Homework 5

Theory of Numbers (Fall 2014)
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Due Date: Monday, December 1, 2014

Questions

1. Consider the sequence A_n defined as follows:
   - $A_0 = 0$, $A_1 = 3$.
   - $A_n = 7A_{n-1} - 10A_{n-2}$, for each $n \geq 2$.

   Use generating functions to find a formula for A_n.

2. Let a, n be natural numbers. Suppose $a^n - 1$ is a prime.
   - Show that $a = 2$.
   - Show that n must be prime.

3. The goal of this problem is to give a new proof of Fermat's little theorem.
   - Suppose p is prime, and that k is an integer with $1 < k < p$.
     Prove that $\binom{p}{k}$ is divisible by p.
   - Suppose p is prime. Prove, by induction on n, that
     $$n^p \equiv n \mod p.$$
     You will need the previous part of this problem.
   - Deduce that if n is relatively prime to p, then
     $$n^{p-1} \equiv 1 \mod p.$$

4. Let $n = p \cdot q$, where p and q are distinct primes. Show that n does not divide $\binom{n}{p}$.

5. Show that 3 is the only natural number p such that p, $p + 2$ and $p + 4$ are all prime.

6. Show that there are infinitely many natural numbers n such that $n, n + 1, n + 2, \ldots, n + 1000$ are all composite.

7. **BONUS:** Show that if $n \equiv 3 \mod 4$, then there must exists some prime p dividing n such that $p \equiv 3 \mod 4$.

   Use this to show the infinitude of primes $\equiv 3 \bmod 4$ as follows. Suppose there were only finitely many such primes $p_1, \ldots, p_t$. Then consider the number $4 \cdot p_1 \cdot p_2 \cdot \ldots \cdot p_t + 3$.