This is the fourth homework for Math 114s in the winter quarter of 2016. It is due Monday February 8th in class.

(1) Recall that a cardinal is a set of the form $|A|$ for some set A. If κ is a cardinal, then we write κ^+ for the least cardinal greater than κ. Read Proposition 12.29 from the book.

In this exercise you will show that for all ordinals α, $|\aleph_\alpha \times \aleph_\alpha| = \aleph_\alpha$.

(a) Check the case when $\alpha = 0$.

(b) Check the limit step, that is assume that the statement is true for all $\beta < \alpha$ and prove it for α.

(c) The successor step. Let $\beta = \alpha + 1$ and assume that $|\aleph_\alpha \times \aleph_\alpha| = \aleph_\alpha$. Define an ordering on $\aleph_\beta \times \aleph_\beta$ by $(\gamma, \delta) < (\gamma', \delta')$ if

$$\begin{aligned}&\text{either } \max\{\gamma, \delta\} < \max\{\gamma', \delta'\},\\ &\text{or } \max\{\gamma, \delta\} = \max\{\gamma', \delta'\} \text{ and } \gamma < \gamma',\\ &\text{or } \max\{\gamma, \delta\} = \max\{\gamma', \delta'\}, \gamma = \gamma' \text{ and } \delta < \delta'\end{aligned}$$

Show that this is a well ordering. By a previous exercise, we know that this well-ordering is isomorphic by a function π to an ordinal η. Show by contradiction that $\eta = \aleph_\beta$. Hint: Use the induction assumption!

(2) In a topological space X, we say that $x_n \to x$ as $n \to \infty$ if for every open U with $x \in U$, there is an $N < \omega$ such that for all $n \geq N$, $x_n \in U$. Suppose that X is metrizable and prove that $x_n \to x$ as $n \to \infty$ if and only if for all $\epsilon > 0$ there is an $N < \omega$ such that for all $n \geq N$, $d(x_n, x) < \epsilon$.

(3) Let X and Y be metrizable topological spaces. Prove that $f : X \to Y$ is continuous (as defined in class) if and only if for every sequence x_n for $n < \omega$ if $x_n \to x$ as $n \to \infty$, then $f(x_n) \to f(x)$ as $n \to \infty$.

(4) In 2^ω consider the open set N_s for some $s \in 2^{<\omega}$, show that N_s is closed.

(5) Find sequence s_n for $n < \omega$ in $2^{<\omega}$ such that for all $n < \omega$, $\mathrm{dom}(s_n) = n$ and $\{s_n{}^\frown \bar{0} \mid n < \omega\}$ is dense. Hint: Work by induction using a bijection between $2^{<\omega}$ and ω.

(6) Note that $2^\omega \subseteq \omega^\omega$. Show that 2^ω is closed in the topology on ω^ω.

(7) Define a function $f : \omega^\omega \to 2^\omega$ by $f(x)$ is the unique $z \in 2^\omega$ with infinitely many 1's such that the n^{th} block of 0's has length $x(n)$. So for example if $x(n) = n$ for all $n < \omega$, then $f(x) = 101001000100001....$ Show that f is continuous and injective. Show that the range of f is a countable intersection of open sets.

(8) Show that the function f from Corollary 2.14 in the book is continuous.