Homework 1

Design and Analysis of Data Structures and Algorithms (Spring 2012)
Rutgers University
Swastik Kopparty

Due Date: February 13, 2012.

Some quick reminders about O, o, Ω, ω, and Θ:

1. We write $f(n) = O(g(n))$ if there are positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.
2. We write $f(n) = o(g(n))$ if, for *any* constant c, there is a constant n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.
3. We write $f(n) = \Omega(g(n))$ if there are positive constants c and n_0 such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$.
4. We write $f(n) = \omega(g(n))$ if, for *any* constant c, there is a constant n_0 such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$.
5. We write $f(n) = \Theta(g(n))$ if there are positive constants c_1, c_2, and n_0 such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$.
6. For every constants $c, d > 0$,
$$n^d = o(2^{cn}),$$
$$n^d = \omega(\log^c n).$$

Questions

1. For each of the following pairs of functions, indicate whether the function on the left is O, o, Ω, ω, or Θ of the function on the right.

n^2	n^3
$\log(n^2)$	$\log(n^3)$
$2n$	$3n$
2	3
$2n + \sin(n)$	$n + 3$
n^2	$2^{\sqrt{n}}$
$n^{\log_2 3}$	$3^{\log_2 n + 5}$
$\log n$	$n^{\frac{1}{\log \log n}}$
$\log n$	$\log \log(n^n)$
$2^{\sqrt{\log n}}$	$(\log n)^{10}$
$2^{2^{\sqrt{\log n}}}$	2^n

2. Show that $2^n = o(n!)$ and $n! = o(n^n)$.

3. If $f(n) = O(\log n)$, then does it follow that $2^{f(n)} = O(n)$? Why or why not?

   If yes, prove it. If not, what should be the condition on $f(n)$ to get this conclusion?

4. (a) Suppose $T : \mathbb{N} \to \mathbb{R}$ satisfies

   $$T(1) = 10$$
   $$T(n) \leq T\left(\lfloor n/3 \rfloor\right) + n$$

   Prove by induction on n that $T(n) \leq 100n$.

   (b) Suppose $T(n) \leq 3T(\lfloor n/2 \rfloor) + n$. Prove by induction on n that $T(n) \leq O(n^{log_2 3})$.

5. Given an n digit integer N as input, how quickly can you decide whether N is a perfect power or not (i.e., do there exist integers $M \geq 1$ and $k \geq 2$ such that $N = M^k$)?

6. You are given a two dimensional $k \times (26^k - 1)$ array, and you are told that it contains every possible string of capital letters of length k, **except for one of them**.

   It is easy to find the missing one by reading all $\Theta(k \cdot 26^k)$ entries of the array. Show that this task can in fact be done while reading only $\Theta(26^k)$ entries of the array.