Homework 3

Theory of Numbers (Fall 2014)
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Due Date: Monday, October 20, 2014

Questions

1. Compute $7^{713} \bmod 4$. Compute $7^{713} \bmod 25$.

   Use this to compute the last two digits of 7^{713} (in base 10). Show your work.

2. Find all $n \in \mathbb{Z}$ which satisfy:
$$n \equiv 2 \mod 7.$$
$$n \cdot 4 \equiv 5 \mod 9.$$
$$n \cdot 7 \equiv 6 \mod 23.$$

   Show your work.

3. Compute $\phi(425)$. Show your work.

4. There are infinitely many lightbulbs $B_1, B_2, B_3, \ldots$. Each bulb can be either on or off. Initially they are all off.

   Every time-step, some of them will get toggled (they change from on to off or from off to on), according to the following rules.

   - At the first time-step, all the bulbs $B_1, B_2, B_3, \ldots$ are toggled. (So they are all on after this time step.)
   - At the second time-step, the bulbs $B_2, B_4, B_6, \ldots$ are toggled. (So only the odd numbered bulbs are on after this time step.)
   - $\ldots$ (and so on ...)
   - In general, at the ith time-step, the bulbs $B_i, B_{2i}, B_{3i}, \ldots$ are toggled.
   - $\ldots$ (and so on ...)

   This process goes on forever.

   Which bulbs eventually stay on forever? Prove your answer.

5. Compute a few values, and then guess and prove formulas for:

   - $0^n \bmod 7$
   - $1^n \bmod 7$
   - $2^n \bmod 7$
   - $3^n \bmod 7$

- $4^n \bmod 7$
- $5^n \bmod 7$
- $6^n \bmod 7$

6. BONUS: How would you choose $n \leq 1000$ so as to minimize $\frac{\phi(n)}{n}$?