

such groups as symmetry. We imagine the crystal to be infinitely large. Then the fact that the molecules are arranged regularly implies that they form an array having three independent translational symmetries. It has been shown that there are 230 types of crystallographic groups, analogous to the 17 lattice groups (4.15). This is too long a list to be very useful, and so crystals have been classified more crudely into seven *crystal systems*. For more about this, and for a discussion of the 32 crystallographic point groups, look in a book on crystallography.

*Un bon héritage vaut mieux que le plus joli problème de géométrie,
parce qu'il tient lieu de méthode générale,
et sert à résoudre bien des problèmes.*

Gottfried Wilhelm Leibnitz

EXERCISES

1. Symmetry of Plane Figures

1. Prove that the set of symmetries of a figure F in the plane forms a group.
2. List all symmetries of (a) a square and (b) a regular pentagon.
3. List all symmetries of the following figures.
(a) (1.4) (b) (1.5) (c) (1.6) (d) (1.7)
4. Let G be a finite group of rotations of the plane about the origin. Prove that G is cyclic.

2. The Group of Motions of the Plane

1. Compute the fixed point of $t_a\rho_\theta$ algebraically.
2. Verify the rules (2.5) by explicit calculation, using the definitions (2.3).
3. Prove that \mathbf{O} is not a normal subgroup of M .
4. Let m be an orientation-reversing motion. Prove that m^2 is a translation.
5. Let SM denote the subset of orientation-preserving motions of the plane. Prove that SM is a normal subgroup of M , and determine its index in M .
6. Prove that a linear operator on \mathbb{R}^2 is a reflection if and only if its eigenvalues are 1 and -1 , and its eigenvectors are orthogonal.
7. Prove that a conjugate of a reflection or a glide reflection is a motion of the same type, and that if m is a glide reflection then the glide vectors of m and of its conjugates have the same length.
8. Complete the proof that (2.13) is a homomorphism.
9. Prove that the map $M \longrightarrow \{1, r\}$ defined by $t_a\rho_\theta \rightsquigarrow 1$, $t_a\rho_\theta r \rightsquigarrow r$ is a homomorphism.
10. Compute the effect of rotation of the axes through an angle η on the expressions $t_a\rho_\theta$ and $t_a\rho_\theta r$ for a motion.

11. (a) Compute the eigenvalues and eigenvectors of the linear operator $m = \rho_\theta r$.
 (b) Prove algebraically that m is a reflection about a line through the origin, which subtends an angle of $\frac{1}{2}\theta$ with the x -axis.
 (c) Do the same thing as in (b) geometrically.
12. Compute the glide vector of the glide $t_a \rho_\theta r$ in terms of a and θ .
13. (a) Let m be a glide reflection along a line ℓ . Prove geometrically that a point x lies on ℓ if and only if $x, m(x), m^2(x)$ are colinear.
 (b) Conversely, prove that if m is an orientation-reversing motion and x is a point such that $x, m(x), m^2(x)$ are distinct points on a line ℓ , then m is a glide reflection along ℓ .
14. Find an isomorphism from the group SM to the subgroup of $GL_2(\mathbb{C})$ of matrices of the form $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, with $|a| = 1$.
15. (a) Write the formulas for the motions (2.3) in terms of the complex variable $z = x + iy$.
 (b) Show that every motion has the form $m(z) = \alpha z + \beta$ or $m(z) = \alpha \bar{z} + \beta$, where $|\alpha| = 1$ and β is an arbitrary complex number.

3. Finite Groups of Motions

1. Let D_n denote the dihedral group (3.6). Express the product $x^2 y x^{-1} y^{-1} x^3 y^3$ in the form $x^i y^j$ in D_n .
2. List all subgroups of the group D_4 , and determine which are normal.
3. Find all proper normal subgroups and identify the quotient groups of the groups D_{13} and D_{15} .
4. (a) Compute the cosets of the subgroup $H = \{1, x^5\}$ in the dihedral group D_{10} explicitly.
 (b) Prove that D_{10}/H is isomorphic to D_5 .
 (c) Is D_{10} isomorphic to $D_5 \times H$?
5. List the subgroups of $G = D_6$ which do not contain $N = \{1, x^3\}$.
6. Prove that every finite subgroup of M is a conjugate subgroup of one of the standard subgroups listed in Corollary (3.5).

4. Discrete Groups of Motions

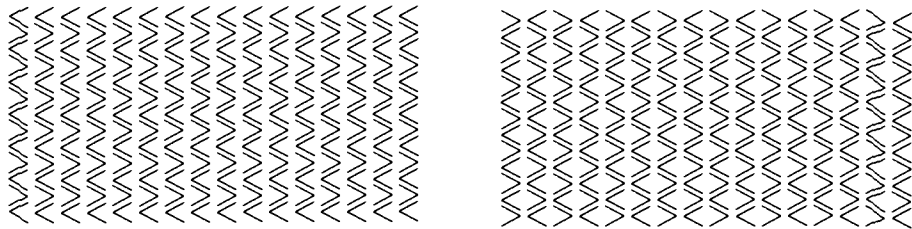
1. Prove that a discrete group G consisting of rotations about the origin is cyclic and is generated by ρ_θ where θ is the smallest angle of rotation in G .
2. Let G be a subgroup of M which contains rotations about two different points. Prove algebraically that G contains a translation.
3. Let (a, b) be a lattice basis of a lattice L in \mathbb{R}^2 . Prove that every other lattice basis has the form $(a', b') = (a, b)P$, where P is a 2×2 integer matrix whose determinant is ± 1 .
4. Determine the point group for each of the patterns depicted in Figure (4.16).
5. (a) Let B be a square of side length a , and let $\epsilon > 0$. Let S be a subset of B such that the distance between any two points of S is $\geq \epsilon$. Find an explicit upper bound for the number of elements in S .
 (b) Do the same thing for a box B in \mathbb{R}^n .

- 6. Prove that the subgroup of \mathbb{R}^+ generated by 1 and $\sqrt{2}$ is dense in \mathbb{R}^+ .
- 7. Prove that every discrete subgroup of \mathbf{O} is finite.
- 8. Let G be a discrete subgroup of M . Prove that there is a point p_0 in the plane which is not fixed by any point of G except the identity.
- 9. Prove that the group of symmetries of the frieze pattern



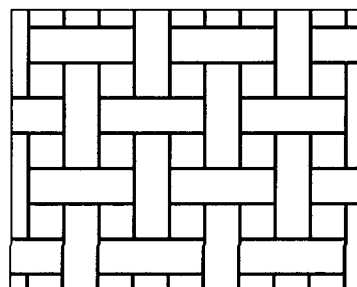
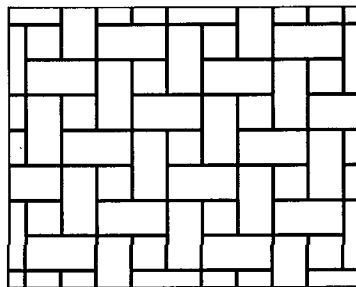
is isomorphic to the direct product $C_2 \times C_\infty$ of a cyclic group of order 2 and an infinite cyclic group.

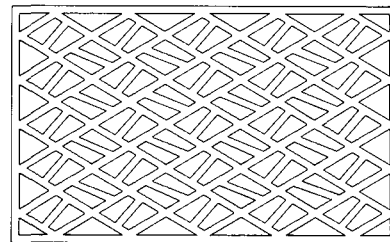
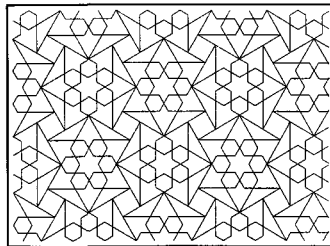
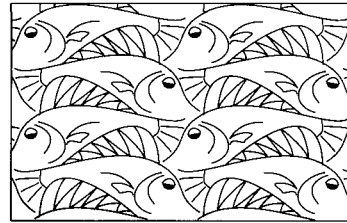
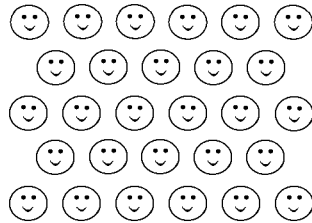
- 10. Let G be the group of symmetries of the frieze pattern $\dots \text{L} \text{R} \text{L} \text{R} \text{L} \text{R} \text{L} \text{R} \dots$
 - (a) Determine the point group \bar{G} of G .
 - (b) For each element $\bar{g} \in \bar{G}$, and each element $g \in G$ which represents \bar{g} , describe the action of g geometrically.
 - (c) Let H be the subgroup of translations in G . Determine $[G:H]$.
- 11. Let G be the group of symmetries of the pattern



Determine the point group of G .

- 12. Let G be the group of symmetries of an equilateral triangular lattice L . Find the index in G of the subgroup $T \cap G$.
- 13. Let G be a discrete group in which every element is orientation-preserving. Prove that the point group \bar{G} is a cyclic group of rotations and that there is a point p in the plane such that the set of group elements which fix p is isomorphic to \bar{G} .
- 14. With each of the patterns shown, find a pattern with the same type of symmetry in (4.16).





15. Let N denote the group of rigid motions of the line $\ell = \mathbb{R}^1$. Some elements of N are

$$t_a: x \rightarrow x + a, \quad a \in \mathbb{R}, \quad s: x \rightarrow -x.$$

- (a) Show that $\{t_a, t_a s\}$ are all of the elements of N , and describe their actions on ℓ geometrically.
- (b) Compute the products $t_a t_b, s t_a, s s$.
- (c) Find all discrete subgroups of N which contain a translation. It will be convenient to choose your origin and unit length with reference to the particular subgroup. Prove that your list is complete.

*16. Let N' be the group of motions of an infinite ribbon

$$R = \{(x, y) \mid -1 \leq y \leq 1\}.$$

It can be viewed as a subgroup of the group M . The following elements are in N' :

$$t_a: (x, y) \rightarrow (x + a, y)$$

$$s: (x, y) \rightarrow (-x, y)$$

$$r: (x, y) \rightarrow (x, -y)$$

$$\rho: (x, y) \rightarrow (-x, -y).$$

- (a) Show that these elements generate N' , and describe the elements of N' as products.
- (b) State and prove analogues of (2.5) for these motions.
- (c) A frieze pattern is any pattern on the ribbon which is periodic and not degenerate, in the sense that its group of symmetries is discrete. Since it is periodic, its group of symmetries will contain a translation. Some sample patterns are depicted in the text (1.3, 1.4, 1.6, 1.7). Classify the symmetry groups which arise, identifying those which differ only in the choice of origin and unit length on the ribbon. I suggest that you begin by trying to make patterns with different kinds of symmetry. Please make

a careful case analysis when proving your results. A suitable format would be as follows: Let G be a discrete subgroup containing a translation.

Case 1: Every element of G is a translation. Then . . . ,

Case 2: G contains the rotation ρ but no orientation-reversing symmetry. Then . . . , and so on.

- *17. Let L be a lattice of \mathbb{R}^2 , and let a, b be linearly independent vectors lying in L . Show that the subgroup $L' = \{ma + nb \mid m, n \in \mathbb{Z}\}$ of L generated by a, b has finite index, and that the index is the number of lattice points in the parallelogram whose vertices are $0, a, b, a + b$ and which are not on the “far edges” $[a, a + b]$ and $[b, a + b]$. (So, 0 is included, and so are points which lie on the edges $[0, a], [0, b]$, except for the points a, b themselves.)
18. (a) Find a subset F of the plane which is not fixed by any motion $m \in M$.
 (b) Let G be a discrete group of motions. Prove that the union S of all images of F by elements of G is a subset whose group of symmetries G' contains G .
 (c) Show by an example that G' may be larger than G .
 *(d) Prove that there exists a subset F such that $G' = G$.
- *19. Let G be a lattice group such that no element $g \neq 1$ fixes any point of the plane. Prove that G is generated by two translations, or else by one translation and one glide.
- *20. Let G be a lattice group whose point group is $D_1 = \{1, r\}$.
 (a) Show that the glide lines and the lines of reflection of G are all parallel.
 (b) Let $L = L_G$. Show that L contains nonzero vectors $a = (a_1, 0)^t, b = (0, b_2)^t$.
 (c) Let a and b denote the smallest vectors of the type indicated in (b). Then either (a, b) or (a, c) is a lattice basis for L , where $c = \frac{1}{2}(a + b)$.
 (d) Show that if coordinates in the plane are chosen so that the x -axis is a glide line, then G contains one of the elements $g = r$ or $g = t_{\frac{1}{2}a}r$. In either case, show that $G = L \cup Lg$.
 (e) There are four possibilities described by the dichotomies (c) and (d). Show that there are only three different kinds of group.
21. Prove that if the point group of a lattice group G is C_6 , then $L = L_G$ is an equilateral triangular lattice, and G is the group of all rotational symmetries of L about the origin.
22. Prove that if the point group of a lattice group G is D_6 , then $L = L_G$ is an equilateral triangular lattice, and G is the group of all symmetries of L .
- *23. Prove that symmetry groups of the figures in Figure (4.16) exhaust the possibilities.

5. Abstract Symmetry: Group Operations

- Determine the group of automorphisms of the following groups.
 (a) C_4 (b) C_6 (c) $C_2 \times C_2$
- Prove that (5.4) is an equivalence relation.
- Let S be a set on which G operates. Prove that the relation $s \sim s'$ if $s' = gs$ for some $g \in G$ is an equivalence relation.
- Let $\varphi: G \rightarrow G'$ be a homomorphism, and let S be a set on which G' operates. Show how to define an operation of G on S , using the homomorphism φ .

5. Let $G = D_4$ be the dihedral group of symmetries of the square.
- What is the stabilizer of a vertex? an edge?
 - G acts on the set of two elements consisting of the diagonal lines. What is the stabilizer of a diagonal?
6. In each of the figures in exercise 14 of Section 4, find the points which have nontrivial stabilizers, and identify the stabilizers.
- *7. Let G be a discrete subgroup of M .
- Prove that the stabilizer G_p of a point p is finite.
 - Prove that the orbit O_p of a point p is a discrete set, that is, that there is a number $\epsilon > 0$ so that the distance between two distinct points of the orbit is at least ϵ .
 - Let B, B' be two bounded regions in the plane. Prove that there are only finitely many elements $g \in G$ so that $gB \cap B'$ is nonempty.
8. Let $G = GL_n(\mathbb{R})$ operate on the set $S = \mathbb{R}^n$ by left multiplication.
- Describe the decomposition of S into orbits for this operation.
 - What is the stabilizer of e_1 ?
9. Decompose the set $\mathbb{C}^{2 \times 2}$ of 2×2 complex matrices for the following operations of $GL_2(\mathbb{C})$:
- Left multiplication
 - Conjugation
10. (a) Let $S = \mathbb{R}^{m \times n}$ be the set of real $m \times n$ matrices, and let $G = GL_m(\mathbb{R}) \times GL_n(\mathbb{R})$. Prove that the rule $(P, Q), A \rightsquigarrow PAQ^{-1}$ defines an operation of G on S .
- Describe the decomposition of S into G -orbits.
 - Assume that $m \leq n$. What is the stabilizer of the matrix $[I \mid 0]$?
11. (a) Describe the orbit and the stabilizer of the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ under conjugation in $GL_n(\mathbb{R})$.
- Interpreting the matrix in $GL_2(\mathbb{F}_3)$, find the order (the number of elements) of the orbit.
12. (a) Define automorphism of a field.
- Prove that the field \mathbb{Q} of rational numbers has no automorphism except the identity.
 - Determine $\text{Aut } F$, when $F = \mathbb{Q}[\sqrt{2}]$.

6. The Operation on Cosets

- What is the stabilizer of the coset aH for the operation of G on G/H ?
- Let G be a group, and let H be the cyclic subgroup generated by an element x of G . Show that if left multiplication by x fixes every coset of H in G , then H is a normal subgroup.
- (a) Exhibit the bijective map (6.4) explicitly, when G is the dihedral group D_4 and S is the set of vertices of a square.

 - Do the same for D_n and the vertices of a regular n -gon.

- (a) Describe the stabilizer H of the index 1 for the action of the symmetric group $G = S_n$ on $\{1, \dots, n\}$ explicitly.

 - Describe the cosets of H in G explicitly for this action.
 - Describe the map (6.4) explicitly.

5. Describe all ways in which S_3 can operate on a set of four elements.
6. Prove Proposition (6.5).
7. A map $S \rightarrow S'$ of G -sets is called a *homomorphism* of G -sets if $\varphi(gs) = g\varphi(s)$ for all $s \in S$ and $g \in G$. Let φ be such a homomorphism. Prove the following:
 - (a) The stabilizer $G_{\varphi(s)}$ contains the stabilizer G_s .
 - (b) The orbit of an element $s \in S$ maps onto the orbit of $\varphi(s)$.

7. The Counting Formula

1. Use the counting formula to determine the orders of the group of rotational symmetries of a cube and of the group of rotational symmetries of a tetrahedron.
2. Let G be the group of rotational symmetries of a cube C . Two regular tetrahedra Δ, Δ' can be inscribed in C , each using half of the vertices. What is the order of the stabilizer of Δ ?
3. Compute the order of the group of symmetries of a dodecahedron, when orientation-reversing symmetries such as reflections in planes, as well as rotations, are allowed. Do the same for the symmetries of a cube and of a tetrahedron.
4. Let G be the group of rotational symmetries of a cube, let S_e, S_v, S_f be the sets of vertices, edges, and faces of the cube, and let H_v, H_e, H_f be the stabilizers of a vertex, an edge, and a face. Determine the formulas which represent the decomposition of each of the three sets into orbits for each of the subgroups.
5. Let $G \supset H \supset K$ be groups. Prove the formula $[G : K] = [G : H][H : K]$ without the assumption that G is finite.
6. (a) Prove that if H and K are subgroups of finite index of a group G , then the intersection $H \cap K$ is also of finite index.
 (b) Show by example that the index $[H : H \cap K]$ need not divide $[G : K]$.

8. Permutation Representations

1. Determine all ways in which the tetrahedral group T (see (9.1)) can operate on a set of two elements.
2. Let S be a set on which a group G operates, and let $H = \{g \in G \mid gs = s \text{ for all } s \in S\}$. Prove that H is a normal subgroup of G .
3. Let G be the dihedral group of symmetries of a square. Is the action of G on the vertices a faithful action? on the diagonals?
4. Suppose that there are two orbits for the operation of a group G on a set S , and that they have orders m, n respectively. Use the operation to define a homomorphism from G to the product $S_m \times S_n$ of symmetric groups.
5. A group G operates faithfully on a set S of five elements, and there are two orbits, one of order 3 and one of order 2. What are the possibilities for G ?
6. Complete the proof of Proposition (8.2).
7. Let $F = \mathbb{F}_3$. There are four one-dimensional subspaces of the space of column vectors F^2 . Describe them. Left multiplication by an invertible matrix permutes these subspaces. Prove that this operation defines a homomorphism $\varphi: GL_2(F) \rightarrow S_4$. Determine the kernel and image of this homomorphism.

- *8. For each of the following groups, find the smallest integer n such that the group has a faithful operation on a set with n elements.
 (a) the quaternion group H (b) D_4 (c) D_6

9. Finite Subgroups of the Rotation Group

- Describe the orbits of poles for the group of rotations of an octahedron and of an icosahedron.
 - Identify the group of symmetries of a baseball, taking the stitching into account and allowing orientation-reversing symmetries.
 - Let O be the group of rotations of a cube. Determine the stabilizer of a diagonal line connecting opposite vertices.
 - Let $G = O$ be the group of rotations of a cube, and let H be the subgroup carrying one of the two inscribed tetrahedra to itself (see exercise 2, Section 7). Prove that $H = T$.
 - Prove that the icosahedral group has a subgroup of order 10.
 - Determine all subgroups of the following groups:
 (a) T (b) I
 - Explain why the groups of symmetries of the cube and octahedron, and of the dodecahedron and icosahedron, are equal.
- *8. (a) The 12 points $(\pm 1, \pm \alpha, 0), (0, \pm 1, \pm \alpha), (\pm \alpha, 0, \pm 1)$ form the vertices of a regular icosahedron if α is suitably chosen. Verify this, and determine α .
 (b) Determine the matrix of the rotation through the angle $2\pi/5$ about the origin in \mathbb{R}^2 .
 (c) Determine the matrix of the rotation of \mathbb{R}^3 through the angle $2\pi/5$ about the axis containing the point $(1, \alpha, 0)$.
- *9. Prove the crystallographic restriction for three-dimensional crystallographic groups: A rotational symmetry of a crystal has order 2, 3, 4, or 6.

Miscellaneous Problems

- Describe completely the following groups:
 (a) $\text{Aut } D_4$ (b) $\text{Aut } H$, where H is the quaternion group
 - (a) Prove that the set $\text{Aut } G$ of automorphisms of a group G forms a group.
 (b) Prove that the map $\varphi: G \rightarrow \text{Aut } G$ defined by $g \rightsquigarrow$ (conjugation by g) is a homomorphism, and determine its kernel.
 (c) The automorphisms which are conjugation by a group element are called *inner automorphisms*. Prove that the set of inner automorphisms, the image of φ , is a normal subgroup of $\text{Aut } G$.
 - Determine the quotient group $\text{Aut } H / \text{Int } H$ for the quaternion group H .
- *4. Let G be a lattice group. A *fundamental domain* D for G is a bounded region in the plane, bounded by piecewise smooth curves, such that the sets gD , $g \in G$ cover the plane without overlapping except along the edges. We assume that D has finitely many connected components.
- Find fundamental domains for the symmetry groups of the patterns illustrated in exercise 14 of Section 4.
 - Show that any two fundamental domains D, D' for G can be cut into finitely many congruent pieces of the form $gD \cap D'$ or $D \cap gD'$ (see exercise 7, Section 5).

- (c) Conclude that D and D' have the same area. (It may happen that the boundary curves intersect infinitely often, and this raises some questions about the definition of area. Disregard such points in your answer.)
- *5. Let G be a lattice group, and let p_0 be a point in the plane which is not fixed by any element of G . Let $S = \{gp_0 \mid g \in G\}$ be the orbit of p_0 . The plane can be divided into polygons, each one containing a single point of S , as follows: The polygon Δ_p containing p is the set of points q whose distance from p is the smallest distance to any point of S :

$$\Delta_p = \{q \in \mathbb{R}^2 \mid \text{dist}(q, p) \leq \text{dist}(q, p') \text{ for all } p' \in S\}.$$

- (a) Prove that Δ_p is a polygon.
 (b) Prove that Δ_p is a fundamental domain for G .
 (c) Show that this method works for all discrete subgroups of M , except that the domain Δ_p which is constructed need not be a bounded set.
 (d) Prove that Δ_p is bounded if and only if the group is a lattice group.
- *6. (a) Let $G' \subset G$ be two lattice groups. Let D be a fundamental domain for G . Show that a fundamental domain D' for G' can be constructed out of finitely many translates gD of D .
 (b) Show that $[G : G'] < \infty$ and that $[G : G'] = \text{area}(D')/\text{area}(D)$.
 (c) Compute the index $[G : L_G]$ for each of the patterns (4.16).
- *7. Let G be a finite group operating on a finite set S . For each element $g \in G$, let S^g denote the subset of elements of S fixed by g : $S^g = \{s \in S \mid gs = s\}$.
 (a) We may imagine a true–false table for the assertion that $gs = s$, say with rows indexed by elements of G and columns indexed by elements. Construct such a table for the action of the dihedral group D_3 on the vertices of a triangle.
 (b) Prove the formula $\sum_{s \in S} |G_s| = \sum_{g \in G} |S^g|$.
 (c) Prove *Burnside's Formula*:

$$|G| \cdot (\text{number of orbits}) = \sum_{g \in G} |S^g|.$$

8. There are $70 = \binom{8}{4}$ ways to color the edges of an octagon, making four black and four white. The group D_8 operates on this set of 70, and the orbits represent equivalent colorings. Use Burnside's Formula to count the number of equivalence classes.
9. Let G be a group of order n which operates nontrivially on a set of order r . Prove that if $n > r!$, then G has a proper normal subgroup.