This is a sketchy write up of selected homework exercises from the quarter. Many of the solutions have details that you will need to fill in. The upshot is that you'll need to fill in the details if I ask you the question on the exam.

Homework 1 problem 7. Our notation for the set of finite sequence of natural numbers is $\omega^{<\omega}$. We can write $\omega^{<\omega} = \bigcup_{n<\omega} \omega^n$ where ω^n is the set of sequences of natural numbers of length n. (We're not using ordinal exponentiation here.) It is enough to show that ω^n is countable for every $n < \omega$.

We can do this by induction. We know that ω is countable. So assume that ω^n is countable for some $n < \omega$. ω^{n+1} has the same cardinality as $\omega^n \times \omega$. The latter set can be written as $\bigcup_{k<\omega} \omega^n \times \{k\}$, hence is countable.

Homework 1 problem 8. Let P be the set of polynomials with integer coefficients. Define a function $f : \omega^{<\omega} \to P$ by $f(s) = s(0) + s(1)x + s(2)x^2 + \ldots s(n-1)x^{n-1}$ where $s : n \to \omega$. Clearly f is a surjection, so P is countable. For each $p \in P$, let R_p be the finite set of roots of p. Now $\{x \mid x \text{ is algebraic } \} = \bigcup_{p\in P} R_p$, hence it is countable.

Notice that in the last two problems we used that a countable union of countable sets is countable a lot.

Homework 2 problem 4. Suppose that $A \subseteq W_0 \times W_1$ is nonempty. Consider $A_1 = \{w \in W_1 \mid \text{there is } w' \in W_0 \text{ such that } (w', w) \in A\}$. A_1 is a nonempty subset of W_1 hence has a $<_1$-least element. We call it w_1. Consider $A_0 = \{w \in W_0 \mid (w, w_1) \in A\}$. A_0 is a nonempty subset of W_0, hence it has a $<_0$-least element which we call w_0. Now check that (w_0, w_1) is the least element of A.

Homework 3 problem 3. We will prove the special case when $(W, <)$ is a well-ordering. This shows that every well-ordered set is isomorphic to an ordinal. Let $(W, <)$ be a well ordering and go by transfinite induction to define a function f with domain W. Suppose that we have defined $f \restriction W^{<x}$ for some $x \in W$. Define $f(x) = \{f(y) \mid y < x\}$. This finishes the definition of f. Now prove by transfinite induction that for all $x \in W$, $f(x)$ is an ordinal. Further show that the range of f is an ordinal α. This shows that $(W, <) \simeq (\alpha, \in)$.

Homework 4 problem 1. It is a little tricky to prove that the described ordering is a well-ordering without a picture. Try to draw one for yourself as you read the proof. Let $A \subseteq \aleph_\beta \times \aleph_\beta$ be nonempty. Consider the set $\{max(a) \mid a \in A\}$ (recall that elements of A are pairs and $\max(a)$ just picks the coordinate which is bigger). This has a least element, call it m. Now let $A_0 = \{a \in A \mid \max(a) = m\}$. We know that the least element of A must come from A_0. Now consider the set

$\{\gamma \mid \gamma$ is the first coordinate of some $a \in A_0\}$. This set is nonempty hence has a least element, which we call γ^*. Now we let $A_1 = \{a \in A_0 \mid$ the first coordinate of a is $\gamma^*\}$. Next we consider the set $\{\delta \mid \delta$ is the second coordinate of some $a \in A_1\}$. This set has a least element which we call δ^*. Now prove that (δ^*, γ^*) is the least element of A.

For the rest of the exercise, we focus on part (c). Suppose that $\pi : \aleph_\beta \times \aleph_\beta \to \eta$ is an isomorphism and assume for a contradiction that $\eta > \aleph_\beta$. Then there is a pair $(\gamma, \delta) \in \aleph_\beta \times \aleph_\beta$ such that $\pi(\gamma, \delta) = \aleph_\beta$. Since π is a bijection it follows that $\{(\gamma', \delta') \mid (\gamma', \delta') < (\gamma, \delta)\}$ has cardinality $\aleph_\beta$. Prove that by the induction hypothesis the size of this set is $\aleph_\alpha$ where $\beta = \alpha + 1$. This is a contradiction.

Homework 4 problem 7. Recall that a function f is continuous if and only if for every open subset U of the codomain, $f^{-1}U$ is open in the domain. The f we are considering goes from ω^ω to 2^ω. It is enough to show that for every $s \in 2^{<\omega}$, $f^{-1}N_s$ is open in ω^ω. Suppose that $x \in f^{-1}N_s$. We need to find $t \in \omega^{<\omega}$ such that $x \in N_t$ and $N_t \subseteq f^{-1}N_s$. Note $f(x)$ is determined bit by bit from x, so choose n large enough so that the sum of the first n natural numbers appearing in x is greater than the length of s. Now let $t = x \restriction n$ and prove that $x \in N_t$ and $N_t \subseteq f^{-1}N_s$. I will leave the proof that f is injective to you.

Homework 5 problem 3. Let $[a, b]$ be an interval so that $\{n \mid x_n \in [a, b]\} \in U$. Consider the intervals $I_0 = [a, \frac{a+b}{2}]$ and $I_1 = [\frac{a+b}{2}, b]$. Now either $\{n \mid x_n \in I_0\} \in U$ or $\{n \mid x_n \in I_1\} \in U$ since U is an ultrafilter. So we managed to shrink the interval by $1/2$ while maintaining that the set of indices of x_n in the interval is in U. Use induction to repeat this ω many times and find shrinking sequence of closed intervals. Show that these intervals converge to a point x and show that x is $\lim_U x_n$.

Homework 5 problem 4. If $U = \{A \subseteq \omega \mid k \in A\}$, then $\lim_U x_n = x_k$.

Homework 5 problem 5. (a) is easy. (b) follows from (c) and (a) with $A = \emptyset$. For (c) you essentially need to show that if $x_n \leq y_n$ for all $n < \omega$, then $\lim_U x_n \leq \lim_U y_n$. For (d) you need show that if x_n and y_n are sequences, then $\lim_U x_n + y_n = \lim_U x_n + \lim_U y_n$.

Homework 5 problem 6. There were too many n's in this problem. Suppose that $k \in \mathbb{N}$. Consider the following line of algebra:

$$\lim_U \frac{|A \cap [-n, n]|}{|[-n, n]|} - \lim_U \frac{|A + k \cap [-n, n]|}{|[-n, n]|} = \lim_U \frac{|A \cap [-n, n]| - |A + k \cap [-n, n]|}{|[-n, n]|}$$

This uses the fact about sums of sequences you proved before. Now notice that (the absolute value of) the top of the final fraction is bounded above by k. This is because shifting A by k can only move k elements in to (or possibly out of) $[-n, n]$. So the limit is roughly of the form

$\frac{k}{2n+1}$ which goes to zero as n goes to infinity. Hence it goes to zero "along the ultrafilter".

Homework 5 problem 7. Suppose that there is such a partition $A_1, \ldots A_n, B_1, \ldots B_m$. Now we have the following using properties of μ:

$$1 = \mu(\mathbb{Z}) = \sum_{i=1}^{n} \mu(A_i) + \sum_{i=1}^{m} \mu(B_i)$$

So the sum of the measures of the A's is strictly between 0 and 1. On the other hand, by the previous problem we have

$$1 = \mu(\mathbb{Z}) = \sum_{i=1}^{n} \mu(a_i + A_i) = \sum_{i=1}^{n} \mu(A_i)$$

So the sum of the measure of the A's must be 1. This is a contradiction.

Homework 6 problem 1. We did this using ultrafilters, so see if you can give the ultrafilter proof. For the König infinity lemma proof, let $T = \{f \mid f : n \to k \text{ is a } k\text{-coloring of } G_n \text{ for some } n < \omega\}$ and order T by end extension. In particular $f < g$ if $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and for all $i \in \mathrm{dom}(f)$, $f(i) = g(i)$. Prove that T is an infinite tree where every element has finitely many immediate successors. Show that an infinite path in T corresponds to a k-coloring of G.

Homework 6 problem 2. Assume $KIL(\kappa)$ and for all $\mu < \kappa$, $2^\mu < \kappa$. Let $f : [\kappa]^2 \to 2$. Construct a function $G : \kappa \to 2^{<\kappa}$ by transfinite induction. Define $G(0) = \emptyset$. Suppose that we have constructed $G \restriction \gamma$ for some $\gamma < \kappa$. Define $G(\gamma)$ to be the unique $s \in 2^{<\kappa}$ such that for all $i \in \mathrm{dom}(s)$, $s(i)$ is $f(\{G^{-1}(s \restriction i), \gamma\})$. You should check that such an s actually exists.

Prove that $T = G``\kappa$ is a subtreeof $2^{<\kappa}$ ordered by end extension. Prove that for all $\mu < \kappa$, $2^\mu < \kappa$ implies that the levels of T have size less than κ. By $KIL(\kappa)$ there is a cofinal branch b through T. Let $B : \kappa \to 2$ be the function determined by b. As for the proof of infinite Ramsey theorem, prove that there is an $i \in 2$ so that $A = \{\alpha < \kappa \mid B(\alpha) = i\}$ has size κ. Prove $\{G^{-1}(B \restriction \alpha) \mid \alpha \in A\}$ is our desired set.

Homework 6 problem 3. For (a) consider the set $S = \{s \in 2^{<\omega} \mid N_s \subseteq U_i \text{ for some } i\}$. Prove that S works.

For (b), let $S^* = \{t \mid t \text{ extends } s \text{ for some } s \in S\}$. Prove that if there is a finite $S_0^* \subseteq S^*$ such that $\bigcup_{s \in S_0^*} N_S = 2^\omega$, then there is a finite $S_0 \subseteq S$ such that $\bigcup_{s \in S_0} N_s = 2^\omega$. So just rename S^* as S.

For (c), assume that T is finite and prove that there is a finite $S_0 \subseteq S$ as above.

If you've understood everything so far, then (d) should be straightforward.

Homework 7 problem 1. Prove using homework 4 problem 1 that for any cardinal κ, the union of κ many sets of cardinality κ has cardinality κ. To show that $\mathrm{cf}(\kappa^+) = \kappa^+$, let $f : \kappa \to \kappa^+$. Prove that $\bigcup_{\alpha<\kappa} f(\alpha)$ is an ordinal of cardinality κ. Call this ordinal γ. Prove that f is not cofinal as witnessed by γ.

Homework 7 problem 2. Prove that if $\alpha < \beta < \gamma$ and $f : \alpha \to \beta$ and $g : \beta \to \gamma$ are increasing and cofinal, then $g \circ f$ is a cofinal function from α to γ. Use this to finish the problem.

Homework 7 problem 3. Consider the set $T = \bigsqcup_{\alpha<\kappa} \alpha \cup \{\star\}$ (remember homework 2?). Define an ordering on T where $\star$ is the least element and $\delta <_T \gamma$ if and only if γ and δ "come from the same ordinal in the disjoint union" and $\delta < \gamma$. Prove that T has no cofinal branch.

Homework 7 problem 4. There was a small correction on this exercise. κ should be a regular cardinal. Let T^* be the set of finite sequences s where if n is the domain of s, then for all $i < n-1$, $s(i)$ is an ordinal less than κ and $s(n-1)$ is an element of T from the previous exercise. For two such sequences s and t we write $s < t$ if and only if $\mathrm{dom}(s) < \mathrm{dom}(t)$, for all $i < \mathrm{dom}(s) - 1$, $s(i) = t(i)$ and $s(\mathrm{dom}(s) - 1)$ is either $\star$ or it is in the $t(\mathrm{dom}(s) - 1)$-th ordinal of the disjoint union defining T.

Prove that this tree is normal and has height κ.

Homework 7 problem 5. $s < t$ means that $\mathrm{dom}(s)$ is a proper subset of $\mathrm{dom}(t)$ and for all $\alpha \in \mathrm{dom}(s)$, $s(\alpha) = t(\alpha)$. In particular if γ_s is the maximum element of the domain of s and γ_t is the maximum element of the domain of t, then $\gamma_s < \gamma_t$ and $\max(s) = s(\gamma_s) = t(\gamma_s) < t(\gamma_t) = \max(t)$.

Homework 7 problem 6. Assume the continuum hypothesis. Let X be a set of size ω_1. We claim that the set $Y = \{A \subseteq X \mid A \text{ is countable } \}$ has size ω_1. It is enough to find an injection from Y into a set of cardinality ω_1. Let $Z = \{(\alpha, A) \mid A \subseteq \alpha\}$. Prove that Z has size ω_1 using the continuum hypothesis. Define an injection $f : Y \to Z$ by $f(A) = (\alpha, A)$ where α is the least upper bound of A.

For the construction, Suppose that L is a linear order of size ω_1. Using the claim we just proved, there is an enumeration (A_α, B_α) for $\alpha < \omega_1$ of all pairs (A, B) where A and B are countable subsets of L where every element of A is below every element of B.

Define a linear order L' with $L' \supseteq L$ having the property that for every $\alpha < \omega_1$ there is an $l' \in L'$ such that l' is above every element of A_α and below every element of B_α.

Now go by transfinite induction. Let L_0 be any linear order of size $\aleph_1$. Assuming that we have defined L_α for some $\alpha < \omega_1$, let $L_{\alpha+1} = (L_\alpha)'$.

For the limit step, if we have L_α for all $\alpha < \gamma$ for some limit γ, then we set $L_\gamma = \bigcup_{\alpha<\gamma}$ with the natural ordering.

Prove that L_{ω_1} is a linear order of size ω_1 with the desired property.

Homework 7 problem 9. Don't worry about this one.

Homework 7 problem 10. Let $C = \{\gamma < \kappa \mid f``\gamma \subseteq \gamma\}$. Closed is straightforward. Suppose that $C \cap \gamma$ is unbounded. We want to prove that $f``\gamma \subseteq \gamma$. Let $\alpha < \gamma$ and find $\beta \in C \cap \gamma$ above α. It follows that $f(\alpha) < \beta$ by the definition of C. So $f``\gamma \subseteq \gamma$.

Let $\alpha < \kappa$. We want to find $\gamma > \alpha$ with $\gamma \in C$. Let $\alpha_0 = \alpha$. Suppose that we have defined α_n for some $n < \omega$. Let α_{n+1} be the least upper bound for the set $f``\alpha_n$. Why is $\alpha_{n+1} < \kappa$?

Let γ be the least upperbound for $\{\alpha_n \mid n < \omega\}$. Prove that $\gamma \in C$.

Homework 7 problem 11. Suppose that for every ordinal $\alpha < \omega_1$, the train is not empty at stage α. Define a function $f : \omega_1 \to \omega_1$ by $f(\alpha)$ is the unique station (ordinal) where the passenger getting off at stage α got on the train. Clearly for all $\alpha < \omega_1$, $f(\alpha) < \alpha$. By Fodor's lemma there is a stationary set $S \subseteq \omega_1$ on which f is constant. Get a contradiction from this.