This is the seventh homework for Math 114s in the winter quarter of 2016. It is due Friday March 11th in class.

(1) Prove that for a cardinal κ, κ^+ is regular.
(2) Prove that $\mathrm{cf}(\alpha)$ is a regular cardinal, that is $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$.
(3) Prove that if κ is singular, then $KIL(\kappa)$ is false.
(4) Let κ be an infinite cardinal. Prove that there is a tree of height κ with no cofinal branch.
(5) We say that a tree T is *normal* if for every $\alpha < \beta$ below the height of T and every $t \in Lev_\alpha T$, there is an $s \in Lev_\beta T$ such that $s > t$. Let $\kappa > \omega$ be a regular cardinal. Prove that there is a normal tree of height κ with no cofinal branch.
(6) Let T be the tree of height ω_1 with countable levels and no cofinal branch that we constructed in class. Define a function $f : T \to \mathbb{Q}$ by $f(t) = \max t$. Show that if $s < t$ in T, then $f(s) < f(t)$.
(7) Let T be a tree of height ω_1. Suppose that there is a function $f : T \to \mathbb{Q}$ such that if $s < t$, then $f(s) < f(t)$. Prove that T has no cofinal branch.
(8) Assume the continuum hypothesis. Prove that there is a linear order $(L, <)$ of size ω_1 such that for any countable A, B subsets of L such that every element of A is less than every element of B, there is an $l \in L$ which is above every element in A and below every element in B.
(9) Assume the continuum hypothesis. Prove that $KIL(\omega_2)$ is false. Hint: Try to repeat the argument from proving that $KIL(\omega_1)$ is false, but using the linear order from the previous exercise in place of $\mathbb{Q}$. Hint 2: Break up the limit step into cases based on the cofinality of the limit ordinal.
(10) Let $f : \kappa \to \kappa$. Show that the set $\{\gamma < \kappa \mid f``\gamma \subseteq \gamma\}$ is a club.
(11) Suppose that there is a train which stops at stations indexed by each ordinal $\alpha < \omega_1$ in order. At each stop 1 person gets off and ω many people get on. Show that there is an ordinal $\gamma < \omega_1$ where the train is empty when it stops (before the passenger exchange) at station γ.