A short proof of the Halpern-Lauchli theorem

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November 27, 2025

A write up of this proof is available at:
`www.math.toronto.edu/sunger/halpern-lauchli.pdf`

Definitions

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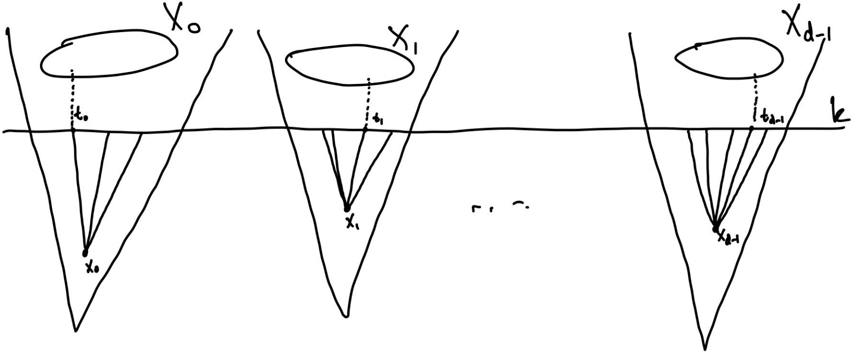
3. $ht(t)$ is the height of node in the tree.
4. $T(k)$ is the k^{th} level of the tree.
5. We use $\prod_{i<d} T_i$ and $\prod_{i<d}^{lev} T_i$ to distinguish between the product and the level product.

Definitions, continued

For $\bar{x} \in \prod_{i<d} T_i$ and $k \in \mathbb{N}$ with $k > \max(ht(x_i))$, sets $X_i \subseteq T_i$ for $i < d$ form a k-$\bar{x}$-dense matrix if for all $\bar{t} \in \prod_{i<d} T_i(k)$ above $\bar{x}$ there is $\bar{y} \in \prod_{i<d} X_i$ above $\bar{t}$.



HL involving dense matrices

Definition (SD_d)

For every coloring $c : \prod_{i<d} T_i \to r$, there are $\bar{x}$ and $k \in \mathbb{N}$ such that there is a monochromatic k-$\bar{x}$-dense matrix.

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HL involving strong subtrees

For a tree T, a subset $S \subseteq T$ is a strong subtree if there is $A \subseteq \mathbb{N}$ infinite such that:

1. For all $s \in S$, $s \in T(n)$ for some $n \in A$ and for all $n \in A$, $S \cap T(n) \neq \emptyset$.
2. If $m < n$ are consecutive elements of A and $s \in S \cap T(m)$, then every immediate successor of s in T has a unique extension in $S \cap T(n)$.

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Definition (SS_d)

For every coloring $c : \prod_{i<d}^{lev} T_i \to r$, there are strong subtrees $S_i \subseteq T_i$ with the same level set such that c is constant on $\prod_{i<d}^{lev} S_i$.

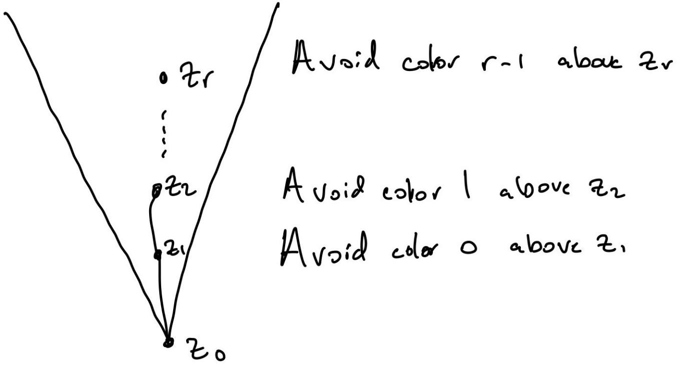
Workflow of the proof

1. Prove SD_1.
2. For all d, SD_d implies DS_d.
3. For all d, DS_d implies SS_d.
4. For all d, DS_d implies SD_{d+1}.

As listed it looks like there is some redundancy but SS_d will be used in the proof of item (4).

Proof of SD_1

Otherwise, for all $x \in T$ and for all $i < r$ there is $y > x$ such that c avoids color i above x.



SD_d implies SD_d for level matrices

A matrix is *level* if all sets in the matrix are contained in the same level.

1. By compactness, for a fixed r there is l such that for all $c : \prod_{i<d} T_i \upharpoonright l \to r$ there is a k-$\bar{x}$-dense matrix.

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2. For each $t \in T_i$, pick $h_i(t) \in T_i(l)$ above t. Define $c^* : \prod_{i<d} T_i \upharpoonright l \to r$, by $c^*(t_i \mid i < d) = c(h(t_i) \mid i < d)$.

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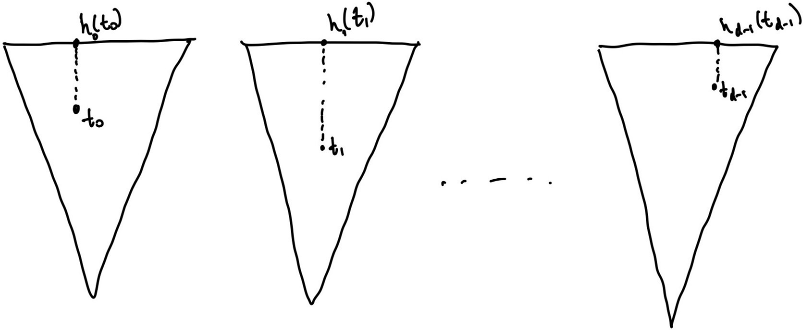
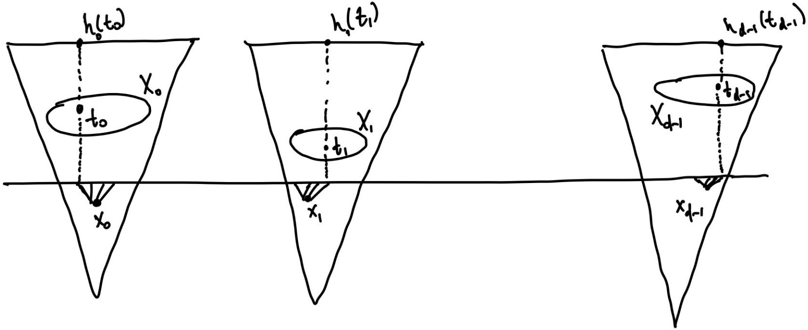


SD_d implies SD_d for level matrices

It follows that if X_i is a monochromatic k-$\bar{x}$-dense matrix for c^*, then $h_i[X_i]$ is a monochromatic k-$\bar{x}$-dense level matrix for c.



SD_d implies DS_d

1. We argue by contradiction.
   Suppose that for every $\bar{x}$ there is $k_{\bar{x}}$ such that for all $k \geq k_{\bar{x}}$ there is no k-$\bar{x}$-dense matrix.

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2. Construct by induction a sequence of levels ℓ_n with the property that $\ell_{n+1} > k_{\bar{x}}$ for all $\bar{x}$ with levels at most ℓ_n.

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3. Let $L = \{\ell_n \mid n \in \mathbb{N}\}$ and apply SD_d to the obvious restriction of the coloring of $\prod_{i<d} T_i$ to levels in L. Any ℓ-$\bar{x}$-dense matrix for this coloring, gives a dense matrix for the original coloring that contradicts the definition of $k_{\bar{x}}$.

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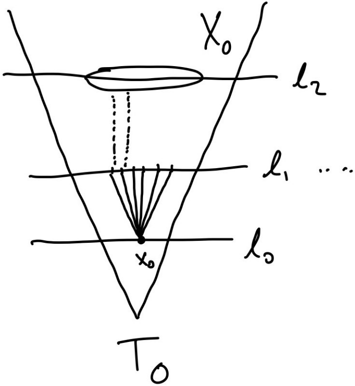

DS_d implies SS_d

A straightforward inductive construction. Let $\bar{x}$ witness DS_d with level matrices for a coloring $\prod_{i<d}^{lev} T_i \to r$. Then for each k we get a color for the k-$\bar{x}$-dense matrix. We can assume this color is constant for all k.

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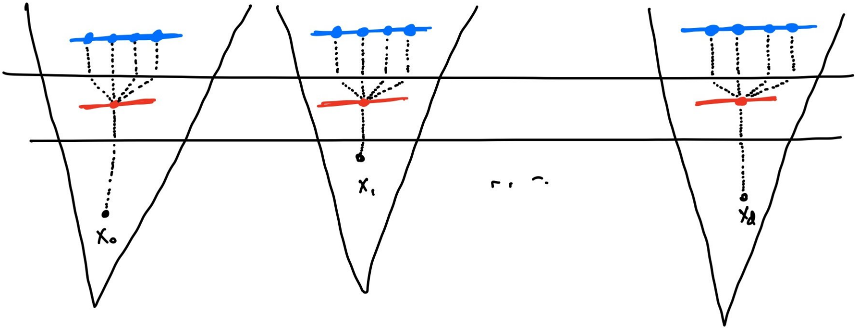
Construct inductively as follows:



DS_d implies SD_{d+1}

Let $c : \prod_{i \leq d} T_i \to r$ be a coloring. For each $t \in T_d$, define $c_t : \prod_{i<d} T_i \to r$ by $c_t(\bar{x}) = c(\bar{x} \frown t)$. By a fusion argument arrange the following

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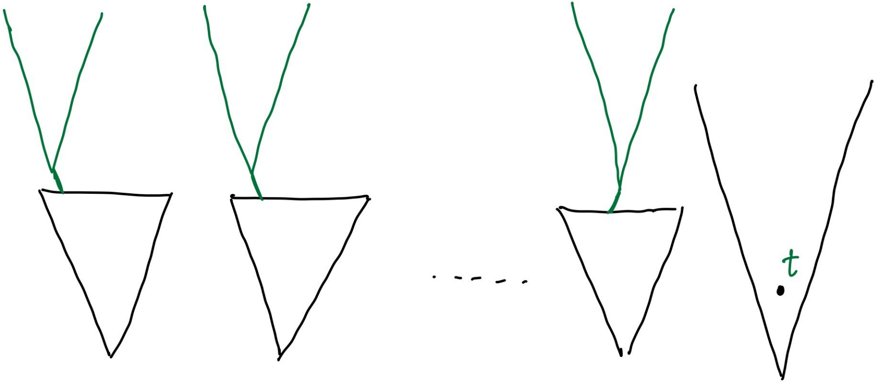
Claim

There are strong subtrees $S_i \subseteq T_i$ with a common level set such that for all $t \in T_d$ and all $\bar{x} \in \prod_{i<d}^{lev} S_i$ with $ht(\bar{x}) \geq ht(t)$, c_t is constant on the product of S_i above $\bar{x}$.

Picture of the proof of the claim

At the inductive step we have fixed d finite strong subtrees and we want to restrict the trees above.



Iterate over choices of green trees and t to make c_t constant on each product.

Pause for a lemma

Lemma

Let T *be an infinite, finitely branching tree and* $g : [T] \to \mathbb{N}$ *be a coloring. There are* $t \in T$, *a dense subset* $D \subseteq T \upharpoonright t$ *and* $j \in \mathbb{N}$ *such that for all* $s \in D$, *there is* $b \in [T]$ *such that* $s \in b$ *and* $g(b) = j$.

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Lemma

Let T be an infinite, finitely branching tree and $g : [T] \to \mathbb{N}$ be a coloring. There are $t \in T$, a dense subset $D \subseteq T \restriction t$ and $j \in \mathbb{N}$ such that for all $s \in D$, there is $b \in [T]$ such that $s \in b$ and $g(b) = j$.

Otherwise, we can inductively build a branch which is not colored.

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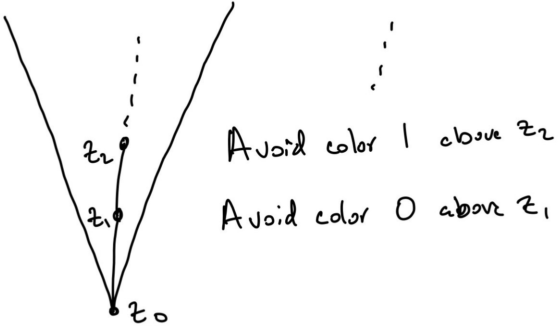

DS_d implies SD_{d+1}, continued

Define a family of colorings $f_b : \prod_{i<d}^{lev} S_i \to r$ for $b \in [T_d]$, by $f_b(\bar{x}) = c(\bar{x} \frown y)$ where y is the unique element of b of height $ht(\bar{x})$.

DS_d implies SD_{d+1}, continued

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For each $b \in [T_d]$, using DS_d with level matrices we have $\bar{x}_b$ and $i_b < r$. Now $b \mapsto (\bar{x}_b, i_b)$ is a countable coloring, so by the previous lemma we have $t \in T_d$ and dense $D \subseteq T_d \restriction t$ together with constant value $\bar{x}^*$ and $i^* < r$.

The end of the proof

We construct a monochromatic k-$(\bar{x}^* \frown t)$-dense matrix where k is the maximum of the heights of x_i^* and t.

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