This is the fifth homework for Math 114s in the winter quarter of 2016. It is due Wednesday February 17th in class.

Do the following problems.

(1) If U is an ultrafilter and $A \cup B \in U$, then either $A \in U$ or $B \in U$.
(2) Let U be an ultrafilter on X and $f : X \to Y$. Show that $f_*(U) = \{A \subseteq Y \mid f^{-1}(A) \in U\}$ is an ultrafilter.
(3) Let U be an ultrafilter on ω and let $\langle x_n \mid n < \omega \rangle$ be a sequence of real numbers. We write $\lim_U x_n = x$ if and only if for every $\epsilon > 0$ there is $A \in U$ such that for all $n \in A$, $|x_n - x| < \epsilon$.

    Prove that for every bounded sequence $\langle x_n \mid n < \omega \rangle$ there is a unique x such that $\lim_U x_n = x$.
(4) Suppose that U is a principal ultrafilter on ω and $\langle x_n \mid n < \omega \rangle$ is a sequence of real numbers. What is $\lim_U x_n$?
(5) Let U be a nonprincipal ultrafilter on ω. For $A \subseteq \mathbb{Z}$ define
$$\mu(A) = \lim_U \frac{|A \cap [-n, n]|}{|[-n, n]|}.$$
    Prove the following properties about μ.
    (a) $\mu(\emptyset) = 0$ and $\mu(\mathbb{Z}) = 1$.
    (b) For all $A \subseteq \mathbb{Z}$, $\mu(A) \geq 0$.
    (c) If $A \subseteq B \subseteq \mathbb{Z}$, then $\mu(A) \leq \mu(B)$.
    (d) If $A, B \subseteq \mathbb{Z}$ and $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

    Hint: Be careful with the properties of $\lim_U$!
(6) Let μ be as in the previous exercise. For $n \in \mathbb{Z}$ and $A \subseteq \mathbb{Z}$ define $n + A = \{n + a \mid a \in A\}$. Prove that for all $n \in \mathbb{Z}$ and $A \subseteq \mathbb{Z}$, $\mu(A) = \mu(n + A)$.
(7) Use the properties of μ that you just proved to show that there is no partition $\{A_1, \ldots A_n, B_1, \ldots B_m\}$ of $\mathbb{Z}$ with associated integers $a_1, \ldots a_n$ and $b_1, \ldots b_m$ such that $\mathbb{Z} = a_1 + A_1 \cup \cdots \cup a_n + A_n = b_1 + B_1 \cup \cdots \cup b_m + B_m$.

    Recall that a $\{A_1, \ldots A_n, B_1, \ldots B_m\}$ partitions $\mathbb{Z}$ if any two of the sets are disjoint and the union of all of them is $\mathbb{Z}$.