Homework 5

Abstract Algebra (Fall 2017)
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Due Date: Dec 4, 2017.

The following problems are from Chapter 6 of Artin (2nd edition, see link on webpage).

1. Subsection 6: 2, 3, 4, 5, 7, 9, 11, 14 (a), 15, 16. From Subsection 6, you may skip writing (but do think about) solutions to any 3 problems.
2. Subsection 7: 1
3. Subsection 8: 1, 6, 7(a), 7(b), 9, 14
4. Consider the set G of all nonconstant linear maps $f : \mathbb{R} \to \mathbb{R}$,

   $$f_{a,b}(X) = aX + b,$$

   where $a, b \in \mathbb{R}$ and $a \neq 0$,

   (a) Show that G is a group under the operation of composition.
   (b) Consider the set N of all $f_{a,b}$ with $a = 1$. Show that N is a normal subgroup of G.
   (c) What group is G/N isomorphic to?
   (d) Let H be the set of all $f_{a,b}$ with $b = 0$. Show that H is not a normal subgroup of G.
   (e) Consider the action of G on $\mathbb{R}$, where for $g \in G$ and $x \in \mathbb{R}$, $g \star x = g(x)$. What is the stabilizer of 0? What is the stabilizer of 1? Show that they are conjugate subgroups.
   (f) Bonus: define a homomorphism ϕ from H to $\mathsf{Aut}(N)$, and show that G is isomorphic to the semidirect product $N \rtimes H$.
5. Show that $\mathbb{Z}_4 \times \mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ are isomorphic. Are they isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$? Justify your answer.
6. Draw the multiplication table of a nonabelian group of size 21.