This is the first homework for Math 114s in the winter quarter of 2016. It is due Monday January 11th at the beginning of class.

(1) Prove or disprove the following statements.
  (a) For every $x \in \mathbb{R}$ there is a $y\mathbb{R}$ such that $y^2 = x$.
  (b) There is an $x \in \mathbb{R}$ such that for every $y \in \mathbb{R}$, $y^2 = x$.

(2) If f is a function and A is a subset of its domain then we write $f[A]$ (or sometimes $f``A$) for the set $\{f(a) \mid a \in A\}$. If f is a function and B is a subset of its codomain then we write $f^{-1}B$ for the set $\{x \mid f(x) \in B\}$. Prove or provide counterexamples for the following claims about a function $f : X \to Y$:
  (a) For all $A_1, A_2 \subseteq X$, $f[A_1 \cup A_2] = f[A_1] \cup f[A_2]$.
  (b) For all $A_1, A_2 \subseteq X$, $f[A_1 \cap A_2] = f[A_1] \cap f[A_2]$.
  (c) For all $B_1, B_2 \subseteq Y$, $f^{-1}(B_1 \cup B_2) = f^{-1}B_1 \cup f^{-1}B_2$.
  (d) For all $B_1, B_2 \subseteq Y$, $f^{-1}(B_1 \cap B_2) = f^{-1}B_1 \cap f^{-1}B_2$.

(3) Prove that if $f : X \to Y$ is an injection and X is not countable, then Y is not countable.

(4) Prove that for any $a, b \in \mathbb{R}$, (a, b) and $[a, b]$ are uncountable.

(5) Prove that the set of straight lines in the plane is uncountable.

(6) Prove that the set of intervals $(a, b) \subseteq \mathbb{R}$ such that a, b are rational is countable.

(7) Prove that the set of finite sequences of natural numbers is countable. Hint: Write the set of finite sequences as a countable union where the sequences are organized by length.

(8) You probably know from 131A or just calculus that an n^{th} degree polynomial has at most n roots. We say that a number is *algebraic* if it is the root of a polynomial with integer coefficients. In this exercise you will show that the set $\{\alpha \in \mathbb{R} \mid \alpha \text{ is algebraic}\}$ is countable.
  (a) Show that the set of polynomials with integer coefficients is countable. Hint: Use the previous exercise.
  (b) Use the previous part to conclude that the set of algebraic numbers is countable.
  (c) Conclude that there are numbers which are not algebraic.