}^{n-1}$

$S_0' = S_0$ with ith coord dropped

$S_1' = S_1$ with ith coord dropped

Let $U_0' =$ Prefix of simplex order on $\{0,1\}^{n-1}$ of size $|S_0|$

View U_0' as a subset of H_0 (by adding a 0 coordinate)

Call this subset U_0

Let U_1' = Prefix of simplex order on $\{0,1\}^{n-1}$ of size $|S_1|$

View U_1' as a subset of H_1 (by adding a 1 coordinate)

Call this subset U_1

Set $U = U_0 \cup U_1$

Define $C_i(S) = U$.

<u>Claim</u>

$$|N(U)| \leq |N(S)|$$

<u>Proof</u>

$$|N(U)| = |N(U) \cap H_0| + |N(U) \cap H_1|$$

$$= |N(U_0') \cup U_1'| + |N(U_1') \cup U_0'|$$

$$|N(S)| = |N(S_0') \cup S_1'| + |N(S_1') \cup S_0'|$$

By induction on n,

$$|N(U_0')| \leq |N(S_0')|$$

Also $|U_1'| = |S_1'|$

NOW CRUCIAL FACT: $N(U_0')$ is a prefix of the simplex order

So one of $N(U_0')$ and U_1' contains the other.

So $|N(U_0') \cup U_1'| \leq |N(S_0') \cup S_1'|$

Similarly $|N(U_1') \cup U_0'| \leq |N(S_1') \cup S_0'|$.

So $N(U) \leq N(S)$.

Wrapping up the proof:

Operation C_i either reduces $\sum_{x \in S} wt(x)$,

or else it reduces $\left(- \sum_{x \in S} \sum_{j \text{ s.t. } x_j = 1} 2^{n-j} \right)$, or else

it does not modify S.

So repeated application of C_i's will stabilize.

Finally we get S s.t.
$C_i(S) = S \quad \forall i$.

Claim: Such an S is either:

① a Prefix of simplex order

② One other exception

(for which we can find $N(S)$ explicitly and verify Harper's Thm)

Proof: Suppose $y < x$ in simplex order, and $y \notin S$, $x \in S$. Let x_{-i} denote x with i^{th} coord dropped.

Then ① if $\exists\, i$ s.t. $x_i = y_i = 1$,

then $x_{-i} > y_{-i}$ in simplex order, $\Rightarrow C_i(S) \neq S$

② if $\exists\, i$ s.t. $x_i = y_i = 0$

then $x_{-i} > y_{-i}$ in simplex order $\Rightarrow C_i(S) \neq S$

③ So $x = y^c$.

So there can be at most one such x, y pair! Further, if $S \neq$ prefix of simplex order, we must have

$S = \{z : z \leq x\} - \{y\}$ in simplex order

where y = predecessor of x and $y = x^c$. $\Longrightarrow$ (By checking cases only one such S).

Exercise: For each n, find this exceptional $S \subseteq \{0,1\}^n$.

Find $N(S)$ for this example and verify Harper's theorem for this S.

(Hint: It depends on n being odd or even.)

Notes

1. Harper's theorem implies Kruskal-Katona

(For $S \subseteq \binom{[n]}{k}$, with $|S| = M$

$|\partial_u S|$ is minimized by prefix of the lex order on $\binom{[n]}{k}$.

Upper Shadow

2. Define $N_t(S) = \{ y : \Delta(x,y) \le t$ for some $x \in S \}$

Harper $\Rightarrow$ If $|S| \geq \binom{n}{0} + \cdots + \binom{n}{k}$

then $|N_t(S)| \geq \binom{n}{0} + \cdots + \binom{n}{k+t}$

In particular, if $|S| = \frac{1}{2} \cdot 2^n$ and $t = \epsilon n$

then $|N_t(S)| \geq (1 - \exp(-n)) \cdot 2^n$