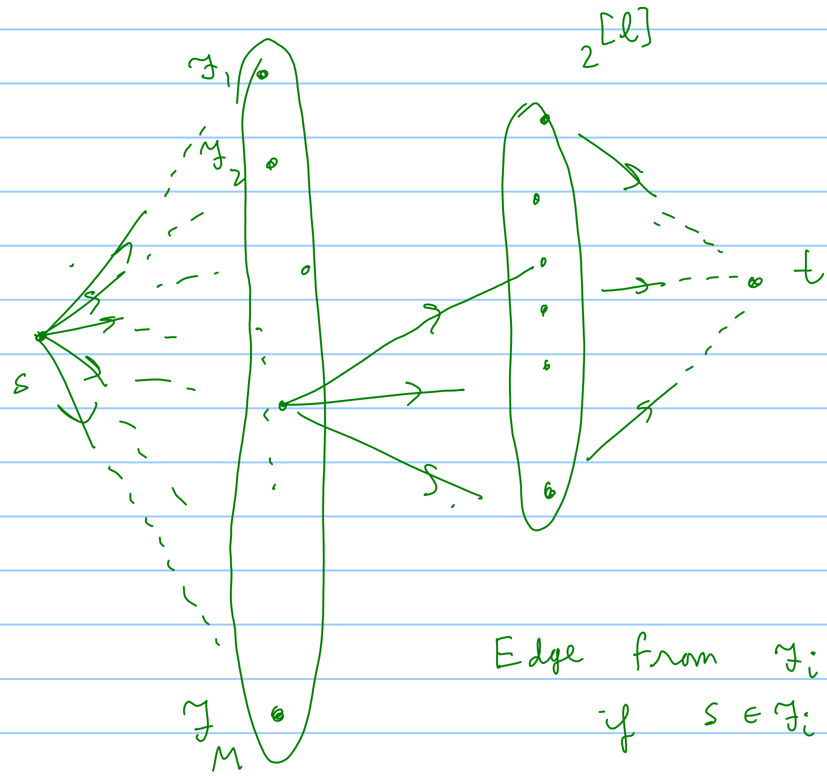
Last step of proof of Baranyai's Theorem

Note Title 12/10/2012



Edge from $\mathcal{F}_i$ to S if $S \in \mathcal{F}_i$

Each $\mathcal{F}_i$ is a partition of $[\ell]$ consisting of $\frac{n}{k}$ parts (ϕ possibly repeated)

We know: $\forall S \subseteq [\ell]$

$$\# i \text{ s.t. } S \in \mathcal{F}_i = \binom{n-\ell}{k-|S|}$$

Capacity of $s \longrightarrow \mathcal{F}_i = 1$

capacity of $\mathcal{F}_i \to S = 1$ if $S \in \mathcal{F}_i$

Capacity of $S \to t = \binom{n-(\ell+1)}{k-(|S|+1)}$

Max flow $\leq M$.

Consider the following flow

$$f(s \to \mathcal{J}_i) = 1$$

$$f(\mathcal{J}_i \to S) = \frac{k - |S|}{n - \ell}$$

$$f(S \to t) = \binom{n-(\ell+1)}{k-(|S|+1)}$$

Why is this a flow?

Constraint at $\mathcal{J}_i$

Inflow $= 1$

$$\text{Out flow} = \sum_{S \in \mathcal{J}} \frac{k-|S|}{n-\ell}$$

$$= \frac{1}{n-\ell} \cdot \sum_{S \in \mathcal{J}} (k - |S|)$$

$$= \frac{1}{n-\ell} \left(\sum_{S \in \mathcal{J}} k - \sum_{S \in \mathcal{J}} |S| \right)$$

$$= \frac{1}{n-\ell} \cdot \left(\frac{n}{k} \cdot k - \ell \right)$$

$$= 1$$

Constraint at $|S|$.

$$\text{Inflow} = \frac{k-|S|}{n-\ell} \cdot \binom{n-\ell}{k-|S|}$$

$$\text{Out flow} = \binom{n-(\ell+1)}{k-(|S|+1)} = \frac{k-|S|}{n-\ell} \cdot \binom{n-\ell}{k-|S|}$$

So it is a flow.

Value $= M$.

So there is an integral flow with value $= M$. (!)

That flow gives a choice each $\mathcal{F}_i$, a set $S \in \mathcal{F}_i$ s.t. each set is chosen $\binom{n-(\ell+1)}{k-(|S|+1)}$ times, as we wanted.