This is the third homework for Math 114s in the winter quarter of 2016. It is due Monday February 1st in class.

(1) In this exercise you will show that for any countable ordinal α, there is a set $A \subseteq \mathbb{Q}$ such that $(A, <)$ (where $<$ is the usual order on $\mathbb{Q}$) is isomorphic to $(\alpha, \in)$.

To do it show the following stronger statement by induction on ordinals $\alpha < \omega_1$. Let $P(\alpha)$ be the statement "For every interval (a, b) with rational endpoints, there is an $A \subseteq (a, b)$ such that $(A, <)$ is isomorphic to $(\alpha, \in)$."

(2) We say that a binary relation E on a set W is well-founded if every nonempty subset X of W has an E-minimal element.

This exercise gives a more general transfinite induction theorem. Let (W, E) be a well-founded binary relation. Prove the following statement: If for all $y \in W$ ((for all xEy, $P(x)$) implies $P(y)$), then $P(x)$ holds for every $x \in W$.

We also have a similar way to define functions by transfinite recursion(induction) on well-founded binary relations. In particular, if G is a function and (W, E) is a well-founded binary relation, then there is a unique function F with domain W such that $F(x) = G(F \restriction W^{Ex})$. Recall that $W^{Ex} = \{y \in W \mid yEx\}$.

(3) Let W be a set and E be a binary relation on W satisfying the following properties:

(a) For all $x, y \in W$, if for every $z \in W$, $z \in x$ if and only if $z \in y$, then $x = y$.

(b) E is well-founded

Prove the following claims:

(a) There are a transitive set M and function π with domain W such that π is an isomorphism from (W, E) to $(M, \in)$. Hint: Consider the definition $\pi(x) = \{\pi(y) \mid yEx\}$. Be very careful verifying that this definition works.

(b) The M and π from the previous part are unique.

Finally, if (W, E) is a well-ordering, then what is the transitive set M from the above claims?

(4) Prove Zorn's lemma from the Axiom of Choice. Zorn's lemma is the following statement: For every partial order $(P, <)$ if every chain has an upper bound then, $(P, <)$ has a maximal element.

A chain C in a partial order $(P, <)$ is a subset of P which is linearly ordered by $<$. x is an upperbound for a chain C if $x \geq y$ for all $y \in C$.