1 Ramsey Theory for regular graphs

**Theorem 1.** Take the complete bipartite graph $K(n,n)$ with $n$ vertices, $n \geq 1000$, on either side. Color all edges red or blue arbitrarily. Then, there is a monochromatic $K_{2,2}$.

*Proof.* Number of red edges or number of blue is $1/2n^2$. So by theorem from last time and since $1/2n^2 \geq O(n^3/2)$, there is $K_{2,2}$ in the more popular color. \( \square \)

**Theorem 2.** For $n \geq 1000$. Take $K_n$ (the complete graph on $n$ vertices). Color all the edges red or blue arbitrarily. Then, there is a monochromatic $K_3$ (triangle).

Q: Does $\geq 1/2\binom{n}{2}$ edges in an $n$-vertex graph guarantee a triangle?

A: No. $K_{2,2}$ has $n^2/4$ edges but has no triangle. So previous reasoning doesn’t work as most popular color can be in $K_{2,2}$.

*Proof.* For $n = 6$.

Take a vertex $v$. $v$ has 5 edges (in $K_6$). $\geq 3$ edges are of the same color, say red. Let $u_1, u_2, u_3$ be 3 neighbors of $v$ such that $vu_i$ is red. Then, if any $u_iu_j$ is red then $vu_ivj$ is a red triangle. Else, $u_1u_2u_3$ is a blue triangle. \( \square \)

**Theorem 3.** Ramsey’s theorem: For any number of colors $c$ and for any size of clique $k$, there exists $n_0$ such that for every $n \geq n_0$ and for any $c$-coloring of the edges of $K_n$, there exists some monochromatic $K_k$.

*Proof.* Assume $n \geq \text{TBD}$.

Pick any vertex $v$. There are $n - 1$ edges from $v$. So $\geq \frac{n-1}{c}$ edges of the same color, $\alpha$.

Let $S = \{ \text{vertices joined to } v \text{ by color } \alpha \}$.

$|S| \geq \frac{n-1}{c}$.

Claim: $P(k_1, ..., k_c) = \forall k_1, ..., k_c, \exists n_0(k_1, ..., k_c)$ s.t. $\forall n \geq n_0$, any $c$-coloring of edges of $K_n$ has either

- $K_{k_1}$ in color 1.
- $K_{k_2}$ in color 2.
- $\vdots$
- $K_{k_c}$ in color $c$. “
If $|S| \geq n_0(k_1, ..., k_{\alpha-1}, k_{\alpha} - 1, k_{\alpha+1}, ..., k_c)$ then there exists either:

- $K_{k_1}$ in color 1 (done)
- $K_{k_2}$ in color 2 (done)
  
- $K_{k_{\alpha - 1}}$ in color $\alpha - 1$ (done).
- $K_{k_{\alpha - 1}}$ in color $\alpha$. (done with $v$ included $\Rightarrow$ gives $K_{k_{\alpha}}$).
  
- $K_{k_c}$ in color $c$ (done).

Value of $n_0(k_1, ..., k_c) = 1 + \max_\alpha n_0(k_1, ..., k_{\alpha-1}, k_{\alpha} - 1, k_{\alpha+1}, ..., k_c)$.

Can take $n_0(k_1, ..., k_c) = 1 + \sum_\alpha n_0(k_1, ..., k_{\alpha-1}, k_{\alpha} - 1, k_{\alpha+1}, ..., k_c)$\hfill $\square$

**Theorem 4.** Infinite (countable) version of Ramsey’s Theorem: For a complete graph on countable infinite number of vertices and for $c \in \mathbb{N}$ colors, there exists an infinite countable monochromatic complete subset of the graph.

**Proof.** Start with a vertex $v_0$. $\exists$ some color $\alpha_0$ s.t. there are infinitely many edges of $v_0$ colored $\alpha_0$. Let $S_0$ be the set of neighbors of $v_0$ with edges of color $\alpha_0$.

Now move into $S_0$. Pick $v_1 \in S_0$. There are infinitely many edges from $v_1$ to $S_0$ and at least infinitely many are of the same color, say $\alpha_2$. Let $S_1 = \{ u \in S_0 \ | \ v_1 u \ \text{is colored} \ \alpha_1 \}$, $|S_1| = \infty$.

Repeat.

We get sequences:

- $v_0, v_1, ..., $ with $v_i \in S_{i-1}$.
- $\alpha_0, \alpha_1, ..., $ s.t. $\alpha_i \in [c]$.
- $S_0, S_1, ..., $ with $S_j \subset S_i$ for $j > i$.

Also note that color $(v_i, v_j) = \begin{cases} \alpha_i, & \text{if } i < j \\ \alpha_j, & \text{if } i < i \end{cases}$

Since there are finitely many $\alpha_i$, some color $\beta$ appears as $\alpha_i$ for infinitely many $i$.

Let $I = \{ i \ \text{s.t. } \alpha_i = \beta \}$.

Let $V = \{ v_i \ \text{s.t. } i \in I \}$.

$\text{color}(v_i, v_j)$ for $i, j \in I$ is equal to $\alpha_i$ or $\alpha_j$ which is $\beta$. \hfill $\square$
2 Ramsey for a 3-uniform hypergraph

Definition 5. **Hypergraph**: \((V, E)\) where the set \(E\) is composed of hyperedges, \(E \subseteq P(V)\)

Definition 6. **3-Uniform Hypergraph**: \((V, E)\) where the hyperedges are \(E \subseteq \binom{V}{3}\)

Definition 7. **\(K^{(3)}_n\)**: the complete 3-uniform hypergraph on \(n\) vertices

Theorem 8. For all \(c, k\) there exists an \(n_0\) such that for all \(n \geq n_0\) any colouring of the edges of **\(K^{(3)}_n\)** has a monochromatic **\(K^{(3)}_k\)**.

### 2.1 Proof 1: Using Graph Ramsey into Pigeonhole Principle

**Proof.** Create graph \(G_0\) on \([n] \setminus v_0\) colour ab in **\(K^{(3)}_n\)**. By Ramsey’s theorem for graphs, if \(n - 1\) is big enough then there exists some \(S_0\) such that all edges in \(S_0\) are coloured some colour \(\alpha_0\). So every hyperedge \(v_0ab\) with \(a, b \in S_0\), where \(|S_0| \approx O(\log n)\) and the hyperedge \(v_0ab\) has colour \(\alpha_0\). Take any \(v_1 \in S_0\) create a colouring of \(K_{S_0 \setminus v_0}\) by colouring in **\(K^{(3)}_k\)** of \(vab\). There exists a monochromatic clique \(S_1\) by graph Ramsey with colour \(\alpha_1\). So any hyperedge \(v_1ab\) with \(a, b \in S_1\) has colour \(\alpha_1\). With \(|S_1| \approx \log |S_0|\). Repeating many times, we get:

\[
\begin{array}{cccc}
v_0 & v_1 & v_2 & \ldots \text{vertices} \\
\alpha_0 & \alpha_1 & \alpha_2 & \ldots \text{colours} \\
S_0 & S_1 & S_2 & \ldots |S_i| \approx O(\log |S_{i-1}|)
\end{array}
\]

1. \(S_j \subseteq S_i, \forall i < j\)
2. \(v_i \in S_{i-1}\)
3. colour\((v_iab) = \alpha_i, \forall a, b \in S_i\)

These 3 facts together means that colour\((v_iv_jv_k) = \alpha_i, \forall i < j < k\). Repeat this process \(ck\) times, some colours appear as \(\alpha_i\) for at least \(k\) choices of \(i\). \(I = \{i \text{ such that } \alpha_i \in \beta\}\). Then if \(|I| \geq k\), colour\((v_iv_jv_k) = \beta, \forall i, j, k \in I\) This means that \(v_i : i \in I\) is the desired **\(K^{(3)}_k\)**. Now because
$|S_i| \approx \mathcal{O}(\log |S_{i-1}|)$ we took $\log_2 c^k$ times in order to get the desired $K^{(3)}_k$ and we need to reverse that in order to get the upper limit. This means

$$R^{(3)}(K) \leq 2^{2^{2\ldots}}$$

Where $R^{(3)}(K)$ is the minimum size of a graph that a random $c$ colouring will have a monochromatic $K^{(3)}_k$ and $2^{2^{2\ldots}}$ is a power tower of height $c^k$. \hfill \square

Here we used graph Ramsey at every stage to refine the set then used the pigeonhole principle at the end. It’s possible to use the pigeonhole principle to refine and graph Ramsey at the end.

2.2 Proof 2: Using Pigeonhole Principle into Graph Ramsey

Proof. Start with a pair $v_0v_1$ consider $v_0v_1a$. Some colour $\alpha_0$ is most popular. Zoom into that:

$$S_0 = \{a : \text{colour}(v_0v_1a) = \beta\}$$

$$|S_0| \geq \frac{n-2}{c}$$

Pick $v_2 \in S_1$. For each $b \in S_0$ consider the tuple of colours $(v_0v_2b, v_1v_2b) \in [C]^2$. Let $(\alpha_{02}, \alpha_{12}) \in [C]^2$ be the popular colours seen and let $S_2 = \{b : \text{colour}(v_0v_2b) = \alpha_{02}, \text{colour}(v_1v_2b) = \alpha_{12}\}$$

$$|S_2| \geq \frac{|S_1|}{c^2}$$

Why $c^2$? Because there are $c^2$ possible colours.

$$\begin{align*}
v_0 & \quad v_1 & \quad v_2 & \quad \ldots & \quad v_{l-1} \\
\alpha_{01} & \quad \alpha_{02} & \quad \alpha_{12} & \quad \ldots & \quad \alpha_{l-1} \\
S_1 & \quad S_2 & \quad S_3 & \quad \ldots & \quad S_{l-1}
\end{align*}$$

$$\text{colour}(v_iv_jb) = \alpha_{ij}, \forall b \in S_j, i < j$$

Pick $v_l \in S_{l-1}$. For each $b \in S_{l-1}$ consider colour vector

$$\bar{w}(b) = (\text{colour}(v_0v_1b), \text{colour}(v_1v_1b), \text{colour}(v_2v_1b), \ldots, \text{colour}(v_{i-1}v_1b))$$

Take the most popular colour vector

$$(\alpha_{0l}, \alpha_{1l}, \alpha_{2l}, \ldots, \alpha_{l-1,l})$$

Let

$$S_l = \{b : \bar{w}(b) = (\alpha_{0l}, \alpha_{1l}, \alpha_{2l}, \ldots, \alpha_{l-1,l})\}$$

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Repeat: we have
\[ v_0, v_1, ..., v_i, ..., v_m, \alpha_{ij} \in [C], m \geq R(k) \]
Apply Graph Ramsey
\[ \text{colour}(v_i v_j v_k) = \alpha_{ij} \in [C], i < j < k \]
If \( V \) is the mono clique of size \( k \) then \( \text{colour}(v_i v_j v_k) \) is the same colour which means that we have a monochromatic \( K_k^{(3)} \). \( |S_i| = \frac{|S_{i-1}|}{c^i} \), we need \( i \) to go into \( R_c(k) \)
\[ |S_{R_c(k)}| = \frac{n}{c^{0+1+2+3+...+R(k)}} = \frac{n}{c^{R_c(k)}} > 1 \]
\[ n \geq c^{R_c(k)} = c^{c^{ck}} \]

2.3 Minimum size of \( R_2(k) \)

**Definition 9.** \( R_c(k) = \text{smallest } n \text{ such that any } c \text{ colouring of } K_n \text{ has a monochromatic } K_k \)

**Definition 10.** \( R_c^{(3)}(k) = \text{smallest } n \text{ such that any } c \text{ colouring of } K_n^{(3)} \text{ has a monochromatic } K_k^{(3)} \)

We saw: \( R_c(k) \leq c^{O(ck)} \) and \( R_c^{(3)}(k) \leq c^{O(ck)} \). In the 1930s there was a conjecture by Erdős and Szekeres that \( R_2(k) \leq k^2 \). In the 1940s Erdős showed that \( R_2(k) \geq (\sqrt{2})^k \)

**Theorem 11.** \( R_2(k) \geq (\sqrt{2})^k \)

**Proof.** Proof that \( R_2(k) \geq (\sqrt{2})^k \): Set \( n \geq TBD \), take a random 2-colouring of \( K_n \). We will show with probability greater than 0 that the resulting colouring has no monochromatic k-clique.

\[
P[\text{There exists a monochromatic red k-clique}] \leq \sum_{S \subseteq \binom{[n]}{k}} P[S \text{ is a red clique}] \\
\leq \binom{n}{k} \cdot \frac{1}{2^{\binom{k}{2}}} 
\]

Choose \( n \) so that \( 2^{\binom{k}{2}} < 1/10 \)

\[
\binom{n}{k} < \frac{2^{\binom{k}{2}}}{10}
\]
Using a useful inequality: \( \binom{n}{k} < \left( \frac{en}{k} \right)^k \)

\[
\left( \frac{en}{k} \right)^k < \frac{2^{\binom{k}{2}}}{10}
\]

\[
\frac{en}{k} < \frac{2^{\frac{k-1}{2}}}{10}
\]

\[
n < \frac{k \cdot 2^{\frac{k-1}{2}}}{10e} < O(k \cdot e^{k/2})
\]

\[
P[\text{There exists a monochromatic clique}] \leq P[\text{There exists red k-clique}] + P[\text{There exists blue k-clique}]
\leq \frac{1}{10} + \frac{1}{10} < 1
\]

\[\square\]