map $\overline{\varphi}$ which sends the coset $\overline{a} = aN$ to $\varphi(a)$:

$$\overline{\varphi}(\overline{a}) = \varphi(a).$$

This is our fundamental method of identifying quotient groups. For example, the absolute value map $\mathbb{C}^\times \longrightarrow \mathbb{R}^\times$ maps the nonzero complex numbers to the positive real numbers, and its kernel is the unit circle U. So the quotient group $\mathbb{C}^\times/U$ is isomorphic to the multiplicative group of positive real numbers. Or, the determinant is a surjective homomorphism $GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^\times$, whose kernel is the special linear group $SL_n(\mathbb{R})$. So the quotient $GL_n(\mathbb{R})/SL_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^\times$.

Proof of the First Isomorphism Theorem. According to Proposition (5.13), the nonempty fibres of φ are the cosets aN. So we can think of $\overline{G}$ in either way, as the set of cosets or as the set of nonempty fibres of φ. Therefore the map we are looking for is the one defined in (5.10) for any map of sets. It maps $\overline{G}$ bijectively onto the image of φ, which is equal to G' because φ is surjective. By construction it is compatible with multiplication: $\overline{\varphi}(\overline{ab}) = \varphi(ab) = \varphi(a)\varphi(b) = \overline{\varphi}(\overline{a})\overline{\varphi}(\overline{b})$. □

Es giebt alſo ſehr viel verſchiedene Arten von Größen,
welche ſich nicht wohl herzehlen laßen;
und daher entſtehen die verſchiedene Theile der Mathematic,
deren eine jegliche mit einer beſondern Art von Größen beſchäftiget iſt.

Leonhard Euler

EXERCISES

1. The Definition of a Group

1. (a) Verify (1.17) and (1.18) by explicit computation.
   (b) Make a multiplication table for S_3.
2. (a) Prove that $GL_n(\mathbb{R})$ is a group.
   (b) Prove that S_n is a group.
3. Let S be a set with an associative law of composition and with an identity element. Prove that the subset of S consisting of invertible elements is a group.
4. Solve for y, given that $xyz^{-1}w = 1$ in a group.
5. Assume that the equation $xyz = 1$ holds in a group G. Does it follow that $yzx = 1$? That $yxz = 1$?
6. Write out all ways in which one can form a product of four elements a, b, c, d in the given order.
7. Let S be any set. Prove that the law of composition defined by $ab = a$ is associative.
8. Give an example of 2×2 matrices such that $A^{-1}B \neq BA^{-1}$.
9. Show that if $ab = a$ in a group, then $b = 1$, and if $ab = 1$, then $b = a^{-1}$.
10. Let a, b be elements of a group G. Show that the equation $ax = b$ has a unique solution in G.
11. Let G be a group, with multiplicative notation. We define an *opposite group* G^0 with law of composition $a \circ b$ as follows: The underlying set is the same as G, but the law of composition is the opposite; that is, we define $a \circ b = ba$. Prove that this defines a group.

2. Subgroups

1. Determine the elements of the cyclic group generated by the matrix $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ explicitly.
2. Let a, b be elements of a group G. Assume that a has order 5 and that $a^3b = ba^3$. Prove that $ab = ba$.
3. Which of the following are subgroups?
   (a) $GL_n(\mathbb{R}) \subset GL_n(\mathbb{C})$.
   (b) $\{1, -1\} \subset \mathbb{R}^{\times}$.
   (c) The set of positive integers in $\mathbb{Z}^{+}$.
   (d) The set of positive reals in $\mathbb{R}^{\times}$.
   (e) The set of all matrices $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, with $a \neq 0$, in $GL_2(\mathbb{R})$.
4. Prove that a nonempty subset H of a group G is a subgroup if for all $x, y \in H$ the element xy^{-1} is also in H.
5. An nth root of unity is a complex number z such that $z^n = 1$. Prove that the nth roots of unity form a cyclic subgroup of $\mathbb{C}^{\times}$ of order n.
6. (a) Find generators and relations analogous to (2.13) for the Klein four group.
   (b) Find all subgroups of the Klein four group.
7. Let a and b be integers.
   (a) Prove that the subset $a\mathbb{Z} + b\mathbb{Z}$ is a subgroup of $\mathbb{Z}^{+}$.
   (b) Prove that a and $b + 7a$ generate the subgroup $a\mathbb{Z} + b\mathbb{Z}$.
8. Make a multiplication table for the quaternion group H.
9. Let H be the subgroup generated by two elements a,b of a group G. Prove that if $ab = ba$, then H is an abelian group.
10. (a) Assume that an element x of a group has order rs. Find the order of x^r.
    (b) Assuming that x has arbitrary order n, what is the order of x^r?
11. Prove that in any group the orders of ab and of ba are equal.
12. Describe all groups G which contain no proper subgroup.
13. Prove that every subgroup of a cyclic group is cyclic.
14. Let G be a cyclic group of order n, and let r be an integer dividing n. Prove that G contains exactly one subgroup of order r.
15. (a) In the definition of subgroup, the identity element in H is required to be the identity of G. One might require only that H have an identity element, not that it is the same as the identity in G. Show that if H has an identity at all, then it is the identity in G, so this definition would be equivalent to the one given.
    (b) Show the analogous thing for inverses.
16. (a) Let G be a cyclic group of order 6. How many of its elements generate G?
    (b) Answer the same question for cyclic groups of order 5, 8, and 10.
    (c) How many elements of a cyclic group of order n are generators for that group?
17. Prove that a group in which every element except the identity has order 2 is abelian.
18. According to Chapter 1 (2.18), the elementary matrices generate $GL_n(\mathbb{R})$.
    (a) Prove that the elementary matrices of the first and third types suffice to generate this group.
    (b) The *special linear group* $SL_n(\mathbb{R})$ is the set of real $n \times n$ matrices whose determinant is 1. Show that $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$.

*(c) Use row reduction to prove that the elementary matrices of the first type generate $SL_n(\mathbb{R})$. Do the 2×2 case first.

19. Determine the number of elements of order 2 in the symmetric group S_4.

20. (a) Let a, b be elements of an abelian group of orders m, n respectively. What can you say about the order of their product ab?
*(b) Show by example that the product of elements of finite order in a nonabelian group need not have finite order.

21. Prove that the set of elements of finite order in an abelian group is a subgroup.

22. Prove that the greatest common divisor of a and b, as defined in the text, can be obtained by factoring a and b into primes and collecting the common factors.

3. Isomorphisms

1. Prove that the additive group $\mathbb{R}^+$ of real numbers is isomorphic to the multiplicative group P of positive reals.
2. Prove that the products ab and ba are conjugate elements in a group.
3. Let a, b be elements of a group G, and let $a' = bab^{-1}$. Prove that $a = a'$ if and only if a and b commute.
4. (a) Let $b' = aba^{-1}$. Prove that $b'^n = ab^na^{-1}$.
(b) Prove that if $aba^{-1} = b^2$, then $a^3ba^{-3} = b^8$.
5. Let $\varphi: G \longrightarrow G'$ be an isomorphism of groups. Prove that the inverse function φ^{-1} is also an isomorphism.
6. Let $\varphi: G \longrightarrow G'$ be an isomorphism of groups, let $x, y \in G$, and let $x' = \varphi(x)$ and $y' = \varphi(y)$.
(a) Prove that the orders of x and of x' are equal.
(b) Prove that if $xyx = yxy$, then $x'y'x' = y'x'y'$.
(c) Prove that $\varphi(x^{-1}) = x'^{-1}$.
7. Prove that the matrices $\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix}$ are conjugate elements in the group $GL_2(\mathbb{R})$ but that they are not conjugate when regarded as elements of $SL_2(\mathbb{R})$.
8. Prove that the matrices $\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ & 2 \end{bmatrix}$ are conjugate in $GL_2(\mathbb{R})$.
9. Find an isomorphism from a group G to its opposite group G^0 (Section 2, exercise 12).
10. Prove that the map $A \rightsquigarrow (A^t)^{-1}$ is an automorphism of $GL_n(\mathbb{R})$.
11. Prove that the set Aut G of automorphisms of a group G forms a group, the law of composition being composition of functions.
12. Let G be a group, and let $\varphi: G \longrightarrow G$ be the map $\varphi(x) = x^{-1}$.
(a) Prove that φ is bijective.
(b) Prove that φ is an automorphism if and only if G is abelian.
13. (a) Let G be a group of order 4. Prove that every element of G has order 1, 2, or 4.
(b) Classify groups of order 4 by considering the following two cases:
(i) G contains an element of order 4.
(ii) Every element of G has order < 4.
14. Determine the group of automorphisms of the following groups.
(a) $\mathbb{Z}^+$, (b) a cyclic group of order 10, (c) S_3.

15. Show that the functions $f = 1/x$, $g = (x - 1)/x$ generate a group of functions, the law of composition being composition of functions, which is isomorphic to the symmetric group S_3.

16. Give an example of two isomorphic groups such that there is more than one isomorphism between them.

4. *Homomorphisms*

1. Let G be a group, with law of composition written $x \# y$. Let H be a group with law of composition $u \circ v$. What is the condition for a map $\varphi: G \longrightarrow H'$ to be a homomorphism?

2. Let $\varphi: G \longrightarrow G'$ be a group homomorphism. Prove that for any elements $a_1, \ldots, a_k$ of G, $\varphi(a_1 \cdots a_k) = \varphi(a_1) \cdots \varphi(a_k)$.

3. Prove that the kernel and image of a homomorphism are subgroups.

4. Describe all homomorphisms $\varphi: \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$, and determine which are injective, which are surjective, and which are isomorphisms.

5. Let G be an abelian group. Prove that the nth power map $\varphi: G \longrightarrow G$ defined by $\varphi(x) = x^n$ is a homomorphism from G to itself.

6. Let $f: \mathbb{R}^+ \longrightarrow \mathbb{C}^\times$ be the map $f(x) = e^{ix}$. Prove that f is a homomorphism, and determine its kernel and image.

7. Prove that the absolute value map $| \; |: \mathbb{C}^\times \longrightarrow \mathbb{R}^\times$ sending $\alpha \rightsquigarrow |\alpha|$ is a homomorphism, and determine its kernel and image.

8. **(a)** Find all subgroups of S_3, and determine which are normal.
(b) Find all subgroups of the quaternion group, and determine which are normal.

9. **(a)** Prove that the composition $\varphi \circ \psi$ of two homomorphisms φ, ψ is a homomorphism.
(b) Describe the kernel of $\varphi \circ \psi$.

10. Let $\varphi: G \longrightarrow G'$ be a group homomorphism. Prove that $\varphi(x) = \varphi(y)$ if and only if $xy^{-1} \in \ker \varphi$.

11. Let G, H be cyclic groups, generated by elements x, y. Determine the condition on the orders m, n of x and y so that the map sending $x^i \rightsquigarrow y^i$ is a group homomorphism.

12. Prove that the $n \times n$ matrices M which have the block form $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ with $A \in GL_r(\mathbb{R})$ and $D \in GL_{n-r}(\mathbb{R})$ form a subgroup P of $GL_n(\mathbb{R})$, and that the map $P \longrightarrow GL_r(\mathbb{R})$ sending $M \rightsquigarrow A$ is a homomorphism. What is its kernel?

13. **(a)** Let H be a subgroup of G, and let $g \in G$. The *conjugate subgroup* gHg^{-1} is defined to be the set of all conjugates ghg^{-1}, where $h \in H$. Prove that gHg^{-1} is a subgroup of G.
(b) Prove that a subgroup H of a group G is normal if and only if $gHg^{-1} = H$ for all $g \in G$.

14. Let N be a normal subgroup of G, and let $g \in G$, $n \in N$. Prove that $g^{-1}ng \in N$.

15. Let φ and ψ be two homomorphisms from a group G to another group G', and let $H \subset G$ be the subset $\{x \in G \mid \varphi(x) = \psi(x)\}$. Prove or disprove: H is a subgroup of G.

16. Let $\varphi: G \longrightarrow G'$ be a group homomorphism, and let $x \in G$ be an element of order r. What can you say about the order of $\varphi(x)$?

17. Prove that the center of a group is a normal subgroup.

18. Prove that the center of $GL_n(\mathbb{R})$ is the subgroup $Z = \{cI \mid c \in \mathbb{R}, c \neq 0\}$.

19. Prove that if a group contains exactly one element of order 2, then that element is in the center of the group.

20. Consider the set U of real 3×3 matrices of the form

$$\begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix}.$$

(a) Prove that U is a subgroup of $SL_n(\mathbb{R})$.
(b) Prove or disprove: U is normal.
*(c) Determine the center of U.

21. Prove by giving an explicit example that $GL_2(\mathbb{R})$ is not a normal subgroup of $GL_2(\mathbb{C})$.

22. Let $\varphi: G \longrightarrow G'$ be a surjective homomorphism.
(a) Assume that G is cyclic. Prove that G' is cyclic.
(b) Assume that G is abelian. Prove that G' is abelian.

23. Let $\varphi: G \longrightarrow G'$ be a surjective homomorphism, and let N be a normal subgroup of G. Prove that $\varphi(N)$ is a normal subgroup of G'.

5. *Equivalence Relations and Partitions*

1. Prove that the nonempty fibres of a map form a partition of the domain.

2. Let S be a set of groups. Prove that the relation $G \sim H$ if G is isomorphic to H is an equivalence relation on S.

3. Determine the number of equivalence relations on a set of five elements.

4. Is the intersection $R \cap R'$ of two equivalence relations $R, R' \subset S \times S$ an equivalence relation? Is the union?

5. Let H be a subgroup of a group G. Prove that the relation defined by the rule $a \sim b$ if $b^{-1}a \in H$ is an equivalence relation on G.

6. (a) Prove that the relation x conjugate to y in a group G is an equivalence relation on G.
(b) Describe the elements a whose conjugacy class (= equivalence class) consists of the element a alone.

7. Let R be a relation on the set $\mathbb{R}$ of real numbers. We may view R as a subset of the (x, y)-plane. Explain the geometric meaning of the reflexive and symmetric properties.

8. With each of the following subsets R of the (x, y)-plane, determine which of the axioms (5.2) are satisfied and whether or not R is an equivalence relation on the set $\mathbb{R}$ of real numbers.
(a) $R = \{(s,s) \mid s \in \mathbb{R}\}$.
(b) $R =$ empty set.
(c) $R =$ locus $\{y = 0\}$.
(d) $R =$ locus $\{xy + 1 = 0\}$.
(e) $R =$ locus $\{x^2y - xy^2 - x + y = 0\}$.
(f) $R =$ locus $\{x^2 - xy + 2x - 2y = 0\}$.

9. Describe the smallest equivalence relation on the set of real numbers which contains the line $x - y = 1$ in the (x, y)-plane, and sketch it.

10. Draw the fibres of the map from the (x,z)-plane to the y-axis defined by the map $y = zx$.

11. Work out rules, obtained from the rules on the integers, for addition and multiplication on the set (5.8).
12. Prove that the cosets (5.14) are the fibres of the map φ.

6. Cosets

1. Determine the index $[\mathbb{Z} : n\mathbb{Z}]$.
2. Prove directly that distinct cosets do not overlap.
3. Prove that every group whose order is a power of a prime p contains an element of order p.
4. Give an example showing that left cosets and right cosets of $GL_2(\mathbb{R})$ in $GL_2(\mathbb{C})$ are not always equal.
5. Let H, K be subgroups of a group G of orders 3,5 respectively. Prove that $H \cap K = \{1\}$.
6. *Justify (6.15) carefully.*
7. **(a)** Let G be an abelian group of odd order. Prove that the map $\varphi: G \longrightarrow G$ defined by $\varphi(x) = x^2$ is an automorphism.
   (b) Generalize the result of (a).
8. Let W be the additive subgroup of $\mathbb{R}^m$ of solutions of a system of homogeneous linear equations $AX = 0$. Show that the solutions of an inhomogeneous system $AX = B$ form a coset of W.
9. Let H be a subgroup of a group G. Prove that the number of left cosets is equal to the number of right cosets **(a)** if G is finite and **(b)** in general.
10. **(a)** Prove that every subgroup of index 2 is normal.
    (b) Give an example of a subgroup of index 3 which is not normal.
11. Classify groups of order 6 by analyzing the following three cases.
    (a) G contains an element of order 6.
    (b) G contains an element of order 3 but none of order 6.
    (c) All elements of G have order 1 or 2.
12. Let G, H be the following subgroups of $GL_2(\mathbb{R})$:

$$G = \left\{\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\right\},\ H = \left\{\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}\right\},\ x > 0.$$

An element of G can be represented by a point in the (x, y)-plane. Draw the partitions of the plane into left and into right cosets of H.

7. Restriction of a Homomorphism to a Subgroup

1. Let G and G' be finite groups whose orders have no common factor. Prove that the only homomorphism $\varphi: G \longrightarrow G'$ is the trivial one $\varphi(x) = 1$ for all x.
2. Give an example of a permutation of even order which is odd and an example of one which is even.
3. **(a)** Let H and K be subgroups of a group G. Prove that the intersection $xH \cap yK$ of two cosets of H and K is either empty or else is a coset of the subgroup $H \cap K$.
   (b) Prove that if H and K have finite index in G then $H \cap K$ also has finite index.

4. Prove Proposition (7.1).
5. Let H, N be subgroups of a group G, with N normal. Prove that $HN = NH$ and that this set is a subgroup.
6. Let $\varphi: G \longrightarrow G'$ be a group homomorphism with kernel K, and let H be another subgroup of G. Describe $\varphi^{-1}(\varphi(H))$ in terms of H and K.
7. Prove that a group of order 30 can have at most 7 subgroups of order 5.

*8. Prove the *Correspondence Theorem:* Let $\varphi: G \longrightarrow G'$ be a surjective group homomorphism with kernel N. The set of subgroups H' of G' is in bijective correspondence with the set of subgroups H of G which contain N, the correspondence being defined by the maps $H \rightsquigarrow \varphi(H)$ and $\varphi^{-1}(H') \leftarrow\!\!\sim H'$. Moreover, normal subgroups of G correspond to normal subgroups of G'.

9. Let G and G' be cyclic groups of orders 12 and 6 generated by elements x, y respectively, and let $\varphi: G \longrightarrow G'$ be the map defined by $\varphi(x^i) = y^i$. Exhibit the correspondence referred to the previous problem explicitly.

8. Products of Groups

1. Let G, G' be groups. What is the order of the product group $G \times G'$?
2. Is the symmetric group S_3 a direct product of nontrivial groups?
3. Prove that a finite cyclic group of order rs is isomorphic to the product of cyclic groups of orders r and s if and only if r and s have no common factor.
4. In each of the following cases, determine whether or not G is isomorphic to the product of H and K.
   (a) $G = \mathbb{R}^\times$, $H = \{\pm 1\}$, $K = \{\text{positive real numbers}\}$.
   (b) $G = \{\text{invertible upper triangular } 2 \times 2 \text{ matrices}\}$, $H = \{\text{invertible diagonal matrices}\}$, $K = \{\text{upper triangular matrices with diagonal entries } 1\}$.
   (c) $G = \mathbb{C}^\times$ and $H = \{\text{unit circle}\}$, $K = \{\text{positive reals}\}$.
5. Prove that the product of two infinite cyclic groups is not infinite cyclic.
6. Prove that the center of the product of two groups is the product of their centers.
7. (a) Let H, K be subgroups of a group G. Show that the set of products $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup if and only if $HK = KH$.
   (b) Give an example of a group G and two subgroups H, K such that HK is not a subgroup.
8. Let G be a group containing normal subgroups of orders 3 and 5 respectively. Prove that G contains an element of order 15.
9. Let G be a finite group whose order is a product of two integers: $n = ab$. Let H, K be subgroups of G of orders a and b respectively. Assume that $H \cap K = \{1\}$. Prove that $HK = G$. Is G isomorphic to the product group $H \times K$?
10. Let $x \in G$ have order m, and let $y \in G'$ have order n. What is the order of (x, y) in $G \times G'$?
11. Let H be a subgroup of a group G, and let $\varphi: G \longrightarrow H$ be a homomorphism whose restriction to H is the identity map: $\varphi(h) = h$, if $h \in H$. Let $N = \ker \varphi$.
    (a) Prove that if G is abelian then it is isomorphic to the product group $H \times N$.
    (b) Find a bijective map $G \longrightarrow H \times N$ without the assumption that G is abelian, but show by an example that G need not be isomorphic to the product group.

9. Modular Arithmetic

1. Compute $(7 + 14)(3 - 16)$ modulo 17.
2. (a) Prove that the square a^2 of an integer a is congruent to 0 or 1 modulo 4.
   (b) What are the possible values of a^2 modulo 8?
3. (a) Prove that 2 has no inverse modulo 6.
   (b) Determine all integers n such that 2 has an inverse modulo n.
4. Prove that every integer a is congruent to the sum of its decimal digits modulo 9.
5. Solve the congruence $2x \equiv 5$ (a) modulo 9 and (b) modulo 6.
6. Determine the integers n for which the congruences $x + y \equiv 2$, $2x - 3y \equiv 3$ (modulo n) have a solution.
7. Prove the associative and commutative laws for multiplication in $\mathbb{Z}/n\mathbb{Z}$.
8. Use Proposition (2.6) to prove the *Chinese Remainder Theorem:* Let m, n, a, b be integers, and assume that the greatest common divisor of m and n is 1. Then there is an integer x such that $x \equiv a$ (modulo m) and $x \equiv b$ (modulo n).

10. Quotient Groups

1. Let G be the group of invertible real upper triangular 2×2 matrices. Determine whether or not the following conditions describe normal subgroups H of G. If they do, use the First Isomorphism Theorem to identify the quotient group G/H.
   (a) $a_{11} = 1$ (b) $a_{12} = 0$ (c) $a_{11} = a_{22}$ (d) $a_{11} = a_{22} = 1$
2. Write out the proof of (10.1) in terms of elements.
3. Let P be a partition of a group G with the property that for any pair of elements A, B of the partition, the product set AB is contained entirely within another element C of the partition. Let N be the element of P which contains 1. Prove that N is a normal subgroup of G and that P is the set of its cosets.
4. (a) Consider the presentation (1.17) of the symmetric group S_3. Let H be the subgroup $\{1, y\}$. Compute the product sets $(1H)(xH)$ and $(1H)(x^2H)$, and verify that they are not cosets.
   (b) Show that a cyclic group of order 6 has two generators satisfying the rules $x^3 = 1$, $y^2 = 1$, $yx = xy$.
   (c) Repeat the computation of (a), replacing the relations (1.18) by the relations given in part (b). Explain.
5. Identify the quotient group $\mathbb{R}^\times/P$, where P denotes the subgroup of positive real numbers.
6. Let $H = \{\pm 1, \pm i\}$ be the subgroup of $G = \mathbb{C}^\times$ of fourth roots of unity. Describe the cosets of H in G explicitly, and prove that G/H is isomorphic to G.
7. Find all normal subgroups N of the quaternion group H, and identify the quotients H/N.
8. Prove that the subset H of $G = GL_n(\mathbb{R})$ of matrices whose determinant is positive forms a normal subgroup, and describe the quotient group G/H.
9. Prove that the subset $G \times 1$ of the product group $G \times G'$ is a normal subgroup isomorphic to G and that $(G \times G')/(G \times 1)$ is isomorphic to G'.
10. Describe the quotient groups $\mathbb{C}^\times/P$ and $\mathbb{C}^\times/U$, where U is the subgroup of complex numbers of absolute value 1 and P denotes the positive reals.
11. Prove that the groups $\mathbb{R}^+/\mathbb{Z}^+$ and $\mathbb{R}^+/2\pi\mathbb{Z}^+$ are isomorphic.

Miscellaneous Problems

1. What is the product of all mth roots of unity in $\mathbb{C}$?
2. Compute the group of automorphisms of the quaternion group.
3. Prove that a group of even order contains an element of order 2.
4. Let $K \subset H \subset G$ be subgroups of a finite group G. Prove the formula $[G : K] = [G : H][H : K]$.

*5. A *semigroup* S is a set with an associative law of composition and with an identity. But elements are not required to have inverses, so the cancellation law need not hold. The semigroup S is said to be generated by an element s if the set $\{1, s, s^2, \ldots\}$ of nonnegative powers of s is the whole set S. For example, the relations $s^2 = 1$ and $s^2 = s$ describe two different semigroup structures on the set $\{1, s\}$. Define isomorphism of semigroups, and describe all isomorphism classes of semigroups having a generator.

6. Let S be a semigroup with finitely many elements which satisfies the Cancellation Law (1.12). Prove that S is a group.

*7. Let $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$ be points in k-dimensional space $\mathbb{R}^k$. A *path* from a to b is a continuous function on the interval $[0, 1]$ with values in $\mathbb{R}^k$, that is, a function $f\colon [0,1] \longrightarrow \mathbb{R}^k$, sending $t \rightsquigarrow f(t) = (x_1(t), \ldots, x_k(t))$, such that $f(0) = a$ and $f(1) = b$. If S is a subset of $\mathbb{R}^k$ and if $a, b \in S$, we define $a \sim b$ if a and b can be joined by a path lying entirely in S.
   (a) Show that this is an equivalence relation on S. Be careful to check that the paths you construct stay within the set S.
   (b) A subset S of $\mathbb{R}^k$ is called *path connected* if $a \sim b$ for any two points $a, b \in S$. Show that every subset S is partitioned into path-connected subsets with the property that two points in different subsets can not be connected by a path in S.
   (c) Which of the following loci in $\mathbb{R}^2$ are path-connected? $\{x^2 + y^2 = 1\}$, $\{xy = 0\}$, $\{xy = 1\}$.

*8. The set of $n \times n$ matrices can be identified with the space $\mathbb{R}^{n \times n}$. Let G be a subgroup of $GL_n(\mathbb{R})$. Prove each of the following.
   (a) If $A, B, C, D \in G$, and if there are paths in G from A to B and from C to D, then there is a path in G from AC to BD.
   (b) The set of matrices which can be joined to the identity I forms a normal subgroup of G (called the *connected component* of G).

*9. (a) Using the fact that $SL_n(\mathbb{R})$ is generated by elementary matrices of the first type (see exercise 18, Section 2), prove that this group is path-connected.
   (b) Show that $GL_n(\mathbb{R})$ is a union of two path-connected subsets, and describe them.

10. Let H, K be subgroups of a group G, and let $g \in G$. The set

$$HgK = \{x \in G \mid x = hgk \text{ for some } h \in H, k \in K\}$$

is called a *double coset*.
   (a) Prove that the double cosets partition G.
   (b) Do all double cosets have the same order?

11. Let H be a subgroup of a group G. Show that the double cosets HgH are the left cosets gH if H is normal, but that if H is not normal then there is a double coset which properly contains a left coset.

*12. Prove that the double cosets in $GL_n(\mathbb{R})$ of the subgroups $H = \{\text{lower triangular matrices}\}$ and $K = \{\text{upper triangular matrices}\}$ are the sets HPK, where P is a permutation matrix.