Now, to prove the bijectivity of $\psi^*$, we use the lemma to construct a map $\psi^*: \mathcal{C} \longrightarrow \mathbf{I}^*$. Consider the composed map $\varphi^*\psi^*: \mathcal{C} \longrightarrow \mathcal{C}$. It sends $H1 \rightsquigarrow H1$. We apply the lemma again, substituting $\mathcal{C}$ for $S$. The uniqueness assertion of the lemma tells us that $\varphi^*\psi^*$ is the identity map. On the other hand, since the operation on $\mathbf{I}^*$ is transitive and since $\psi^*$ is compatible with the operations, $\psi^*$ must be surjective. It follows that $\varphi^*$ and $\psi^*$ are bijective. □

*The axiomatic method has many advantages over honest work.*

Bertrand Russell

## EXERCISES

### 1. The Operations of a Group on Itself

1. Does the rule $g, x \rightsquigarrow xg^{-1}$ define an operation of $G$ on itself?
2. Let $H$ be a subgroup of a group $G$. Then $H$ operates on $G$ by left multiplication. Describe the orbits for this operation.
3. Prove the formula $|G| = |Z| + \sum|C|$, where the sum is over the conjugacy classes containing more than one element and where $Z$ is the center of $G$.
4. Prove the Fixed Point Theorem (1.12).
5. Determine the conjugacy classes in the group $M$ of motions of the plane.
6. Rule out as many of the following as possible as Class Equations for a group of order 10: $1+1+1+2+5$, $1+2+2+5$, $1+2+3+4$, $1+1+2+2+2+2$.
7. Let $F = \mathbb{F}_5$. Determine the order of the conjugacy class of $\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}$ in $GL_2(\mathbb{F}_5)$.
8. Determine the Class Equation for each of the following groups.
   **(a)** the quaternion group, **(b)** the Klein four group, **(c)** the dihedral group $D_5$, **(d)** $D_6$, **(e)** $D_n$, **(f)** the group of upper triangular matrices in $GL_2(\mathbb{F}_3)$, **(g)** $SL_2(\mathbb{F}_3)$.
9. Let $G$ be a group of order $n$, and let $F$ be any field. Prove that $G$ is isomorphic to a subgroup of $GL_n(F)$.
10. Determine the centralizer in $GL_3(\mathbb{R})$ of each matrix.
   **(a)** $\begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix}$
   **(e)** $\begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$ **(f)** $\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$
*11. Determine all finite groups which contain at most three conjugacy classes.
12. Let $N$ be a normal subgroup of a group $G$. Suppose that $|N| = 5$ and that $|G|$ is odd. Prove that $N$ is contained in the center of $G$.

*13. (a) Determine the possible Class Equations for groups of order 8.
(b) Classify groups of order 8.

14. Let $Z$ be the center of a group $G$. Prove that if $G/Z$ is a cyclic group, then $G$ is abelian and hence $G = Z$.

*15. Let $G$ be a group of order 35.
(a) Suppose that $G$ operates nontrivially on a set of five elements. Prove that $G$ has a normal subgroup of order 7.
(b) Prove that every group of order 35 is cyclic.

## 2. The Class Equation of the Icosahedral Group

1. Identify the intersection $I \cap O$ when the dodecahedron and cube are as in Figure (2.7).
2. Two tetrahedra can be inscribed into a cube $C$, each one using half the vertices. Relate this to the inclusion $A_4 \subset S_4$.
3. Does $I$ contain a subgroup $T$? $D_6$? $D_3$?
4. Prove that the icosahedral group has no subgroup of order 30.
5. Prove or disprove: $A_5$ is the only proper normal subgroup of $S_5$.
6. Prove that no group of order $p^e$, where $p$ is prime and $e > 1$, is simple.
7. Prove or disprove: An abelian group is simple if and only if it has prime order.
8. (a) Determine the Class Equation for the group $T$ of rotations of a tetrahedron.
(b) What is the center of $T$?
(c) Prove that $T$ has exactly one subgroup of order 4.
(d) Prove that $T$ has no subgroup of order 6.
9. (a) Determine the Class Equation for the octahedral group $O$.
(b) There are exactly two proper normal subgroups of $O$. Find them, show that they are normal, and show that there are no others.
10. Prove that the tetrahedral group $T$ is isomorphic to the alternating group $A_4$, and that the octahedral group $O$ is isomorphic to the symmetric group $S_4$. Begin by finding sets of four elements on which these groups operate.
11. Prove or disprove: The icosahedral group is not a subgroup of the group of real upper triangular $2 \times 2$ matrices.

*12. Prove or disprove: A nonabelian simple group can not operate nontrivially on a set containing fewer than five elements.

## 3. Operations on Subsets

1. Let $S$ be the set of subsets of order 2 of the dihedral group $D_3$. Determine the orbits for the action of $D_3$ on $S$ by conjugation.
2. Determine the orbits for left multiplication and for conjugation on the set of subsets of order 3 of $D_3$.
3. List all subgroups of the dihedral group $D_4$, and divide them into conjugacy classes.
4. Let $H$ be a subgroup of a group $G$. Prove that the orbit of the left coset $gH$ for the operation of conjugation contains the right coset $Hg$.
5. Let $U$ be a subset of a finite group $G$, and suppose that $|U|$ and $|G|$ have no common factor. Is the stabilizer of $|U|$ trivial for the operation of conjugation?
6. Consider the operation of left multiplication by $G$ on the set of its subsets. Let $U$ be a

subset whose orbit $\{gU\}$ partitions $G$. Let $H$ be the unique subset in this orbit which contains 1. Prove that $H$ is a subgroup of $G$ and that the sets $gU$ are its left cosets.

**7.** Let $H$ be a subgroup of a group $G$. Prove or disprove: The normalizer $N(H)$ is a normal subgroup of the group $G$.

**8.** Let $H \subset K \subset G$ be groups. Prove that $H$ is normal in $K$ if and only if $K \subset N(H)$.

**9.** Prove that the subgroup $B$ of upper triangular matrices in $GL_n(\mathbb{R})$ is conjugate to the group $L$ of lower triangular matrices.

**10.** Let $B$ be the subgroup of $G = GL_n(\mathbb{C})$ of upper triangular matrices, and let $U \subset B$ be the set of upper triangular matrices with diagonal entries 1. Prove that $B = N(U)$ and that $B = N(B)$.

***11.** Let $S_n$ denote the subgroup of $GL_n(\mathbb{R})$ of permutation matrices. Determine the normalizer of $S_n$ in $GL_n(\mathbb{R})$.

**12.** Let $S$ be a finite set on which a group $G$ operates transitively, and let $U$ be a subset of $S$. Prove that the subsets $gU$ cover $S$ evenly, that is, that every element of $S$ is in the same number of sets $gU$.

**13.** **(a)** Let $H$ be a normal subgroup of $G$ of order 2. Prove that $H$ is in the center of $G$.
**(b)** Let $H$ be a normal subgroup of prime order $p$ in a finite group $G$. Suppose that $p$ is the smallest prime dividing $|G|$. Prove that $H$ is in the center $Z(G)$.

***14.** Let $H$ be a proper subgroup of a finite group $G$. Prove that the union of the conjugates of $H$ is not the whole group $G$.

**15.** Let $K$ be a normal subgroup of order 2 of a group $G$, and let $\overline{G} = G/K$. Let $\overline{C}$ be a conjugacy class in $\overline{G}$. Let $S$ be the inverse image of $\overline{C}$ in $G$. Prove that one of the following two cases occurs.
**(a)** $S = C$ is a single conjugacy class and $|C| = 2|\overline{C}|$.
**(b)** $S = C_1 \cup C_2$ is made up of two conjugacy classes and $|C_1| = |C_2| = |\overline{C}|$.

**16.** Calculate the double cosets $HgH$ of the subgroup $H = \{1, y\}$ in the dihedral group $D_n$. Show that each double coset has either two or four elements.

**17.** Let $H, K$ be subgroups of $G$, and let $H'$ be a conjugate subgroup of $H$. Relate the double cosets $H'gK$ and $HgK$.

**18.** What can you say about the order of a double coset $HgK$?

## *4. The Sylow Theorems*

**1.** How many elements of order 5 are contained in a group of order 20?

**2.** Prove that no group of order $pq$, where $p$ and $q$ are prime, is simple.

**3.** Prove that no group of order $p^2q$, where $p$ and $q$ are prime, is simple.

**4.** Prove that the set of matrices $\begin{bmatrix} 1 & a \\ & c \end{bmatrix}$ where $a, c \in \mathbb{F}_7$ and $c = 1, 2, 4$ forms a group of the type presented in (4.9b) (and that therefore such a group exists).

**5.** Find Sylow 2-subgroups in the following cases:
**(a)** $D_{10}$ **(b)** $T$ **(c)** $O$ **(d)** $I$.

**6.** Find a Sylow $p$-subgroup of $GL_2(\mathbb{F}_p)$.

***7.** **(a)** Let $H$ be a subgroup of $G$ of prime index $p$. What are the possible numbers of conjugate subgroups of $H$?
**(b)** Suppose that $p$ is the smallest prime integer which divides $|G|$. Prove that $H$ is a normal subgroup.

*8. Let $H$ be a Sylow $p$-su group of $G$, and let $K = N(H)$. Prove or disprove: $K = N(K)$.

9. Let $G$ be a group of order $p^e m$. Prove that $G$ contains a subgroup of order $p^r$ for every integer $r \leq e$.

10. Let $n = pm$ be an integer which is divisible exactly once by $p$, and let $G$ be a group of order $n$. Let $H$ be a Sylow $p$-subgroup of $G$, and let $S$ be the set of all Sylow $p$-subgroups. How does $S$ decompose into $H$-orbits?

*11. (a) Compute the order of $GL_n(\mathbb{F}_p)$.
 (b) Find a Sylow $p$-subgroup of $GL_n(\mathbb{F}_p)$.
 (c) Compute the number of Sylow $p$-subgroups.
 (d) Use the Second Sylow Theorem to give another proof of the First Sylow Theorem.

*12. Prove that no group of order 224 is simple.

13. Prove that if $G$ has order $n = p^e a$ where $1 \leq a < p$ and $e \geq 1$, then $G$ has a proper normal subgroup.

14. Prove that the only simple groups of order $< 60$ are groups of prime order.

15. Classify groups of order 33.

16. Classify groups of order 18.

17. Prove that there are at most five isomorphism classes of groups of order 20.

*18. Let $G$ be a simple group of order 60.
 (a) Prove that $G$ contains six Sylow 5-subgroups, ten Sylow 3-subgroups, and five Sylow 2-subgroups.
 (b) Prove that $G$ is isomorphic to the alternating group $A_5$.

## 5. *The Groups of Order 12*

1. Determine the Class Equations of the groups of order 12.

2. Prove that a group of order $n = 2p$, where $p$ is prime, is either cyclic or dihedral.

*3. Let $G$ be a group of order 30.
 (a) Prove that either the Sylow 5-subgroup $K$ or the Sylow 3-subgroup $H$ is normal.
 (b) Prove that $HK$ is a cyclic subgroup of $G$.
 (c) Classify groups of order 30.

4. Let $G$ be a group of order 55.
 (a) Prove that $G$ is generated by two elements $x,y$, with the relations $x^{11} = 1$, $y^5 = 1$, $yxy^{-1} = x^r$, for some $r$, $1 \leq r < 11$.
 (b) Prove that the following values of $r$ are not possible: 2, 6, 7, 8, 10.
 (c) Prove that the remaining values are possible, and that there are two isomorphism classes of groups of order 55.

## 6. *Computation in the Symmetric Group*

1. Verify the products (6.9).

2. Prove explicitly that the permutation $(\mathbf{1\,2\,3})(\mathbf{4\,5})$ is conjugate to $(\mathbf{2\,4\,1})(\mathbf{3\,5})$.

3. Let $p, q$ be permutations. Prove that the products $pq$ and $qp$ have cycles of equal sizes.

4. (a) Does the symmetric group $S_7$ contain an element of order 5? of order 10? of order 15?
 (b) What is the largest possible order of an element of $S_7$?

**5.** Show how to determine whether a permutation is odd or even when it is written as a product of cycles.

**6.** Prove or disprove: The order of a permutation is the least common multiple of the orders of the cycles which make it up.

**7.** Is the cyclic subgroup $H$ of $S_n$ generated by the cycle $(\mathbf{1\,2\,3\,4\,5})$ a normal subgroup?

***8.** Compute the number of permutations in $S_n$ which do not leave any index fixed.

**9.** Determine the cycle decomposition of the permutation $\mathbf{i} \rightsquigarrow \mathbf{n{-}i}$.

**10.** (a) Prove that every permutation $p$ is a product of transpositions.

(b) How many transpositions are required to write the cycle $(\mathbf{1\,2\,3} \cdots \mathbf{n})$?

(c) Suppose that a permutation is written in two ways as a product of transpositions, say $p = \tau_1\tau_2 \cdots \tau_m$ and $p = \tau_1'\tau_2' \cdots \tau_n'$. Prove that $m$ and $n$ are both odd or else they are both even.

**11.** What is the centralizer of the element $(\mathbf{1\,2})$ of $S_4$?

**12.** Find all subgroups of order 4 of the symmetric group $S_4$. Which are normal?

**13.** Determine the Class Equation of $A_4$.

**14.** (a) Determine the number of conjugacy classes and the Class Equation for $S_5$.

(b) List the conjugacy classes in $A_5$, and reconcile this list with the list of conjugacy classes in the icosahedral group [see (2.2)].

**15.** Prove that the transpositions $(\mathbf{1\,2}), (\mathbf{2\,3}), \ldots, (\mathbf{n{-}1, n})$ generate the symmetric group $S_n$.

**16.** Prove that the symmetric group $S_n$ is generated by the cycles $(\mathbf{1\,2} \cdots \mathbf{n})$ and $(\mathbf{1\,2})$.

**17.** (a) Show that the product of two transpositions $(\mathbf{i\,j})(\mathbf{k\,l})$ can always be written as a product of 3-cycles. Treat the case that some indices are equal too.

(b) Prove that the alternating group $A_n$ is generated by 3-cycles, if $n \geq 3$.

**18.** Prove that if a proper normal subgroup of $S_n$ contains a 3-cycle, it is $A_n$.

***19.** Prove that $A_n$ is simple for all $n \geq 5$.

***20.** Prove that $A_n$ is the only subgroup of $S_n$ of index 2.

**21.** Explain the miraculous coincidence at the end of the section in terms of the opposite group (Chapter 2, Section 1, exercise 12).

## 7. *The Free Group*

**1.** Prove or disprove: The free group on two generators is isomorphic to the product of two infinite cyclic groups.

**2.** (a) Let $F$ be the free group on $x,y$. Prove that the two elements $u = x^2$ and $v = y^3$ generate a subgroup of $F$ which is isomorphic to the free group on $u, v$.

(b) Prove that the three elements $u = x^2$, $v = y^2$, and $z = xy$ generate a subgroup isomorphic to the free group on $u, v, z$.

**3.** We may define a *closed word* in $S'$ to be the oriented loop obtained by joining the ends of a word. Thus

$$\begin{matrix} & & c\,a^{-1} & & \\ & b & & b^{-1} & \\ a & & & & b, \\ & a & & c & \\ & & bbd & & \end{matrix}$$

represents a closed word, if we read it clockwise. Establish a bijective correspondence between reduced closed words and conjugacy classes in the free group.

4. Let $p$ be a prime integer. Let $N$ be the number of words of length $p$ in a finite set $S$. Show that $N$ is divisible by $p$.

## *8. Generators and Relations*

1. Prove that two elements $a, b$ of a group generate the same subgroup as $bab^2, bab^3$.
2. Prove that the smallest normal subgroup of a group $G$ containing a subset $S$ is generated as a subgroup by the set $\{gsg^{-1} \mid g \in G, s \in S\}$.
3. Prove or disprove: $y^2x^2$ is in the normal subgroup generated by $xy$ and its conjugates.
4. Prove that the group generated by $x, y, z$ with the single relation $yxyz^{-2} = 1$ is actually a free group.
5. Let $S$ be a set of elements of a group $G$, and let $\{r_i\}$ be some relations which hold among the elements $S$ in $G$. Let $F$ be the free group on $S$. Prove that the map $F \longrightarrow G$ (8.1) factors through $F/N$, where $N$ is the normal subgroup generated by $\{r_i\}$.
6. Let $G$ be a group with a normal subgroup $N$. Assume that $G$ and $G/N$ are both cyclic groups. Prove that $G$ can be generated by two elements.
7. A subgroup $H$ of a group $G$ is called *characteristic* if it is carried to itself by all automorphisms of $G$.
   **(a)** Prove that every characteristic subgroup is normal.
   **(b)** Prove that the center $Z$ of a group $G$ is a characteristic subgroup.
   **(c)** Prove that the subgroup $H$ generated by all elements of $G$ of order $n$ is characteristic.
8. Determine the normal subgroups and the characteristic subgroups of the quaternion group.
9. The *commutator subgroup* $C$ of a group $G$ is the smallest subgroup containing all commutators.
   **(a)** Prove that the commutator subgroup is a characteristic subgroup.
   **(b)** Prove that $G/C$ is an abelian group.
10. Determine the commutator subgroup of the group $M$ of motions of the plane.
11. Prove by explicit computation that the commutator $x(yz)x^{-1}(yz)^{-1}$ is in the normal subgroup generated by the two commutators $xyx^{-1}y^{-1}$ and $xzx^{-1}z^{-1}$ and their conjugates.
12. Let $G$ denote the free abelian group $\langle x, y; xyx^{-1}y^{-1}\rangle$ defined in (8.8). Prove the universal property of this group: If $u, v$ are elements of an abelian group $A$, there is a unique homomorphism $\varphi: G \longrightarrow A$ such that $\varphi(x) = u$, $\varphi(y) = v$.
13. Prove that the normal subgroup in the free group $\langle x, y\rangle$ which is generated by the single commutator $xyx^{-1}y^{-1}$ is the commutator subgroup.
14. Let $N$ be a normal subgroup of a group $G$. Prove that $G/N$ is abelian if and only if $N$ contains the commutator subgroup of $G$.
15. Let $\varphi: G \longrightarrow G'$ be a surjective group homomorphism. Let $S$ be a subset of $G$ such that $\varphi(S)$ generates $G'$, and let $T$ be a set of generators of $\ker \varphi$. Prove that $S \cup T$ generates $G$.
16. Prove or disprove: Every finite group $G$ can be presented by a finite set of generators and a finite set of relations.
17. Let $G$ be the group generated by $x, y, z$, with certain relations $\{r_i\}$. Suppose that one of the relations has the form $wx$, where $w$ is a word in $y, z$. Let $r_i'$ be the relation obtained by substituting $w^{-1}$ for $x$ into $r_i$, and let $G'$ be the group generated by $y, z$, with relations $\{r_i'\}$. Prove that $G$ and $G'$ are isomorphic.

## 9. The Todd–Coxeter Algorithm

1. Prove that the elements $x, y$ of (9.5) generate $T$, and that the permutations (9.7) generate $A_4$.
2. Use the Todd–Coxeter Algorithm to identify the group generated by two elements $x, y$, with the following relations.
   (a) $x^2 = y^2 = 1,\ xyx = yxy$
   (b) $x^2 = y^3 = 1,\ xyx = yxy$
   (c) $x^3 = y^3 = 1,\ xyx = yxy$
   (d) $x^4 = y^2 = 1,\ xyx = yxy$
   (e) $x^4 = y^4 = x^2y^2 = 1$
3. Use the Todd–Coxeter Algorithm to determine the order of the group generated by $x, y$, with the following relations.
   (a) $x^4 = 1,\ y^3 = 1,\ xy = y^2x$ (b) $x^7 = 1,\ y^3 = 1,\ yx = x^2y$.
4. Identify the group $G$ generated by elements $x, y, z$, with relations $x^4 = y^4 = z^3 = x^2z^2 = 1$ and $z = xy$.
5. Analyze the group $G$ generated by $x, y$, with relations $x^4 = 1,\ y^4 = 1,\ x^2 = y^2,\ xy = y^3x$.

*6. Analyze the group generated by elements $x, y$, with relations $x^{-1}yx = y^{-1},\ y^{-1}xy = x^{-1}$.

7. Let $G$ be the group generated by elements $x, y$, with relations $x^4 = 1,\ y^3 = 1,\ x^2 = yxy$. Prove that this group is trivial in these two ways.
   (a) using the Todd–Coxeter Algorithm
   (b) working directly with the relations
8. Identify the group $G$ generated by two elements $x, y$, with relations $x^3 = y^3 = yxyxy = 1$.
9. Let $p \le q \le r$ be integers $>1$. The *triangle group* $G^{pqr}$ is defined by generators $G^{pqr} = \langle x, y, z; x^p, y^q, z^r, xyz \rangle$. In each case, prove that the triangle group is isomorphic to the group listed.
   (a) the dihedral group $D_n$, when $p, q, r = 2, 2, n$
   (b) the tetrahedral group, when $p, q, r = 2, 3, 3$
   (c) the octahedral group, when $p, q, r = 2, 3, 4$
   (d) the icosahedral group, when $p, q, r = 2, 3, 5$
10. Let $\Delta$ denote an isosceles right triangle, and let $a, b, c$ denote the reflections of the plane about the three sides of $\Delta$. Let $x = ab,\ y = bc,\ z = ca$. Prove that $x, y, z$ generate a triangle group.
11. (a) Prove that the group $G$ generated by elements $x, y, z$ with relations $x^2 = y^3 = z^5 = 1,\ xyz = 1$ has order 60.
    (b) Let $H$ be the subgroup generated by $x$ and $zyz^{-1}$. Determine the permutation representation of $G$ on $G/H$, and identify $H$.
    (c) Prove that $G$ is isomorphic to the alternating group $A_5$.
    (d) Let $K$ be the subgroup of $G$ generated by $x$ and $yxz$. Determine the permutation representation of $G$ on $G/K$, and identify $K$.

## Miscellaneous Problems

1. (a) Prove that the subgroup $T'$ of $O_3$ of all symmetries of a regular tetrahedron, including orientation-reversing symmetries, has order 24.

**(b)** Is $T'$ isomorphic to the symmetric group $S_4$?
**(c)** State and prove analogous results for the group of symmetries of a dodecahedron.

**2.** **(a)** Let $U = \{1, x\}$ be a subset of order 2 of a group $G$. Consider the graph having one vertex for each element of $G$ and an edge joining the vertices $g$ to $gx$ for all $g \in G$. Prove that the vertices connected to the vertex 1 are the elements of the cyclic group generated by $x$.
**(b)** Do the analogous thing for the set $U = \{1, x, y\}$.

***3.** **(a)** Suppose that a group $G$ operates transitively on a set $S$, and that $H$ is the stabilizer of an element $s_0 \in S$. Consider the action of $G$ on $S \times S$ defined by $g(s_1, s_2) = (gs_1, gs_2)$. Establish a bijective correspondence between double cosets of $H$ in $G$ and $G$-orbits in $S \times S$.
**(b)** Work out the correspondence explicitly for the case that $G$ is the dihedral group $D_5$ and $S$ is the set of vertices of a 5-gon.
**(c)** Work it out for the case that $G = T$ and that $S$ is the set of edges of a tetrahedron.

***4.** Assume that $H \subset K \subset G$ are subgroups, that $H$ is normal in $K$, and that $K$ is normal in $G$. Prove or disprove: $H$ is normal in $G$.

***5.** Prove the *Bruhat decomposition*, which asserts that $GL_n(\mathbb{R})$ is the union of the double cosets $BPB$, where $B$ is the group of upper triangular matrices and $P$ is a permutation matrix.

**6.** **(a)** Prove that the group generated by $x, y$ with relations $x^2, y^2$ is an infinite group in two ways:
  **(i)** It is clear that every word can be reduced by using these relations to the form $\cdots xyxy \cdots$. Prove that every element of $G$ is represented by exactly one such word.
  **(ii)** Exhibit $G$ as the group generated by reflections $r, r'$ about lines $\ell, \ell'$ whose angle of intersection is not a rational multiple of $2\pi$.
**(b)** Let $N$ be any proper normal subgroup of $G$. Prove that $G/N$ is a dihedral group.

**7.** Let $H, N$ be subgroups of a group $G$, and assume that $N$ is a normal subgroup.
**(a)** Determine the kernels of the restrictions of the canonical homomorphism $\pi: G \longrightarrow G/N$ to the subgroups $H$ and $HN$.
**(b)** Apply the First Isomorphism Theorem to these restrictions to prove the *Second Isomorphism Theorem:* $H/(H \cap N)$ is isomorphic to $(HN)/N$.

**8.** Let $H, N$ be normal subgroups of a group $G$ such that $H \supset N$, and let $\overline{H} = H/N$, $\overline{G} = G/N$.
**(a)** Prove that $\overline{H}$ is a normal subgroup of $\overline{G}$.
**(b)** Use the composed homomorphism $G \longrightarrow \overline{G} \longrightarrow \overline{G}/\overline{H}$ to prove the *Third Isomorphism Theorem:* $G/H$ is isomorphic to $\overline{G}/\overline{H}$.