PRIKRY FORCINGS, UNGER'S LECTURES

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1. January 5

I contend that the hedgehog and the cactus should be, respectively, the spirit animal and spirit plant of this topic. They are both prickly.

Continuum Problem:[1] want to understand the behavior of the continuum function $\kappa \mapsto 2^\kappa$ (the size of powerset of κ).

Easy Facts:

(1) $\kappa < 2^\kappa$ (Cantor).
(2) $\kappa < \lambda \longrightarrow 2^\kappa \leq 2^\lambda$.
(3) $\text{cf}(2^\kappa) > \kappa$, since $\kappa^{\text{cf}(\kappa)} > \kappa$ (Ḱonig).

Theorem 1.1. (Easton) *Subject to the three constraints above, any function on the regular cardinals can be realized as the continuum function of some model of* ZFC.

To get any further, we need more axioms! But large cardinals are not likely to decide CH, as 2^ω can be changed with small forcings, which in particular, fix large cardinal properties.

Remaining questions in ZFC are about singular cardinals. Recall that GCH says $2^\kappa = \kappa^+$ for all κ. We get failures of GCH at singular μ using Easton's theorem (by failing "badly" beforehand).

The Singular Cardinals hypothesis (one version, at least, and "the most classical, in a sense" according to S.) at singular μ says: if μ is a strong limit cardinal, then $2^\mu = \mu^+$.

What is different about singular cardinals?

(1) ZFC bounds exist for 2^μ with μ singular. These use PCF-theoretic techniques. The most famous example is $\aleph_\omega$ strong limit implies $2^{\aleph_\omega} < \aleph_{\omega_4}$ (Shelah).
(2) Consistency results that require large cardinals. The consistency of the failure of SCH requires large cardinals.

We'll be working towards a proof of the following (possibly addressing optimal hypotheses later):

Theorem 1.2. (Shelah) Con($\exists$ a supercompact) $\implies$ Con($\aleph_\omega$ strong limit + $2^{\aleph_\omega} = \aleph_{\alpha+1}$ for some $\alpha < \omega_1$)

[1] "Prikry forcing is motivated by one of the best things you can be motivated by in set theory." S.

Prikry Forcing

Let U be a normal measure on κ. We define a poset $\mathbb{P}$, called "Prikry forcing:" conditions are pairs (s, A) where s is a finite set of inaccessibles below κ and $A \in U$. The ordering is as follows: we declare $(s, A) \leq (t, B)$ if s end-extends t, $s \backslash t \subseteq B$, and $A \subseteq B$.

The idea is that we can add things to the top of s, which are to be taken from the old measure one set, the "control", which then decreases. Given a condition (s, A), we will often call s the *lower part* or *stem* and A the *upper part* or *constraint*.

If $A^*, A \in U$ and $A^* \subseteq A$, then we say that (s, A^*) is a *direct extension* of (s, A); we will write $(s, A) \leq^* (t, B)$ to mean that (s, A) is a direct extension of (t, B) (in particular, $s = t$). Direct extensions are key in Prikry forcing(s).

We will refer to the *length* of a condition (s, A), as $l(s) := |s|$.

The forcing $\mathbb{P}$ has the following properties:

- $\mathbb{P}$ has κ^+-c.c.
- $\mathbb{P}$ satisfies the Prikry lemma: given φ in the forcing language and (s, A), there is a direct extension $(s, A^*) \leq (s, A)$ deciding φ.
- As a corollary to the Prikry lemma, $\mathbb{P}$ doesn't add any bounded subsets to κ.
- (Strong Prikry Lemma) For every dense open $D \subseteq \mathbb{P}$ and (s, A), there is $n < \omega$ and $A^* \in U$ such that every n-step extension of (s, A^*) is in D (i.e., whenever we take n-many points from A^* and add them on top of s, we land in D).
- (Characterization of Genericity) $\vec{\alpha} = \langle \alpha_n : n < \omega \rangle$ generates a $\mathbb{P}$-generic filter iff the sequence $\vec{\alpha}$ is eventually in every measure-one set in U. More precisely, $\vec{\alpha}$ generates a $\mathbb{P}$-generic filter iff

$$(\forall A \in U)\ (\exists n)\ (\forall m \geq n)\ [\alpha_m \in A].$$

Claim: $\mathbb{P}$ has κ^+-c.c.

Proof. First note that if two conditions (s, A) and (s, B) have the same lower part, then they are compatible; indeed, $(s, A \cap B)$ is a condition below each. Observe that there are just κ-many lower parts (since κ-many finite subsets of κ). Thus if $\mathcal{A} \subseteq \mathbb{P}$ is of size κ^+, then there is a subset $\mathcal{A}' \subseteq \mathcal{A}$ for which all lower parts are the same. Hence $\mathcal{A}$ is not an antichain. $\square$

The following might be described as a "capturing lemma." It says that we can refine a condition (s, A) via a direct extension (s, A^*) such that any further extension $(t, B) \leq (s, A^*)$ which is in D is such that $(t, A^* \backslash (\max(t) + 1))$, was already in D (i.e., "captured").

Claim: For all dense open $D \subseteq \mathbb{P}$ and conditions (s, A), there is $A^* \in U$ such that for all $(t, B) \leq (s, A^*)$, if $(t, B) \in D$, then $(t, A^* \backslash (\max(t) + 1)) \in D$.

Proof. Let t end-extend s. Let A_t be such that $(t, A_t) \in D$, if possible.[2] We define a diagonal intersection ΔA_t of the A_t above as follows: set $\alpha \in \Delta A_t$ iff for each lower part $t \sqsupseteq s$ such that $t \cup \{\alpha\} \sqsupseteq t$, we have $\alpha \in A_t$.[3] In symbols,

$$\Delta A_t = \{\alpha < \kappa : \forall t \sqsupseteq s\ ((t, A_t) \in D \wedge t \cup \{\alpha\} \sqsupseteq t \longrightarrow \alpha \in A_t)\}.$$

Note that it is implicit in the notation "$t \cup \{\alpha\} \sqsupseteq t$," that $\alpha > \max(t)$.

Now set $A *^= \Delta A_t$. We show that $A^* \in U$. Let $j : V \longrightarrow \mathsf{Ult}(V, U)$ be the ultrapower embedding given by the normal measure U. Recall that since U is normal, κ is represented in the ultrapower by $[\mathrm{id}_k]$, where id_κ is the identity function on κ. Thus we have

$$U = \{X \subseteq \kappa : \kappa \in j(X)\}.$$

So to check that $A^* \in U$, we check that $\kappa \in j(A^*)$. Now by elementarity, $\kappa \in j(A^*)$ iff for each lower part t in $M \cap j(\mathbb{P})$ with $\kappa > \max(t)$, we have $\kappa \in j(A_t)$. However, the lower parts $t \in M$ with $\max(t) < \kappa$ are just the lower parts t (in V) in $\mathbb{P}$ with $\max(t) < \kappa$.[4] As $A_t \in U$ for each such t, we have $\kappa \in j(A_t)$ for each such t, and so indeed $\kappa \in j(A^*)$. Thus $A^* \in U$.

Now we check that (s, A^*) is as required. Suppose $(t, B) \leq (s, A^*)$ and $(t, B) \in D$. Then we answered "yes" to t above, i.e., $(t, A_t) \in D$. As $A^* \setminus (\max(t) + 1) \subseteq A_t$ (by definition of A^*), we have $(t, A^* \setminus (\max(t) + 1)) \leq (t, A_t) \in D$, and so $(t, A^* \setminus (\max(t) + 1)) \in D$, since D is dense open. □

A few comments about the above proof are in order. We ranged over each possible candidate t, and chose an appropriate $A_t \in U$ if possible. We then took the diagonal intersection of these A_t. Then taking a condition (t, B) satisfying the assumptions of the claim, we must have answered "yes" at "stage t." Then we applied that D was dense open. This is a common technique in Prikry forcings; the details become substantially harder, but the core idea stays the same.

2. January 7

Note to the fastidious reader: we finished the proof of the above claim on this date, though it is written in full under the previous date.

Recall that the notation $p \| \varphi$ means that p *decides* φ, i.e., $p \Vdash \varphi$ or $p \Vdash \neg\varphi$.

Claim: (Weaker Prikry Lemma) For all φ in the forcing language and all $(s, A) \in \mathbb{P}$, there is a direct extension of (s, A) that decides φ.[5]

Proof. Apply the "capturing" claim to the dense open set $D := \{p \in \mathbb{P} : p \| \varphi\}$ to get $(s, A^*) \leq (s, A)$ as in the capturing claim. For each stem t, partition $A^* \setminus (\max(t)+1)$ as follows:

$$B_t^0 := \{\alpha \in A^* : (t \cup \{\alpha\}, A^*) \Vdash \varphi\} \qquad B_t^1 := \{\alpha \in A^* : (t \cup \{\alpha\}, A^*) \Vdash \neg\varphi\},$$

and

$$B_t^2 := (A^* \setminus (\max(t) + 1)) \setminus (B_t^0 \cup B_t^1).$$

[2] We will, somewhat colloquially, say that "we answered yes to t" if there is some $A_t \in U$ such that $(t, A_t) \in D$.

[3] This is, therefore, "diagonal" in the sense that to check membership of $\alpha \in \Delta A_t$, we check membership of α in A_t for all t "below" α.

[4] This is because $j \restriction V_\kappa = \mathrm{id}_{V_k}$.

[5] "This is the advantage of writing down a lot of terminology. You get to use it." -S.

For each appropriate stem t, one of the B_t^i is measure one; let's call it B_t. Set

$$A^{**} := \Delta B_t.$$

(Note A^{**} is in U as it is a diagonal intersection of sets in U.)

We now claim that (s, A^{**}) decides φ. Let $(t, B) \leq (s, A^{**})$ decide φ and be of minimal length. Suppose for a contradiction that $l(t) > l(s)$. To simplify notation, let $t^+ := \max(t)$ and $t^- := t \setminus \{t^+\}$. First observe that (t, A^*) decides φ since $(t, B) \leq (s, A^{**}) \leq (s, A^*)$ and capturing jointly imply $(t, A^*) \in D := \{p : p \| \varphi\}$. For simplicity, and without loss of generality, let's suppose that $(t, A^*) \Vdash \varphi$. Now since $(t, B) \leq (s, A^{**})$ and $l(t) > l(s)$, we know $t^+ \in A^{**}$, by definition of the ordering. Hence by definition of A^{**}, we get $t^+ \in B_{t^-}$. But $t^+ \in B_{t^-}$ and $(t^- \cup \{t^+\}, A^*) = (t, A^*) \Vdash \varphi$ implies that $B_{t^-} = B^0_{t^-}$.

Finally, observe that every one-point extension of (t^-, A^{**}) then forces φ, since if $(t^- \cup \{\alpha\}, A') \leq (t^-, A^{**})$, then $\alpha \in A^{**}$, so α is in $B_{t^-} = B^0_{t^-}$. However, this implies that $(t^-, A^{**}) \Vdash \varphi$ since every extension of greater length extends some one-point extension, which we know all force φ.

Since $(t^-, A^{**}) \leq (s, A^{**})$ and (t^-, A^{**}) has smaller length than (t, B), this contradicts the minimality of the length of t. With this the proof is complete. □

Claim: (Stronger Prikry Lemma) Given a dense open $D \subseteq \mathbb{P}$ and $(s, A) \in \mathbb{P}$, there are $n < \omega$ and $A^{**} \in U$ so that every n-step extension of (s, A^{**}) is in D.

Proof. Apply the capturing Lemma to D to get $(s, A^*) \leq (s, A)$ as in the capturing lemma. Define sets Y_m for $m < \omega$ as follows:

$$Y_0 := \{t \sqsupseteq s : (t, A^*) \in D\}, \text{ and } Y_{m+1} = \{t \sqsupseteq s : \exists A \in U\ \forall \alpha \in A\ [t \cup \{\alpha\} \in Y_m]\}.$$

Being in Y_m is, roughly, saying that you are an m-step extension away from being in D.

For each lower part t and each m, if $t \notin Y_{m+1}$, then $\{\alpha < \kappa : t \cup \{\alpha\} \in Y_m\}$ is measure zero; hence its complement is measure one. So for each lower part t and each m, we get an A_t^m witnessing that $t \in Y_{m+1}$ or $t \notin Y_{m+1}$. Define

$$B_t := \bigcap_{m<\omega} A_t^m,$$

(in U by completeness of U) and $A^{**} := \Delta B_t$; so $A^{**} \in U$ as well.

Now let $(t, B) \leq (s, A^{**})$ be in D and set $n := l(t) - l(s)$. We show that $s \in Y_n$. First, $t \in Y_0$. Now we show by induction that $t \restriction l(s)+n-i \in Y_i$ for each $1 \leq i \leq n$. Fix $i < n$ and suppose $u := t \restriction l(x)+n-i \in Y_i$; we show that $u^- := u \setminus \{u^+\} \in Y_{i+1}$ (where $u^+ := \max(u)$). Now $(t, B) \leq (s, A^{**})$ implies that $u^+ \in A^{**}$, and therefore $u^+ \in B_{u^-}$. By our inductive assumption, $u^- \cup \{u^+\} \in Y_i$. Since $u^+ \in A^i_{u^-}$ also, there must be measure one many α such that $u^- \cup \{\alpha\} \in Y_i$. But this is precisely the statement $u^- \in Y_{i+1}$.

Thus we know that $s \in Y_n$. Now let $(t', B') \leq (s, A^{**})$ be an arbitrary n-step extension; we show $(t', B') \in D$. Let $t_1 < \ldots < t_n$ enumerate $t' \setminus s$. Using the definition of A^{**} and an argument similar to the one in the last paragraph, argue by induction that for each $1 \leq i \leq n$, we have $s^\frown\langle t_1 \ldots, t_i\rangle \in Y_{n-i}$. Then we conclude $t \in Y_0$, and so $(t', A^*) \in D$. Hence (t', B') is too, by the openness of D. □

Claim: Forcing with $\mathbb{P}$ doesn't add bounded subsets of κ.

Proof. Fix a name $\dot{X}$ for a subset of (limit) $\mu < \kappa$. For each $\xi < \mu$, let φ_ξ be the forcing-language sentence $\check{\xi} \in \dot{X}$. Given an arbitrary (s, A) build a decreasing sequence $\langle (s, A_\xi) : \xi < \mu \rangle$ of direct extensions of length μ, where at stage $\xi + 1 < \mu$, we have $(s, A_\xi) \| \varphi_\xi$; at limit ξ, we simply take $A_\xi := \bigcap_{\zeta < \xi} A_\zeta$, which is in U by κ-completeness of the measure. Similarly, by the fact $\leq^*$ is κ-closed, we can find (s, A^*) a lower bound for the sequence.

Now set $X = \{\xi < \mu : (s, A^*) \Vdash \varphi_\xi\}$. By definability of forcing, $X \in V$; furthermore, $(s, A^*) \Vdash \check{X} = \dot{X}$. Since every condition can be refined to one which forces that $\dot{X}$ is in the ground model, it is forced that $\dot{X}$ is in the ground model. □

<u>Corollary</u>: κ is preserved.

Proof. κ is a limit cardinal in V. Moreover, since no bounded subsets of κ are added after forcing, we know that all cardinals below κ are still cardinals in $V[G]$. Hence κ is still a limit of cardinals, and in particular, a cardinal. □

Getting the failure of SCH. Start with κ measurable and $2^\kappa = \kappa^{++}$. Force with Prikry forcing.

<u>Exercise</u>: Prove the characterization of genericity for $\mathbb{P}$.

<u>Exercise</u>: The critical sequence of the ω-step iterated ultrapower by U is Prikry generic over M_ω for $j_{0,\omega}(U)$.

3. January 9

Recall $\mathbb{P}$ is Prikry forcing defined from U, a normal measure on κ.
<u>Facts about the extension by Prikry forcing</u>.

We'll study more carefully the combinatorics of the model $V[G]$. First some definitions. Recall that if $\mathrm{cf}(\lambda) > \omega$ and $S \subseteq \lambda$ is stationary, then we say that S *reflects* if there is $\delta < \lambda$ with $\mathrm{cf}(\delta) > \omega$ such that $S \cap \delta$ is a stationary subset of δ. If $\langle S_i : i < \xi \rangle$ is a sequence (finite or infinite) of stationary subsets of λ, then we say $\langle S_i : i < \xi \rangle$ *reflects simultaneously* if there is $\delta < \lambda$ with $\mathrm{cf}(\delta) > \omega$ for which $S_i \cap \delta$ is stationary in δ for each $i < \xi$.

We'll also need a small bit of background in PCF. Let $\langle \tau_n : n < \omega \rangle$ be an increasing sequence of regular cardinals with sup τ. A *scale* of length τ^+ in $\prod_{n<\omega} \tau_n$ is a sequence $\vec{f} = \langle f_\alpha : \alpha < \tau^+ \rangle$ which is increasing and cofinal in $(\prod \tau_n, <^*)$, where $<^*$ is the eventual domination ordering. More explicitly, increasing means

$$(\forall \alpha < \beta < \tau^+)\ (\exists m \in \omega)\ (\forall n \geq m)\ [f_\alpha(n) < f_\beta(n)].$$

Cofinal means that for any $g \in \prod_{n<\omega} \tau_n$, there is some $\alpha < \tau^+$ such that $g <^* f_\alpha$.

An ordinal $\gamma < \tau^+$ is *good* (resp. *very good*) for $\vec{f}$ if there is $A \subseteq \gamma$ unbounded (resp. club) and $m < \omega$ such that $\forall n \geq m$, $\langle f_\alpha(n) : \alpha \in A \rangle$ is strictly increasing. A scale $\vec{f}$ is good (very good) if modulo a club, almost every point of uncountable cofinality is good (very good).

Now we're ready to state some combinatorial results that hold in the generic extension after forcing with $\mathbb{P}$:

- (Cummings-Schimmerling) $\square_{\kappa,\omega}$ holds in $V[\mathbb{P}]$.[6]
- (Cummings-Foreman-Magidor) In V, set $S_0 := \kappa^+ \cap \text{cof}(< \kappa)$ and $S_1 := \kappa^+ \cap \text{cof}(\kappa)$. Then in $V[\mathbb{P}]$:
  - S_1 is a non-reflecting stationary set[7]
  - There is an ω-sequence of stationary subsets of S_0 which don't reflect simultaneously.
  - If κ is κ^+-supercompact, then every finite collection of stationary subsets of S_0 reflects simultaneously.[8]
- (Cummings-Foreman-Magidor) There is a very good scale of length κ^+ in $\prod \kappa_n^+$ where $\langle \kappa_n : n < \omega \rangle$ is $\mathbb{P}$-generic.
- (Moore) MRP (Mapping Reflection Principle) fails.

Notes:

- $\mathbb{P}$ κ^+-c.c. implies S_0, S_1 still stationary in $V[\mathbb{P}]$.
- If $\omega < \text{cf}(\delta) < \kappa$ in $V[\mathbb{P}]$, then $\omega < \text{cf}(\delta) < \kappa$ in V (as we know what happens to the other possible cofinalities).

Claim $S_1 := \kappa^+ \cap \text{cof}(\kappa)$ does not reflect in $V[\mathbb{P}]$.

Proof. In $V[\mathbb{P}]$ let $\delta < \kappa^+$ with uncountable cofinality; in particular, since $\text{cf}^{V[\mathbb{P}]}(\kappa) = \omega$, we have $\omega < \text{cf}^{V[\mathbb{P}]}(\delta) < \kappa$. By the remarks above, we also have $\omega < \text{cf}^V(\delta) < \kappa$. We'll show that $S_1 \cap \delta$ is non-stationary. In V, fix a club $D \subseteq \delta$ such that $\text{ot}(D) = \delta$ and D contains no points of cofinality $\geq \delta$; in particular, D contains no points of cofinality κ (in V). Then $D \cap S_1 \cap \delta = \varnothing$. As D is still a club in $V[\mathbb{P}]$, this suffices. $\square$

Claim In $V[\mathbb{P}]$ there is an ω-sequence of stationary subsets of S_0 which don't reflect simultaneously.

Proof. Apply Solovay Splitting, in V, to S_0 to get a sequence $\langle T_\alpha : \alpha < \kappa \rangle$ of disjoint, stationary subsets of S_0. In $V[\mathbb{P}]$, let $T_n^* = T_{\kappa_n}$ where $\langle \kappa_n : n < \omega \rangle$ is the Prikry-generic sequence.

Suppose for a contradiction that, in $V[\mathbb{P}]$, there is a δ with $\text{cf}(\delta) > \omega$ such that $T_n^* \cap \delta$ is stationary for all $n < \omega$. From the note above, we know that in V, $\omega < \text{cf}(\delta) < \kappa$. Also in V, define $B := \{\alpha < \kappa : T_\alpha \cap \delta \text{ is stationary}\}$. We will show that B is unbounded in κ and that $|B| \leq \delta$. Since $\text{cf}^V(\delta) < \kappa$, this will contradict the regularity of κ in V.

If B was bounded, then there would be an $\alpha < \kappa$ such that for all $\beta \geq \alpha$, $T_\beta \cap \delta$ was nonstationary V, and hence nonstationary in $V[\mathbb{P}]$. But in $V[\mathbb{P}]$ there is an n such that $\kappa_n > \alpha$ and (by our assumption for a contradiction) $T_{\kappa_n} \cap \delta$ is stationary.

Thus we know that B is unbounded. To see that $|B| \leq \delta$, let $D \subseteq \delta$ be a club with $\text{ot}(D) = \delta$; in particular, $|D| = \delta$. Now since the T_α are disjoint, we know that $\langle T_\alpha \cap D : \alpha \in B \rangle$ is a sequence of non-empty, disjoint subsets of D. Since $|D| \leq \delta$, we must have $|B| \leq \delta$. $\square$

Claim If κ is κ^+-supercompact, then every finite collection of stationary subsets of S_0 reflects simultaneously.

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[6] This implies the very good scale mentioned below.

[7] "L has lots of non-reflecting stationary sets. The way to see this is that they come from $\square$. However, lots of stationary set reflection requires large cardinals." -S.

[8] "This is like saying: given a bit more large cardinal strength, you can't get more reflection from S_0." -S.

Proof. Suppose not, for a contradiction. Fix $n < \omega$, and assume that

$$\Vdash_{\mathbb{P}} \text{“}\exists \langle \dot{T}_i : i < n \rangle \text{ which is a counterexample.”}$$

Let G be $\mathbb{P}$-generic, and define $G_j := \{p \in G : l(p) = j\}$. Now define

$$T_i^j := \left\{ \alpha \in S_0 : \exists p \in G_j, p \Vdash \alpha \in \dot{T}_i \right\}.$$

We claim that for each $i \leq n$, there is $j < \omega$ such that T_i^j is stationary. Otherwise, for some $i < n$, T_i^j is nonstationary for each j. Let $C_j \subseteq \kappa^+$ be a club witnessing this, and define $C := \bigcap C_j$, which is still club in κ^+. Let $p \in G$ be arbitrary. Since in $V[G]$, $(\dot{T}_i)^G$ is stationary in κ^+, there is $\alpha \in (\dot{T}_i)^G \cap C$. Thus there is a condition $q \in G$ such that $q \Vdash \check{\alpha} \in \dot{T}_i$. Let $r \leq p, q$ with $r \in G$, and set $j := l(r)$. Then $r \in G_j$ and $r \Vdash \alpha \in \dot{T}_i$. Since $\alpha \in C$ also, we have $\alpha \in C \cap T_i^j \subseteq C_j \cap T_i^j = \varnothing$, a contradiction.

Now observe that $T_i^{j+1} \supseteq T_i^j$ for each $i \leq n$ and $j < \omega$, since if $\alpha \in S_0$ and there is $p \in G_j$ with $p \Vdash \alpha \in \dot{T}_i$, then any one-step extension of p also forces $\alpha \in \dot{T}_i$. As shown in the previous paragraph, for each $i < n$, there is $j < \omega$ such that T_i^j is stationary. Since the T_i^j are increasing, we can find a single j such that T_i^j is stationary for each $i \leq n$ (essential use of the finiteness here).

Let s be the first j elements of the Prikry sequence. Define, in V,

$$U_i := \left\{ \alpha \in S_0 : (\exists A \in U) \left[(s, A) \Vdash \alpha \in \dot{T}_i \right] \right\}.$$

Then $T_i^j \subseteq U_i$, so that U_i is stationary, for each i. Since κ is κ^+-supercompact,[9] there is δ with $\kappa > \mathrm{cf}(\delta) > \omega$ such that $U_i \cap \delta$ is stationary for all $i < n$. Let $D \subseteq \delta$ be a club in V with $\mathrm{ot}(D) = \mathrm{cf}(\delta)$. For each $\beta \in D \cap U_i$ we get $A_{\beta,i}$ such that $(s, A_{\beta,i}) \Vdash \beta \in \dot{T}_i$. Let

$$A^* := \bigcap_{i<n} \bigcap_{\beta \in U_i \cap D} A_{\beta,i}.$$

Now $(s, A^*) \Vdash D \cap U_i \subseteq \dot{T}_i$ for $i < n$. Moreover, it forces $D \cap U_i$ is stationary in δ. So it forces each T_i to reflect at δ, a contradiction. □

4. January 12

<u>Claim</u> If $\langle \kappa_n : n < \omega \rangle$ is Prikry-generic, then there is a very good scale of length κ^+ in $\prod \kappa_n^+$.

Proof. Fix functions $f_\alpha : \kappa \longrightarrow \kappa$ for $\alpha < \kappa^+$ such that $[f_\alpha]_U = \alpha$. Define f_α^* with domain ω by: $n \mapsto f_\alpha(\kappa_n)$. We claim that on a tail subset of ω, $f_\alpha^*(n) < \kappa_n^+$. Indeed, $\alpha = [f_\alpha]_U = j(f_\alpha)(\kappa) < \kappa^+$. Since $\kappa^+ = (\kappa^+)^M$ (because M closed under κ-sequences, and hence computes the successor of κ correctly) we get, using the normality of U, that $A_\alpha := \{\beta < \kappa : f_\alpha(\beta) < \beta^+\} \in U$. Since the sequence $\langle \kappa_n : n < \omega \rangle$ is Prikry-generic, for each α there is a tail end of the κ_n such that $\kappa_n \in A$. Hence, for each α, $f_\alpha(\kappa_n) < \kappa_n^+$ on a tail-end, i.e., $f_\alpha^*(n) < \kappa_n^+$ for all large enough n.

Thus we may assume, without loss of generality (i.e., by modifying the f_α on measure zero sets) that $f_\alpha^* \in \prod \kappa_n^+$ for each $\alpha < \kappa^+$.

We claim that $\langle f_\alpha^* : \alpha < \kappa^+ \rangle$ is a scale of length κ^+. We check (1) increasing and (2) cofinal.

[9] See notes for Jan. 15 for review of supercompact cardinals.

For (1), let $\alpha < \alpha' < \kappa^+$. Now $[f_\alpha] = \alpha < \alpha' = [f_{\alpha'}]$. So there is an $A \in U$ such that $\forall \beta \in A$, $f_\alpha(\beta) < f_{\alpha'}(\beta)$. Since $\langle \kappa_n : n < \omega \rangle$ is Prikry-generic, $\kappa_n \in A$ for all large enough n. Hence $f^*_\alpha(n) < f^*_{\alpha'}(n)$ for all large enough n.

(2) For cofinal, let $\dot{f}$ be a name for an element of $\prod \kappa_n^+$. First we make an observation. Let $n < \omega$ and $p \in \mathbb{P}$ with $l(p) > n$. Then there is β_n such that $p \Vdash \dot{\kappa}_n = \beta_n$ (i.e., p decides the value of $\dot{\kappa}_n$), namely, $\beta_n := s_p(n)$ where $p = \langle s_p, A_p \rangle$. Since p also forces $\dot{f}$ in $\prod \kappa_n^+$, we have $p \Vdash \dot{f}(n) < \beta_n^+$ (note that writing β_n^+ here makes sense, since this forcing preserves all cardinals).

For each $n < \omega$, define

$$P_n := \left\{ \beta < \kappa : (\exists p \in \mathbb{P}) \left[l(p) = n+1 \wedge s_p(n) = \beta \right] \right\},$$

so that P_n is the set of $\beta < \kappa$ for which there is a condition of length $n+1$ which has β as it's top Prikry point. Note that since $\kappa \in j(P_n)$, we have $P_n \in U$, for each $n < \omega$.

Fix an $n < \omega$. For each $\beta \in P_n$, say witnessed by $p = \langle s, A \rangle$, we can find $\gamma_{s,n} < \beta^+$ and a direct extension (by the PL) $\langle s, A_s \rangle$ of p such that $\langle s, A_s \rangle \Vdash \dot{f}(n) = \gamma_{s,n}$. (note that $\gamma_{s,n}$ is uniquely determined by s and n, since any two conditions with the same length-$n+1$ stem are compatible). Now there are at most β-many stems of length $n+1$ which have β as a top point. Thus defining $\gamma_{\beta,n} := \sup_s \gamma_{s,n}$, we have $\gamma_{\beta,n} < \beta^+$.

Now consider the function g_n defined on P_n by $\beta \mapsto \gamma_{\beta,n}$, and let $\alpha_n := [g_n]_U$ (note that $[g_n]_U$ is indeed an ordinal since $g_n(\xi)$ is an ordinal for all $\xi \in P_n$, a measure-one set). Since $g_n(\beta) < \beta^+$ for all $\beta \in P_n$, the normality of U implies that $\alpha_n < \kappa^+$. Thus we can take $\alpha^* > \sup_n \alpha_n$ with $\alpha^* < \kappa^+$. Finally set $A^* := \Delta_s A_s$.

We claim that

$$\Vdash \dot{f} <^* f^*_{\alpha^*}.$$

Indeed, take G to be a generic and let B be a measure one set such that $\forall n < \omega$ and $\forall \xi \in B$, $f_{\alpha_n}(\xi) < f_{\alpha^*}(\xi)$. Now let n^* large enough such that for all $n \geq n^*$, $\kappa_n \in B \cap A^*$. Fix $n \geq n^*$ and let s be the unique stem of length $n+1$ in G. Note that $(s, A^*) \in G$ since it is compatible with all elements of G. Then $(s, A^*) \leq (s, A_s)$ by definition of A^* as a diagonal intersection, and $(s, A_s) \Vdash \ulcorner \dot{f}(n) = \gamma_{s,n} \urcorner$. So in $V[G]$ we have $(\dot{f})^G(n) = \gamma_{s,n} \leq \gamma_{\max(s),n} = f_{\alpha_n}(\kappa_n) < f_{\alpha^*}(\kappa_n)$. So $(\dot{f})^G(n) < f_{\alpha^*}(n)$ for all $n \geq n^*$. Hence $(\dot{f})^G <^* f^*_{\alpha^*}$. □

<u>Claim</u> If $\omega < \mathrm{cf}(\gamma) < \kappa$ in $V[\langle \kappa_n : n < \omega \rangle]$ then γ is very good for $\langle f^*_\alpha : \alpha < \kappa^+ \rangle$.

Proof. By a note from last time, $\omega < \mathrm{cf}^V(\gamma) < \kappa$. So fix $D \subseteq \gamma$ club with $\mathrm{ot}(D) = \mathrm{cf}(\gamma)$. Now we claim that

$$A := \{\alpha < \kappa : \langle f_\eta(\alpha) : \eta \in D \rangle \text{ is strictly increasing}\} \in U.$$

We show that $\kappa \in j(A)$; going through the acrobatics of applying j, we must show

$$\kappa \in \{\alpha < j(\kappa) : \langle j(f)_\eta(\alpha) : \eta \in j(D) \rangle \text{ is strictly increasing}\}.$$

Now observe that since $|D| = \mathrm{cf}(\gamma) < \kappa$, we have $j(D) = j''D$. So we show $\langle j(f)_{j(\eta)} : \eta \in D \rangle$ is strictly increasing; this holds iff $\langle j(f_\eta) : \eta \in D \rangle$ is strictly increasing. But recalling that $j(f_\eta)(\kappa) = [j_\eta] = \eta$, we have that this holds iff $\langle \eta : \eta \in D \rangle$ is strictly increasing.

Let n^* be such that $\forall n \geq n^*$, $\kappa_n \in A$. Then n^*, D witness that $\langle f^*_\alpha : \alpha < \kappa^+ \rangle$ is very good at γ since for each $n \geq n^*$, $\kappa_n \in A$, and so $\langle f_\eta(\kappa_n) : \eta \in D \rangle = \langle f^*_\eta(n) : \eta \in D \rangle$ is strictly increasing. □

5. January 14

Supercompactness: a brief interlude.

For $\kappa \leq \lambda$, let's define $P_\kappa(\lambda) := \{X \subseteq \lambda : |X| < \kappa\}$. If you get confused, just look at $\kappa \subseteq P_\kappa(\kappa)$. Let U be an ultrafilter on $P_\kappa(\lambda)$.

- U is κ-complete if $\forall \langle A_\alpha : \alpha < \mu \rangle$ with $\mu < \kappa$, $\bigcap_{\alpha<\mu} A_\alpha \in U$.
- U is fine if $\forall \alpha < \lambda$, $\{X : \alpha \in X\} \in U$.[10]
- U is normal if $\forall F : P_\kappa(\lambda) \longrightarrow \lambda$ such that $\underbrace{\forall X \in \mathrm{dom}(F)\, F(X) \in X}_{\text{Think: regressive}}$, there is $A \in U$ such that F is constant on A.

An ultrafilter U on $P_\kappa(\lambda)$ is a supercompactness measure if it has the three properties above.

We can form $\mathsf{Ult}(V, U)$ as before. We get $j : V \longrightarrow M \cong \mathsf{Ult}(V, U)$. Then $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. Furthermore, ${}^\lambda M \subseteq M$.

Exercise: This embedding has the stated properties.

Fact: $U = \{X \subseteq P_\kappa(\lambda) : j''\lambda \in j(X)\}$.[11]

κ is λ-supercompact iff there is a supercompactness measure on $P_\kappa(\lambda)$ iff there is an embedding as above.

Claim: If κ is λ-supercompact and λ is regular, then any sequence $\langle S_\xi : \xi < \mu \rangle$ for $\mu < \kappa$ of stationary subsets of $\lambda \cap \mathrm{cof}(< \kappa)$ reflect simultaneously.[12]

Proof. Let $j : V \longrightarrow M \cong \mathsf{Ult}(V, U)$ be as above. Let $\gamma := \sup j''\lambda$.[13] We claim that $\gamma < j(\lambda)$. First note that $\langle j(\alpha) : \alpha < \lambda \rangle \in M$ since ${}^\lambda M \subseteq M$. Thus $M \models \mathrm{cf}(\gamma) = \lambda$. Since $\lambda < j(\kappa) < j(\lambda)$ and $M \models$ "$j(\lambda)$ is regular and $\mathrm{cf}(\gamma) = \lambda$" we get $\gamma < j(\lambda)$.

Now we show that for each $\xi < \mu$, $j(S_\xi) \cap \gamma$ is stationary in M. Then

$$M \models \text{“}(\exists \delta < j(\lambda))\ (\forall \xi < \mu)\ j(S_\xi) \cap \delta \text{ is stationary.”}$$

This is therefore enough by elementarity (note we're implicitly using the fact that $j(\langle S_\xi : \xi < \mu \rangle) = \langle j(S_\xi) : \xi < \mu \rangle$ which holds since $j \restriction \kappa = \mathrm{id}_\kappa$)

Indeed, fix a club $C \subseteq \gamma$ in M. Define $D = \{\alpha < \lambda : j(\alpha) \in C\}$. We claim that D is $< \kappa$-club.[14] For unbounded, fix $\xi < \lambda$. Then form a sequence $\langle \langle \beta_n, \gamma_n \rangle : n < \omega \rangle$ such that

- $j(\xi) < \beta_0$
- $\gamma_n = j(\alpha_n)$ for some $\alpha_n < \lambda$
- $\beta_{n+1} > \gamma_n > \beta_n$
- $\beta_n \in C$.

[10] In the case of $U = P_\kappa(\kappa)$, we get that U contains all of the tail sets.

[11] As in the measurable case, we'll use this to test for whether a set is measure one.

[12] "You just hit stuff with j, and good things happen." -S.

[13] "Because $j''\lambda$ is magical, the sup is also magical." -S.

[14] That is, unbounded and $< \kappa$-closed.

Letting $\gamma := \sup_n \gamma_n = \sup_n \beta_n$, we have that $\gamma \in C$ by closure of C and $\gamma = \sup_n j(\alpha_n) = j(\sup_n \alpha_n)$ (with the last following since $j \restriction \kappa = \mathrm{id}_\kappa$). Hence $\sup_n \alpha_n \in D$ is above ξ. That D is $< \kappa$-closed follows from $j \restriction \kappa = \mathrm{id}_\kappa$.

Now since D is $< \kappa$-club, $D \cap S_\xi \neq \varnothing$ for each $\xi < \mu$. So if $\alpha \in D \cap S_\xi$, then $j(\alpha) \in C \cap j(S_\xi)$ as required. □

Bemerkungen:

- $\mathrm{cf}^M(\gamma) = \lambda < j(\kappa)$ implies that S reflects at a point of cf $(< \kappa)$.
- The proof would work for any collection of $< \kappa$-many stationary subsets.

Our next goal is the following Theorem of Magidor: if κ is supercompact and $k < \omega$, then there is a generic extension in which $\kappa = \aleph_\omega$ is strong limit and $2^{\aleph_\omega} = \aleph_{\omega+k+1}$.

Towards a definition of the poset, let U be a supercompactness measure on $P_\kappa(\kappa^{+k})$, and assume that $2^\kappa = \kappa^{+k+1}$.[15] The set

$$Z := \left\{ X \in P_\kappa(\lambda) : \kappa \cap X \text{ is inaccessible} \wedge |X| = (X \cap \kappa)^{+k} \right\}$$

is in U, as $\kappa = j(\kappa) \cap j''\kappa^{+k}$ and hence $j''\kappa^{+k} \in j(Z)$.

- For $X \in Z$, set $\kappa_X := X \cap \kappa$.
- For $X, Y \in Z$ set $X \prec Y$ if $|X| < \kappa_Y$ and $X \subseteq Y$.
- A supercompact Prikry stem is a sequence $\langle X_0, X_1, \ldots, X_{n-1} \rangle$ of elements of Z such that $X_i \prec X_{i+1}$ for all $i < n-1$.

Now define a poset $\mathbb{P}$: conditions are (s, F) where $s = \langle X_0, f_0, X_1, f_1, \ldots, X_{n-1}, f_{n-1} \rangle$ where $\vec{X}$ is a supercompact Prikry stem, where for convenience, X_0 is such that $\kappa_{X_0} = \omega$ and $\forall i < n-1$, $f_i \in \mathbb{C}(\kappa_{X_i}, \kappa_{X_{i+1}})$ where $\mathbb{C}(\alpha, \beta) = \mathrm{Coll}(\alpha^{+k+2}, < \beta)$, and where $f_{n-1} \in \mathbb{C}(\kappa_{X_{n-1}}, \kappa)$. Furthermore, F is a function with $\mathrm{dom}(F) \in U \cap \mathcal{P}(Z)$ and $\forall X \in \mathrm{dom}(F)$, $F(X) \in \mathbb{C}(\kappa_X, \kappa)$.

Now say $s = \langle X_0, f_0, \ldots, X_{n-1}, f_{n-1} \rangle$ and $t = \langle Y_0, g_0, \ldots, Y_{m-1}, g_{m-1} \rangle$. We define $(s, F) \leq (t, G)$ if

1. $\vec{X}$ end-extends $\vec{Y}$.
2. $\forall i < m$, $f_i \leq g_i$ in the poset $\mathbb{C}(\kappa_{X_i}, \kappa_{X_{i+1}})$.
3. For all $i \in [m, n)$ we have $X_i \in \mathrm{dom}(G)$ and $f_i \leq G(X_i)$.
4. $\mathrm{dom}(F) \subseteq \mathrm{dom}(G)$ and $\forall X \in \mathrm{dom}(F)$, $F(X) \leq G(X)$.

6. January 16

Recall what the conditions from last time were of the form $\langle s, F \rangle$, where F is our "constraining function" and the "stem" s is of the form $\langle X_0, f_0, X_1, f_1, \ldots, X_{n-1}, f_{n-1} \rangle$.

Given conditions p, q, we define (direct extension) $p \leq^* q$ if $p \leq q$ and $l(p) = l(q)$.

Claim: (Prikry Lemma) Given $p \in \mathbb{P}$ and φ in the forcing language, there is $q \leq^* p$ such that q decides φ.

Claim: (Round 1) For every dense open set D and every condition (s, F), there is F^* so that $(s, F^*) \leq (s, F)$ and if $(t, G) \leq (s, F^*)$ is in D, then $(t, F^* \restriction \{X : t \prec X\})$. (extending use of symbol $\prec$ here to say that X is a next possible Prikry point; in particular, κ_X should be greater than sup of the ranges of the collapses).

[15] We may come back and address how to get large cardinals κ with 2^κ large.

Proof. Side note: how many stems are there? $|\mathcal{P}_\kappa(\kappa^{+k})| = \kappa^{+k}$ if κ is κ^{+k}-supercompact.

Let $\langle s_\alpha : \alpha < \kappa^{+k}\rangle$ enumerate all possible stems extending s. We construct $\langle F_\alpha : \alpha < \kappa^{+k}\rangle$ such that $F_0 = F$, and $F_{\alpha+1} \leq F_\alpha$ (pointwise). At limit γ we want $[F_\gamma]_U \leq [F_\alpha]_U$ for all $\alpha < \gamma$.

At stage $\alpha+1$, let $F_{\alpha+1} \leq F_\alpha$ such that $(s_\alpha, F_{\alpha+1}) \in D$ if possible.

Why can we take lower bounds at limits? $[F]_U = j(F)(j''\kappa^{+k}) \in \mathbb{C}^{\mathsf{Ult}}(j''\kappa^{+k} \cap j(\kappa), j(\kappa)) = \mathbb{C}^{\mathsf{Ult}}(\kappa, j(\kappa)) := \mathrm{Coll}^{\mathsf{Ult}}(\kappa^{+k+2}, < j(\kappa))$. Now $M \cong \mathsf{Ult}(V, U)$ is closed under κ^{+k}-sequences. This implies that $\mathrm{Coll}^{\mathsf{Ult}}(\kappa^{+k+2}, < j(\kappa))$ is κ^{+k+1}-closed (i.e., $< \kappa^{+k+1}$-sequences), so we can take a lower bound.

Next we capture the measure one sets used by the F_α's. Set $\bar{F}$ to be, mod-U, a lower bound for $\langle F_\alpha : \alpha < \kappa^{+k}\rangle$. Let A_α be the measure one set witnessing $[F_{\alpha+1}]_U > [\bar{F}]_U$.

Let $A^* := \left\{X : \forall\alpha < \kappa^{+k} \text{ if } s_\alpha \prec X, \text{ then } X \in A_\alpha\right\}$. We claim that $A^* \in U$. We check $j''\kappa^{+k} \in j(A^*)$. $j''\kappa^{+k} \in \left\{X : \forall\alpha < j(\kappa^{+k}) \text{ if } j(s)_\alpha \prec X \text{ then } j''\kappa^{+k} \in j(A)_\alpha\right\}$. Which stems in $j(\mathbb{P})$ are $\prec j''\kappa^{+k}$? This is exactly the j-images of stems in $\mathbb{P}$. Note this does it. For such stems, indexed by some $f(\alpha)$, we have $j''\kappa^{+k} \in j(A_{j(\alpha)})$. Now set $F^* := \bar{F} \restriction A^*$.

Let $(t, G) \leq (s, F^*)$ be in D. Then $t = s_\alpha$ for some $\alpha < \kappa^{+k}$. Since every $X \in A^*$ with $X \succ s_\alpha$ is in A_α, we must have $(s_\alpha, F_{\alpha+1}) \in D$. So since F^* is below $F_{\alpha+1}$ on A_α, we get $(s_\alpha, F^*) \in D$ (and $s_\alpha = t$, and probably some restriction of F^* to X's which are above t). □

Important Remark Suppose $\dot{\gamma}$ is a name for an ordinal, and $D = \{p \in \mathbb{P} : p \| \dot{\gamma}\}$. At the end the value that (t, G) decides is the same as the value that (t, F^*) decides.

7. January 21

Claim: (Round 1) For every dense open set D and every condition (s, F), there is F^* so that $(s, F^*) \leq (s, F)$ and if $(t, G) \leq (s, F^*)$ is in D, then $(t, F^* \restriction \{X : t \prec X\})$.

(Recall Claim 1 was proved last time.)

Claim: (Round 2) Let D be dense open, and (s, F^*) be as in Round 1. Then there is a direct extension (s, F^{**}) of (s, F^*) so that if $(t, G) \leq (s, F^{**})$ is in D with X is the top Prikry point of t, then $(t \restriction (l(t)-1)^\frown\langle X, F^{**}(X)\rangle, F^*) \in D$. (reducing top-most collapse to one given by universal constraining function).

Proof. Fix $X \in \mathrm{dom}(F^*)$. Enumerate $\{s : s \prec X\}$, i.e. the stems on top of which X can sit, as $\langle s_\alpha : \alpha < \kappa_X^{+k}\rangle$ (can do this as $(\kappa_X^{+k})^{<\kappa_X}$ has size κ_X^{+k}, by enough GCH and that it holds for κ, and reflection from supercompact.) Construct $\langle f_\alpha : \alpha < \kappa_X^{+k}\rangle$ so that $f_0 = F^*(X) \in \mathbb{C}(\kappa_X, \kappa)$, (recall κ_X^{+k+2}-closed) and $f_{\alpha+1} \leq f_\alpha$ (ordering in $\mathbb{C}(\kappa_X, \kappa)$) is such that $(s_\alpha^\frown\langle X, f_{\alpha+1}\rangle, F^*) \in D$, if there is one. By closure of $\mathbb{C}(\kappa_X, \kappa)$, we can take a lower bound $F^{**}(X)$.

Let $(t, G) \leq (s, F^{**})$ be in D. By Round 1, $(t, F^*) \in D$. Fix α so that $t \restriction l(t)-1 = s_\alpha$, and X is the top Prikry point of t. By construction $(s_\alpha^\frown\langle X, f_{\alpha+1}, F^*) \in D$. Hence so is $(s_\alpha^\frown(X, F^{**}(X)), F^*)$. □

Let (s, F^{**}) be as in Round 2 for $D = \{p : p \| \varphi\}$.

Let t be a stem. Partition $\{X : t \prec X\}$ into $A_t^0 := \{X : (t^\frown(X, F^{**}(X)), F^*) \Vdash \varphi\}$, $A_t^1 := \{X : (t^\frown(X, F^{**}(X), F^*)) \Vdash \neg\varphi\}$, and $A_t^2 := \{\ldots : \text{ doesnt decide } \varphi\}$. Let

$A_t := A_t^i \in U$. Let $F^{***} := F^{**} \restriction \Delta A_t$.

Claim: (Round 3) There is a direct extension of (s, F^{***}) that decides φ.

Proof. Let $(t, G) \leq (s, F^{***})$ be of minimal length deciding φ. Assume for a contradiction that $l(t) > l(s)$. Let $t^- := t \restriction l(t) - 1$ and X be the top point of t. By Rounds 1 and 2, $(t^- \frown \langle X, F^{**}(X) \rangle, F^*)$ decides φ. Assume (since measure one set of things which decide) that it forces φ. Now $X \in A_{t^-}$ by definition of diagonal intersection, so $A_{t^-} = A^0_{t^-}$. Hence (t^-, F^{***}) already forces φ, contradicting the minimality of $l(t)$. $\square$

Exercise: Prove the strong form of the Prikry Lemma (will need Rounds 1 and 2 above and the following claim).

Claim:[16] Let D be dense open and (s, F^{**}) be as in Round 2 above. Then there exist a decreasing sequence of constraints F_n and sets Y_m such that $Y_0 = \{s : (s, F^*) \in D\}$ and $Y_{m+1} := \{s : \exists A \in U \; \forall X \in A \; (s^\frown(X, F_m(X)) \in Y_m)\}$ (trying to capture idea that you're $m+1$ steps away from being in D) and such that if $n \geq 1$ and t is the stem of an extension (s, F_n) with with $t \in Y_n$, then $t \restriction l(t) - 1^\frown(X, F_n(X)) \in Y_n$. As before, X is the top Prikry point of t.

Let $\dot{g}_n$ be a name for the nth generic collapse. We'll prove the following next time:

Corollary: Let $\dot{X}$ be a name for a subset of some $\mu < \kappa$. Then $\Vdash_{\mathbb{P}} \dot{X} \in V[\dot{g}_0 \times \ldots \times \dot{g}_{n-1}]$ for some n. Note that κ is preserved in models of the form $V[\dot{g}_0 \times \ldots \times \dot{g}_{n-1}]$ since it's a generic extension of a product of collapses well below κ.

8. January 26

Recall where we were last time: Magidor's forcing had conditions

$$\langle X_0, f_0, X_1, f_1, \ldots, X_{n-1}, f_{n-1}, F \rangle.$$

The constraining function on the top guides the extensions.

We finished proving the PL.

Corollary 8.1. *For every name $\dot{X}$ for a subset of some $\mu < \kappa$, $\Vdash_{\mathbb{P}} \exists n < \omega \; \dot{X} \in V[\dot{g}_0 \times \dot{g}_1 \times \ldots \times \dot{g}_{n-1}]$ where $\dot{g}_i$ names the $\mathbb{C}(\kappa_{X_i}, \kappa_{X_{i+1}})$-generic added by $\mathbb{P}$.*

Proof. (Natural first try, which doesn't quite work) Let $p = (s, F)$ be a condition so that $\mu < \kappa_{X_{n-1}}$ where the stem s has the usual form $\langle x_0, f_0, \ldots, X_{n-1}, f_{n-1} \rangle$. Look at $\mathbb{P} \restriction p$ with $\leq^*$ (i.e., direct extensions of p). This "is" a product. By diagonalizing over $\vec{h} \in \prod_{i<n} \mathbb{C}(\kappa_{X_i}, \kappa_{X_{i+1}}) \times \mathbb{C}(\kappa_{X_{n-1}}, \kappa)$, we can find (s, F^*) such that if there is G such that $(t, G) \leq (s, F^*)$ decides $\dot{\alpha} \in \dot{X}$, where t is s-strengthened by $\vec{h}$, then so does (t, F^*). ...[17]

(Second try) Let $p = (s, F)$ as before. For $\alpha < \mu$ and $\vec{h} \in \prod_{i<n} \mathbb{C}(\kappa_{X_i}, \kappa_{X_{i+1}})$, set $D_{\alpha, \vec{h}}$ to be the set of (t, G) such that $l(t) = n$, $t = \langle Y_0, g_0, \ldots, Y_{n-1}, g_{n-1} \rangle$, and either $g \restriction n-1 \leq \vec{h}$ and $(t, G) \| \alpha \in \dot{X}$, or $\vec{g} \restriction n-1 \perp \vec{h}$. Diagonalize over $\alpha's$ and $\vec{h}$ to get $(s^*, F^*) \leq^* (s, F)$ so that if $(t, G) \leq^* (s^*, F^*)$ is in the good case, then

[16] "If the claim is incorrect, then part of exercise is to fix it." -S.

[17] Exercise: Convince yourself something went wrong here!

so is $(t \restriction (l(t)-1)^\frown\langle x,\ \text{top collapse of } s^*\rangle, F^*)$. Note the collapses in $t \restriction (l(t)-1)$ are exactly the $\vec{h}'$s. Hence, if compatible with $\vec{h}$, take lower bound, and extend to decide. □

Corollary 8.2. *κ is preserved, strong limit, singular, cofinality ω and cardinals in the interval $(\kappa_{X_i}^{+k+2}, \kappa_{X_{i+1}})$ are collapsed.*

So $\kappa = \aleph_\omega$ in the extension by $\mathbb{P}$. (Un?)fortunately, $\bigcup X_n = \kappa^{+k}$, so $|\kappa^{+k}| = \kappa$ in the extension. Recall we had $2^\kappa = \kappa^{+k+1}$.

For each $i \le k$, we define $V_i = V[\langle X_n \cap \kappa^{+i} : n < \omega\rangle, \langle g_n : n < \omega\rangle]$.

Claim: The model we want is V_0. In particular, κ^{+i} is preserved in V_0 for all $i \le k$.

To do this, we'll show that κ^{+i+1} is preserved in V_i. (preserving these guys in outer models of V_0, and so in V_0 itself).

Proof. Note $|\kappa^{+i}| = \kappa$ in V_i. Assume there is $b : \mu \longrightarrow \kappa^{+i+1}$ a cofinal map with $\mu < \kappa$. Note $\forall j \le i$, $\text{cf}(\kappa^{+j}) = \omega$ in V_i. So $\text{cf}(\kappa^{+i+1})$ must be strictly below κ, if collapsed. Let $\dot{b}$ be a $\mathbb{P}$-name for such a function. □

Digression on Automorphisms:

Let $\mathcal{A}$ be the group of permutations of κ^{+k} which fix the whole set κ^{+i}. Note that each $\Gamma \in \mathcal{A}$ permutes $P_\kappa(\kappa^{+k})$, by taking pointwise images. In fact, each Γ gives an automorphism of the forcing. Given $p = \langle X_0, f_0, \ldots, X_{n-1}, f_{n-1}, F\rangle$, set $\Gamma p = \langle \Gamma X_0, f_0, \Gamma X_1, f_1, \ldots, \Gamma X_{n-1}, f_{n-1}, F \circ \Gamma^{-1}\rangle$. Notice that $\text{dom}(F \circ \Gamma^{-1}) \in U$.

We claim that $\{X : \Gamma X = X\} \in U$. (pointwise). Notice $j(\Gamma)$ fixes $j''\kappa^{+k}$ pointwise. So $j''\kappa^{+k} \in j(\{X : \Gamma X = X\})$. That is $j(\Gamma)(j''\kappa^{+k}) = j''\kappa^{+k}$.

We claim: Let $p, p' \in \mathbb{P}$ with

$$p = \langle X_0, f_0, \ldots, X_{n-1}, f_{n-1}, F\rangle \text{ and } p' = \langle Y_0, f_0, \ldots, Y_{n-1}, f_{n-1}, G\rangle.$$

Suppose that (1) $\forall j < n$, $X_j \cap \kappa^{+i} = Y_j \cap \kappa^{+i}$ and (2) For all $X \in \text{dom}(F) \cap \text{dom}(G)$, $F(X) = F(G)$. Then there is $\Gamma \in \mathcal{A}$ such that $\Gamma p \parallel p'$.

Further, given a $p \in \mathbb{P}$ we can find $(s^*, F^*) = p^* \le p$ such that for all $\alpha < \mu$, if $(t, G) \le (s^*, F^*)$ decides $\dot{b}(\alpha)$, then so does (t, F^*), with the same value (using capturing round of PL over and over for each α).

Claim: $|\left\{\lambda < \kappa^{+i+1} : \exists p^{**} \le p\ p^{**} \Vdash \dot{b}(\alpha) = \lambda\right\}| \le \kappa^{+i}$.

Proof idea: If there are too many λ, then can find coniitions satisfying the conditions of the above claim, which will lead to a contradiction.

Note that this (sub)claim finishes the proof that κ^{+i+1} is preserved, as the above claim covers the range of $\dot{b}$ by a small set.

9. January 28

Recall where we ended last time. We had assumed the existence of $\dot{b}$ a name for a function from $\mu < \kappa$ to κ^{+i+1} which is cofinal. Moreover, $\Vdash \dot{b} \in V_i$, where $V_i := V[\langle X_n \cap \kappa^{+i} : n < \omega\rangle, \langle g_n : n < \omega\rangle]$. We found p^* such that $\forall \alpha < \mu\ \forall q \le p^*$ if $q \Vdash \dot{b}(\alpha) = \lambda$, then so does the condition $\langle \text{stem}(q), \text{constraint}(p^*)\rangle$.

Now we claim that $\left|\left\{\lambda < \kappa^{+i+1} : \exists q \le p^*\ q \Vdash \dot{b}(\alpha) = \lambda\right\}\right| \le \kappa^{+i}$. Let's call this set A. Recall this a contradiction (covering the range of $\dot{b}$ by a small set).

Otherwise, $|A| > \kappa^{+i}$. So fix q_λ for $\lambda \in A$ such that $q_\lambda \Vdash \dot{b}(\alpha) = \lambda$. We can assume that the constraint of q_λ is the constraint of p^* (by our choice of p^*). We can also assume that $l(q_\lambda)$ is fixed on a set of size $> \kappa^{+i}$. Now, there are

κ^{+i} supercompact Prikry stems $\vec{Z}$ with each $Z_j \in \mathcal{P}_\kappa(\kappa^{+i})$.). So there are $q_\lambda = \langle X_0, f_0, X_1, f_1, \ldots, X_{n-1}, f_{n-1}, F^* \rangle$ and $q_{\lambda'} = \langle Y_0, f_0, Y_1, f_1, \ldots, Y_{n-1}, f_{n-1}, F^* \rangle$ (upper part fixed; can also assume that the sequence of the f's are constant; just κ-many; The X_i, Y_i may be different, but their intersections with κ^{+i} are the same) so that $X_j \cap \kappa^{+i} = Y_j \cap \kappa^{+i}$ for all j. By the previous claim, there is a Γ which fixes κ^{+i} so that $\Gamma q_\lambda \| q_{\lambda'}$. But Γ fixes $\dot{b}$. But this contradicts Γq_λ and $q_{\lambda'}$ decide different values for $\dot{b}(\alpha)$.

Other ways of preserving cardinals above κ:

Method 1: List of references.

(1) Foreman-Woodin: GCH fails everywhere.
(2) Foreman: More saturated ideals.
(3) Cummings, Morgan (Charles), Djamonza, ...? Universality numbers of graphs at singulars (note this is not the title).

Sketch (with many exercises that are not too bad). Let $R\mathbb{C}(\alpha, \beta)$ be the regular open algebra for $\mathbb{C}(\alpha, \beta)$. Recall $\mathbb{C}(\alpha, \beta) = \mathrm{Coll}(\alpha^{+k+2}, < \beta)$. Define a measurable version $\bar{\mathbb{P}}$ of the forcing $\mathbb{P}$ we just did. Conditions are just (where $\alpha_0 = \omega$) $\langle \alpha_0, f_0, \alpha_1, f_1, \ldots, \alpha_{n-1}, f_{n-1}, f \rangle$. $\mathrm{dom}(f) \in \bar{U}$ where $\bar{U}$ =projection of U to a normal measure on κ,[18] and for all $\alpha \in \mathrm{dom}(f)$, $f(\alpha) \in R\mathbb{C}(\alpha, \kappa)$.

For a constraint F from $\mathbb{P}$ and $A \in U$, define $b(F, A)$ is a function where the domain is $\{\kappa_X : X \in A \cap \mathrm{dom}(F)\}$. $\forall \alpha \in \mathrm{dom}(b(F, A))$, take

$$b(V, A)(\alpha) = \bigvee_{X \in \mathrm{dom}(F) \cap A, \kappa_X = \alpha} F(X)$$

(Note this is a boolean sup, but a sup of non-zero things, so by completeness, non-zero. That is, it is the common amount of information.)

Bemerkungen: Strengthening F, A strengthens $b(F, A)$. If follows that $\{[b(F, A)]_{\bar{U}} : A \in U\}$ generates a filter on $R\mathbb{C}^{\mathsf{Ult}(V, \bar{U})}(\kappa, j_{\bar{U}}(\kappa))$ (every two members having common refinement). Call this filter $\mathrm{Fil}(F)$. For ease of notation, call this thing (the $R\mathbb{C}$) $\mathbb{B}_0$.

Claim: For all F and all $b \in \mathbb{B}_0$ there is $F^* \leq F$ so that either $b \in \mathrm{Fil}(F)$ or $(\neg b) \in \mathrm{Fil}(F^*)$.

Proof. Exercise. □

Claim For all F, there is $F^* \leq F$ so that $\mathrm{Fil}(F^*)$ is an ultrafilter.

Proof. Exercise. Hint: How many boolean values are in $\mathbb{B}_0$? How closed is decreasing this F modulo U? (κ^{+k+1}-closed). So make sure to count elements and put each one in or out. □

Now restrict $\bar{\mathbb{P}}$ so that $[f]_U \in \mathrm{Fil}(F^*)$[19] which is an ultrafilter. Now $\bar{\mathbb{P}}$ has κ^+-c.c.! We want $\mathbb{P}$ to generate a generic for $\bar{\mathbb{P}}$. There is a natural order preserving map into $\bar{\mathbb{P}}$ (though this is not a projection). Take $X \mapsto X \cap \kappa$ and take $F \mapsto b(F, \mathrm{dom}(F))$.. However, this is still not good enough to induce a generic. We need some kind of density condition.

[18] How do we do this? If $j := j_U : V \longrightarrow M$, then take $\bar{U} := \{X \leq \kappa : \kappa \in j(X)\}$

[19] "Things always live dual lives." -S.

Exercise: (Look at More Saturated Ideals) There is $p^* \in \mathbb{P}$ below which the above map is a "projection" (i.e., may need a different notion of projection). Refer to the paper for more details.

Method 2: Define $\bar{\mathbb{P}}$ as above but $f(\alpha) \in \mathbb{C}(\alpha, \kappa)$. The idea: produce a filter on the $[f]$'s which is generic for $\mathbb{C}^{\mathsf{Ult}(V,\bar{U})}(\kappa, j_{\bar{U}}(\kappa))$, starting with a model of GCH with κ is κ^{+k}-supercompact. Iterate $\mathrm{Add}(\alpha, \alpha^{+k+1})$ for $\alpha \leq \kappa$ and lift an embedding $j : V \longrightarrow M$ in a careful way so that a generic for $\mathbb{C}^{\mathsf{Ult}(V[G],U)}(\kappa, j(\kappa))$ generates a generic for $\mathbb{C}^{\mathsf{Ult}(V[G],\bar{U})}(\kappa, j_{\bar{U}}(\kappa))$. We'll examine this method more next time.

10. January 30

Review: From Magidor, we define $V_0 = V[\langle \kappa_{X_n} : n < \omega\rangle, \langle g_n : n < \omega\rangle]$. Our goal is to describe a forcing for which these objects are generic. To that end we defined $\bar{\mathbb{P}}$ where conditions were $\langle \alpha_0, f_0, \alpha_1, f_1, \ldots, \alpha_{n-1}, f_{n-1}, f\rangle$ where $\mathrm{dom}(f) \in \bar{U}$ (ultrafilter gotten by projecting supercompactness measure) and $f(\alpha) \in \mathbb{C}(\alpha, \kappa)$. This forcing should have the Prikry property, but is also collapses stuff above κ (but for different reasons). We want to "fix" this forcing by altering it to make it κ^+-c.c. In order to preserve cardinals above κ, make the f's come from the filter. We gave two ideas last time:

Method 1: Projection from supercompact version. Get a projection from $\mathbb{P}$ to $\bar{\mathbb{P}}$.

Method 2: Make the f's come from a generic filter. (The filter from first method probably not generic, just some random ultrafilter.) Goal: to build a generic H for $\mathbb{C}^{\mathsf{Ult}(V,\bar{U})}(\kappa, j_{\bar{U}}(\kappa))$ which is generic over $\mathsf{Ult}(V, \bar{U})$ (you'll have no luck trying to do this for V itself; but we only need to meet dense sets in this Ult anyway). If we restrict $\bar{\mathbb{P}}$ to have $f \in H$, we get κ^+-c.c.

Exercise: Prove that this version of $\bar{\mathbb{P}}$ satisfies the PL.

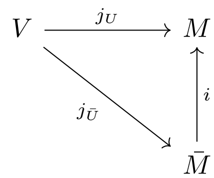
We're going to build this generic today, but in a somewhat backwards way. We'll need the supercompactness to build this measure.

Let U be a supercompactness measure on $\mathcal{P}_\kappa(\kappa^{+k})$. Let $\bar{U}$ be the projection of U to a measure on κ. Let's assume that $2^\kappa = \kappa^{+k+1}$.

Claim: There is H^* which is generic for $\mathbb{C} := \mathbb{C}^{\mathsf{Ult}(V,U)}(\kappa, j_U(\kappa))$ over $\mathsf{Ult}(V, U)$. Note we are still working with the original ultrafilter U, not is projection $\bar{U}$.

Proof. Let $j_U = j : V \longrightarrow M$. $M \models$ "$\mathbb{C}$ is κ^{+k+2}-closed and has $j(\kappa)$ maximal antichains." In V, $\mathbb{C}$ is κ^{+k+1}-closed and has $|j(\kappa)|$ antichains in M. How big is $|j(\kappa)|$? We have $|j(\kappa)| = \left|\{F : \mathcal{P}_\kappa(\kappa^{+k}) \longrightarrow \kappa\}\right| = 2^{\kappa^{+k}} = \kappa^{+k+1}$. Build a decreasing sequence meeting all maximal antichains in M. Then take the filter generated by the decreasing sequence, namely, H^*. □

Recall: ultrapower by $\bar{U}$ embedds naturally into ultrapower by U.



$i(\mathbb{C}^{\bar{M}}(\kappa, j_{\bar{U}}(\kappa))) = \mathbb{C}^M(\kappa, j_U(\kappa))$. Define $H = \{f \in \mathbb{C} : i(f) \in H^*\}$. We hope that H is generic.

Notice: if $\mathrm{crit}(i) > j_U(\kappa)$, then $i \restriction \mathbb{C}^{\bar{M}}$ is the identity (pointwise image).

<u>Claim</u>: Assume GCH.[20] and κ is κ^{+k}-supercompact. If we iterate $\mathrm{Add}(\alpha, \alpha^{+k+1})$ for $\alpha \leq \kappa$ inaccessible, then in the extension, $V[H]$, by this iteration, there is a generic H^* for j(iteration) and an embedding $j : V[H] \longrightarrow M[H^*]$ so that for all $\alpha < j(\kappa)$, there is $f : \kappa \longrightarrow \kappa$ so that $j(f)(\kappa) = \alpha$. (Recalling earlier diagram, $i([f]) = j_U(f)(\kappa)$.) In particular, $\mathrm{crit}(i) > j(\kappa)$.

Proof. First some notation. Let $\mathbb{P}_\kappa$ be the iteration up to κ and $\dot{\mathbb{Q}}_\kappa$ is a $\mathbb{P}_\kappa$-name for $\mathrm{Add}(\kappa, \kappa^{+k+1})$. Fix $j : V \longrightarrow M$ witnessing that κ is κ^{+k}-supercompact. Let $H = H_0 * H_1$ be generic for the iteration $\mathbb{P}_\kappa * \mathbb{Q}_\kappa$. Now $j(\mathbb{P}_\kappa) \restriction (\kappa + 1) \cong \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ (with Easton support). Let $\dot{\mathbb{R}}$ be such that $j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa * \dot{\mathbb{R}}$. In $V[H]$, $\mathbb{R}$ is κ^{+k+1}-closed and has $|j(\kappa)|$-many antichains. Build a generic H_0^* for $j(\mathbb{P}_\kappa)$ (which includes $H_0 * H_1$) as before. This allows us to lift $j : V[H_0] \longrightarrow M[H_0^*]$ in $V[H]$[21]. We want to lift to the extension by $\mathbb{Q}_\kappa$. Note that $\forall \gamma < \kappa^{+k+1}$, $j''(H_1 \restriction \gamma)$ is in $M[H_0^*]$. Moreover, $\bigcup j''(H_1 \restriction \gamma)$ is a condition in $j(\mathbb{Q}_\kappa)$. So as before, $j(\mathbb{Q}_\kappa)$ is κ^{+k+1}-closed and has $|j(\kappa)^{+k+1}| = \kappa^{+k+1}$-many antichains in $V[H]$.

Enumerate the antichains as $\langle A_i : i < \kappa^{+k+1} \rangle$. Each A_i is maximal in $\mathrm{Add}(j(\kappa), j(\xi))$ for some $\xi < \kappa^{+k+1}$ (comes from properties of the embedding). Let $\langle \alpha_i : i < \kappa^{+k+1} \rangle$ increasing so that A_i is maximal in $\mathrm{Add}(j(\kappa), j(\alpha_i))$. Build a decreasing sequence in $j(\mathbb{Q}_\kappa)$, $\langle r_i : i < \kappa^{+k+1} \rangle$ so that

(1) $r_i \in \mathrm{Add}(j(\kappa), j(\alpha_i))$,
(2) r_i is below some member of A_i,
(3) $r_i \leq \bigcup j''(H \restriction \alpha_i)$,
(4) $\forall \alpha < \alpha_i$, $r_i(j(\alpha), \kappa)$ is the α-th element of $j(\kappa)$. (Here view $\mathrm{Add}(j(\kappa), j(\kappa^{+k+1}))$ as partial functions from $j(\kappa)^{+k+1} \times j(\kappa)$ to $j(\kappa)$. This is saying that if f is α-th generic function, then $j(f)(\kappa)$ is αth element of $j(\kappa)$.)

Closure then finishes the proof. □

11. February 2

We'll work towards the following theorem: if κ supercompact and $\alpha < \omega_1$, then there is a model where $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} = \aleph_{\alpha+1}$.

Fix $\alpha < \omega_1$ and δ limit such that $\alpha = \delta + k$ for some $k < \omega$. κ indestructibly supercompact. Let $\mathbb{Q} = \mathrm{Add}(\kappa, \kappa^{+\alpha+1})$. Write $\delta = \bigcup_{n<\omega} D_n$, where $\langle D_n : n < \omega \rangle$ is increasing, with each D_n finite. Set $T_n = \{(\beta, \gamma) : \beta < \gamma \leq \delta \text{ and } \forall \zeta \in D_n, \zeta \notin (\beta, \gamma)\}$.

[20] "Going back to a primordial universe where we have GCH."

[21] Recall the criteria for lifting: pointwise image of the first generic must be contained in second generic.

Then set

$$\mathbb{C}_n := \prod_{(\beta,\gamma)\in T_n} \mathrm{Coll}(\kappa^{+\beta}, \kappa^{+\gamma}).$$

and $\mathbb{R}_n = \mathbb{Q} \times \mathbb{C}_n$. Note that for $n < m$, $\mathbb{R}_n$ projects onto $\mathbb{R}_m$.

Let $\lambda = \kappa^{+\alpha+2}$. Note in $V[\mathbb{R}_n]$, $\lambda = \kappa^{+l}$ for some $l < \omega$. So by indesctructibility there is a supercompactness measure $\dot{U}_n$ on $\mathcal{P}_\kappa(\lambda)$ so that

$$\dot{Z}_n = \left\{X : \kappa_X \text{ is inaccessible and } \mathrm{ot}(X) = \kappa_X^{+l}\right\} \in \dot{U}_n.$$

For $X \in \dot{Z}_n$, let $\lambda_X := \mathrm{ot}(X)$ ($= \kappa^+ l$ (see definition of $\dot{Z}_n$), but don't need to keep track of this l).

We define a forcing $\mathbb{P}$. Conditions are of the form

$$\langle r, X_0, f_0, X_1, f_1, \ldots, X_{n-1}, f_{n-1}, \dot{F}_n, \dot{F}_{n+1}, \ldots\rangle.$$

[22] with

1. $r \in \mathbb{R}_0$.
2. Each X_i, f_i are $\mathbb{R}_i$-names (respectively) for an element of $\dot{Z}_i$ and $\mathrm{Coll}(\lambda^+_{X_i}, < \kappa_{X_{i+1}})$ (if $i < n-1$) or $\mathrm{Coll}(\lambda^+_{X_i}, < \kappa)$ (if $i = n-1$). ($\mathbb{R}_i$-names for elements of V).
3. $X_i \prec X_{i+1}$.
4. $\dot{F}_i$ is an $\mathbb{R}_i$-name for a function with $\mathrm{dom}(F_i) \in \dot{U}_i$, and $\forall x \in \mathrm{dom}(\dot{F}_i)$, $F_i(x) \in \mathrm{Coll}(\lambda^+_X, < \kappa)$.

For a condition $p \in \mathbb{P}$, we write $p = \langle f^p, X^p_0, f^p_0, \ldots, F^p_n, \ldots\rangle$. $l(p)$ is the length of p (which in this case is n). Define $p \leq q$ if

1. $l(p) \geq l(q)$.
2. $r^p \leq r^q$. and $r^p \Vdash$ forces the rest of the following requirements.
3. $X^p_i = X^q_i$ for $i < l(q)$ and
4. $f^p_i \leq f^q_i$ for $i < l(q)$
5. For $i \in [l(q), l(p))$, we have $X^p_i \in \mathrm{dom}(F^q_i)$ and $f^p_i \leq F^q_i(X^p_i)$.
6. For $i \geq l(p)$, $\mathrm{dom}(F^p_i) \subseteq \mathrm{dom}(F^q_i)$ and $\forall x \in \mathrm{dom}(F^p_i)$, $F^p_i(x) \leq F^q_i(x)$.

Exercise: Does $\mathbb{P}$ collapse λ to be countable? Hint 1: Actually check to see if κ is collapsed, since λ collapsed to be size κ. Try to code surjections from ω to κ?. Hint 2: Probably.

Note throughout we are implicitly using projections of $\mathbb{R}_0$ into $\mathbb{R}_i$ since $r^p \in \mathbb{R}_0$, but the things r^p forces are $\mathbb{R}_i$-names.

We take an inner model of the forcing extension. Define $V_0 = V[A, \langle \kappa_{X_n}, g_n : n < \omega\rangle]$ where A is $\mathbb{Q}$-generic, $\langle X_n : n < \omega\rangle$ are the Prikry points and g_n is generic for $\mathrm{Coll}(\lambda^+_{X_n}, < \kappa_{X_{n+1}})$.

12. February 4

Reminder of where we were and the notation we've been using.

κ is indestructibly supercompact. Had $\mathbb{R}_n \cong \mathrm{Add}(\kappa, \kappa^{+\alpha+1}) \times$ Collapses. The point: as n gets larger, collapses in $\mathbb{R}_n$ get less destructive. Conditions look like $\langle r, X_0, f_0, X_1, f_1, \ldots, X_{n-1}, f_{n-1}, \dot{F}_n, \dot{F}_{n+1}, \ldots\rangle$[23] where $r \in \mathbb{R}_0$ and each of the other things are names corresponding to the index (ex: X_1 is an $\mathbb{R}_1$-name; but

[22] In reality, everything besides r is a name (see definition), but we'll be sloppy with notation.

[23] As mentioned in the previous footnote, we'll continue to be sloppy with notation.

projecting of $\mathbb{R}_0$ to $\mathbb{R}_1$, lets us view it is an $\mathbb{R}_0$-name, so that r can tell us something about it). Note further $\Vdash_{\mathbb{R}_n}$ "$\operatorname{dom} \dot{F}_n \in \dot{U}_n$". In the extension by $\mathbb{R}_n$, λ is a finite successor of κ and $\dot{U}_n$ a measure on $\mathcal{P}_k\kappa(\lambda)$. ($\mathbb{R}_n$ tells which finitely many cardinals after Prikry points are preserved; as n grows, we preserve more cardinals). We had $X \in \dot{Z}_n$ then above κ_X the Laver Preperation did a reflection of $\mathbb{R}_n$.

Let's try to get a capturing claim of sorts. More notation first:

(Recall) $V_0 = V[A, \langle \kappa_{X_n}, g_n : n < \omega\rangle]$. ($A$ is generic for $\operatorname{Add}(\kappa, \kappa^{+\alpha+1})$) (How to find out what X_n is? A long enough condition should help, but it is still $\mathbb{R}_n$-name. Thus, we need some fragment of the $\mathbb{R}_n$-generic being added by $\mathbb{R}_0$; similarly, information about g_n is determined by what X_n is. Thus, we have an implicit dependence on $\mathbb{R}_n$.)

Last bit of notation: a V_0-name is a $\mathbb{P}$-name which is fixed by any automorphism which fixes V_0. For $p \in \mathbb{P}$, let $s(p)$, called the *stem*, be

$$s(p) := \langle X_0, f_0, X_1, f_1, \dots, X_{n-1}, f_{n-1}\rangle;$$

Note: r is not part of the stem.

Claim: (Version 1 of a Capturing Claim) Let $\dot{\eta}$ be a V_0-name for an ordinal and $p \in \mathbb{P}$ so that $\mathbb{R}_{l(p)} \cong \mathbb{Q}_0 \times \mathbb{Q}_1$ where $\mathbb{Q}_0$ is μ^+-c.c. and $\mathbb{Q}_1$ is μ^+-closed, for some μ.[24] Then there is $p^* \leq^* p$ [25] such that if $q \leq p^*$ decides $\dot{\eta}$, then $(r^q, s(q), \bar{F}^{p^*} \restriction [l(q), \omega))$ decides $\dot{\eta}$ in the same way.

(Note: A proof was started today, but not finished. See below for a proof.)

13. February 6

We continue with the proof of the claim from last time.

Proof. [26] Let $l(p) = l$. We'll construct sequences $\langle r_n | n < \omega\rangle$ and $\langle\langle F^n_m | m \geq l\rangle | n < \omega\rangle$, forced to be decreasing.

Set $r_0 = r^p$ and $\langle F^0_m | m \geq l\rangle = \langle F^p_m | m \geq l\rangle$. To construct $\langle F^{n+1}_m | m \geq l\rangle$ and r_{n+1} from $\langle F^n_m | m \geq l\rangle$ and r_n first enumerate stems $\langle s_\alpha | \alpha < \mu\rangle$ of length $l+n$ so that every "stem restricted to μ" is the restriction of some s_α. (If a stem is $\langle X_0, f_0, \dots, X_{n-1}, f_{n-1}\rangle$, its restriction to μ is $\langle X_0 \cap \mu, f_0, \dots, X_{n-1} \cap \alpha, f_{n-1}\rangle$.) By the choice of $l = l(p)$, $\mathbb{R}_{l+n} \cong \mathbb{Q}_0 \times \mathbb{Q}_1$ where $\mathbb{Q}_0$ is μ^+-cc and $\mathbb{Q}_1$ is μ^+-closed. Now work through the s_α's, building inductively $\langle F^{n,\alpha}_m | m \geq l+n\rangle$ and $r_{n,\alpha}$ for $\alpha < \mu$. At stage $\alpha + 1$, build a maximal antichain B in $\mathbb{Q}_0$ and a decreasing sequence in $\mathbb{Q}_1$, say with lower bound q_1, such that for all $q \in B$ there exists $\bar{F}_q$ such that $((q, q_1), s_\alpha, \bar{F}_q)$ decides $\dot{\eta}$ if possible. Find names $F^{n,\alpha+1}_m$ and $r_{n,\alpha+1}$ as follows: $r_{n,\alpha+1}$ is $r_{n,\alpha}$ strengthened by q and $r_{n,\alpha+1} \Vdash \bar{F}^{n,\alpha+1} = \bar{F}^q$, where q is the unique $q \in B \cap$ the generic.

By closure of $\mathbb{Q}_1$, we can take a lower bound for the $r_{n,\alpha}$ at limits (as we are only changing the μ^+-closed part of the condition along the way).

Now let r_{n+1} be a lower bound for $r_{n,\alpha}$ and F^{n+1}_m be forced to by a mod U_m lower bound for $F^{n,\alpha}_m$ (here using supercompactness measure and the fact that

[24] Make sure that this can happen for some μ's. There are some other implicit dependences. Hint: take some suitable μ not "overlapped" by $\mathbb{R}_{l(p)}$.

[25] As before $\leq^*$ is a direct extension, where we now are preserving length, but can refine r.

[26] See exercise from next time.

$\lambda^+ > \mu$). Finally let r^{p^*} = lower bound for $\langle r_n | n < \omega\rangle$ and $F_m^{p^*}$ = lower bound for $\langle F_m^n | n < \omega\rangle$. Then let

$$A_m^{n,\alpha} = \left\{ x \in \mathrm{dom}(F_m^{n,\alpha+1}) | F_m^{p^*}(x) \leq F_m^{n,\alpha+1}(x) \right\}$$

and

$$A_m^n = \Delta_{\alpha<\mu} A_m^{n,\alpha} = \{x | \text{ if } s_\alpha \frown x \text{ is a stem, then } x \in A_m^{n,\alpha}\}.$$

Then set

$$A_m^{p^*} = \bigcap_{l+n\leq m} A_m^n.$$

Now set $p^* = (r^{p^*}, s(p), \bar{F}^{p^*} \restriction \langle A_m^{p^*} | m \geq l\rangle)$. If $q \leq p^*$ decides $\dot{\eta}$ then there exists an $al < \mu$ such that the restriction of $s(q)$ to μ is s_α (from stage $l(q) - l(p)$ in the construction), so there is a permutation Γ of λ, fixing μ, so that $\Gamma(s(q)) = s_\alpha$. Since Γ fixes V_0, $\Gamma(q)$ decides $\dot{\eta}$ in the same way as q. Hence, by construction $(r^q, s_\alpha, \bar{F}^{p^*})$ decides $\dot{\eta}$ (in the same way). (Note Γ is not touching the r_0 coordinate and does nothing to the constraining functions modulo a measure one set.) Thus the following holds: $(r^q, s(q), \Gamma^{-1}(\bar{F}^{p^*}))$ decides $\dot{\eta}$ all in the same way. □

14. February 9

Exercise: Double check the $\alpha+1$ stage in the construction from last time, correcting any mistakes.

Claim: Suppose that $\dot{b}$ is a V_0-name for a function from μ to ordinals and $p \in \mathbb{P}$ such that $\mathbb{R}_{l(p)} \cong \mathbb{Q}_0 \times \mathbb{Q}_1$. Then there is a $p^* \leq^* p$ (i.e. a direct extension) so that for all q, if $q \leq p^*$, $i \in \mathrm{top}(s(q))$,[27] and q decides $\dot{b}(i)$, then $(r^q, s(q), \bar{F}^{p^*} \restriction [l(q), \omega))$ decides $\dot{b}(i)$ in the same way.

Proof. As before, but slightly more care taken at each stage. □

Claim: Cardinals above κ are preserved in V_0.

Proof. Note, if μ is a successor above κ, then for all large enough m, $\mathbb{R}_m \cong \mathbb{Q}_0 \times \mathbb{Q}_1$ where $\mathbb{Q}_0$ is μ^+-cc and $\mathbb{Q}_1$ is μ^+-closed (the idea here is that as m grows, we are getting rid of more collapses, so eventually don't overlap μ). Let $\dot{b}$ be a name for a function from μ into On. Choose p so that $\mathbb{R}_{l(p)}$ factors as above. Get p^* from the previous claim (so $p^* \leq^* p$). We want to show that $|A| \leq \mu$, where for any fixed i, $A = \left\{\alpha | \exists q \leq p^*, q \Vdash \dot{b}(i) = \alpha\right\}$. Suppose not, and for $\alpha \in A$, choose $q_\alpha \Vdash \dot{b}(i) = \alpha$. We can assume $i \in \mathrm{top}(s(q_\alpha))$ for each α (just extend more if necessary). By the previous claim, we can also assume that $\bar{F}^{q_\alpha} = \bar{F}^{p^*} \restriction [l(q_\alpha), \omega)$. We can further assume (trimming A as necessary while preserving its size):

- $l := l(q_\alpha) = l(q_{\alpha'})$ for all $\alpha, \alpha' \in A$
- $\forall i < l, x_i^{q_\alpha} \cap \mu = x_i^{q_{\alpha'}} \cap \mu$ and $f_i^{q_\alpha} = f_i^{q_{\alpha'}}$ for all $\alpha, \alpha' \in A$.

What about the r parts (i.e. the parts from $\mathbb{R}_0$)? In fact, we can find α, α' so that $r^{q_\alpha} \| r^{q_{\alpha'}}$.[28]

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[27] Recall that $s(q)$ = the stem of q, and the top of a stem is the top Prikry point.

[28] Exercise: Fill in the details of this fact.

From here, any permutation Γ of λ fixing μ and sending $x_i^{q_\alpha}$ to $x_i^{q_{\alpha'}}$ for $i < l$ (and such permutations do exist) gives a contradiction. In particular, we have $\Gamma(q_\alpha) \| q_{\alpha'}$ but $\Gamma(q_\alpha) \Vdash \dot{b}(i) = \alpha$ while $q_{\alpha'} \Vdash \dot{b}(i) = \alpha'$. $\square$

Notation: let $s^-(q) = s(q)$ without the topmost collapse.

We'll prove the following next time:

Claim: Let $\dot{\eta}$ be a V_0-name (i.e., name fixed by any automorphism that fixes V_0) for an ordinal and $p^* \in \mathbb{P}$ be as in Round 1. Then there is $p^{**} \leq^* p^*$ so that $\forall q$ if $q \leq p^{**}$ decides $\dot{\eta}$, then so does $(r^q, s^-(q)^\frown \langle F_{l(q)-1}^{p^{**}}(\mathrm{top}(s(q)))\rangle, \bar{F}^{p^{**}} \restriction [l(q), \omega))$, and in the same way.

15. February 11

Recall that conditions look like: $\langle r, X_0, f_0, \ldots, X_{n-1}, f_{n-1}, F_n, F_{n+1}, \ldots\rangle$ except everything in sight is an $\mathbb{R}_n$-name, $r \in \mathbb{R}_0$, and you can decide stuff about these names by refining r (using R_0 projects on to $\mathbb{R}_n$).

Claim: Let $\dot{\eta}$ be a V_0-name (i.e., name fixed by any automorphism that fixes V_0) for an ordinal and $p^* \in \mathbb{P}$ be as in Round 1. Then there is $p^{**} \leq^* p^*$ so that $\forall q$ if $q \leq p^{**}$ decides $\dot{\eta}$, then so does $(r^q, s^-(q)^\frown \langle F_{l(q)-1}^{p^{**}}(\mathrm{top}(s(q)))\rangle, \bar{F}^{p^{**}} \restriction [l(q), \omega))$, and in the same way.

Proof. Enumerate all stems (without their topmost collapses) $\langle s_\alpha : \alpha < \lambda\rangle$. Build $\langle F_m^\alpha : m \geq l(p^*)\rangle$ for $\alpha < \lambda$. Let's let $l := l(p^*)$. At the beginning, set $\langle F_m^0 : m \geq l\rangle = \langle F_m^{p^*} : m \geq l\rangle$. At stage $\alpha+1$, consider the condition $(r^{p^*}, s_\alpha^\frown F_{l(s_\alpha)-1}^\alpha(\mathrm{top}(s_\alpha)), \bar{F}^{p^*} \restriction [l(s_\alpha), \omega))$. If there is an extension of r^{p^*} and $F_{l(s_\alpha)-1}^\alpha(\mathrm{top}(s_\alpha))$ so that the strengthened condition decides $\dot{\eta}$, then we only needed to extend r^{p^*} by some condition in $\mathbb{R}_{l(s_\alpha)-1}$. (All true $\mathbb{R}_0$-names are fixed at this point) Build a maximal antichain B in $\mathbb{R}_{l(s_\alpha)-1}$ and conditions f_r for $r \in B$ such that so that strengthening the above condition by r, f_r decides η if possible. We make $F_{l(s_\alpha)-1}^{\alpha+1}(\mathrm{top}(s_\alpha))$ by amalgamating the conditions f_r for $r \in B$ to get a name (if an extension as above doesn't exist, then use the original condition as the name).[29]

At limits γ, we want to know that $F_m^\gamma(y)$ exists for all m and $y \in \mathrm{dom}(F_m^{p^*})$ (strengthened when y is top of some s_α). The sequence $\langle F_m^\alpha(y) : \alpha < \gamma\rangle$ decreases when $\mathrm{top}(s_\alpha) = y$. How many such α's are there? At most λ_Y such α's; this follows from $\lambda_Y^{<\kappa_Y} = \lambda_Y$. (Recall $\lambda_Y = \mathrm{otp}(Y) = \kappa_Y^{+l}$ for some l.) Each $F_m^\alpha(y)$ is forced to be in $\mathrm{Coll}(\lambda_Y^+, < \kappa)$. So pick a name $F_m^\gamma(y)$ for a lower bound.

Let $\langle F_m^{p^{**}} : m \geq l\rangle$ be a lower bound for the whole construction. Let $p^{**} = \langle r^{p^*}, s(p^*), \bar{F}^{p^{**}}\rangle$. If $q \leq p^{**}$ decides $\dot{\eta}$, then let α be such that $s^-(q) = s_\alpha$. Then we get $\langle r^q, s(q), \bar{F}^{p^{**}} \restriction [l(q), \omega)\rangle$ decides $\dot{\eta}$ (as the upper part is universal). This implies that there were extensions deciding $\dot{\eta}$ at stage $\alpha + 1$ of the contruction. Any $r \in B$ with $r \| r^q$ must give the same decision. So we get $\langle r^q, s^-(q)^\frown F_{l(s_\alpha)=1}^{p^{**}}(\mathrm{top}(s_\alpha)), \bar{F}^{p^{**}}\rangle$ decides $\dot{\eta}$. $\square$

Claim: Let $\dot{\eta}$ be a V_0-name for either 0 or 1, and p^{**} as in the previous claim. There is a direct extension p^{**} which decides $\dot{\eta}$.

[29]That is, the name is such that if r is the unique condition in B and the generic, then r forces it to be f_r

Proof. Let $\langle t_\alpha : \alpha < \lambda \rangle$ enumerate all stems. For $x \succ t_\alpha$, define (names for measure one sets) $\dot\eta_{\alpha,x}$ to be an $\mathbb{R}_{l(t_\alpha)}$-name for some ordinal < 3 so that

- $\dot\eta_{\alpha,x} = 0$ if $\exists r \in g_{l(s_\alpha)}$ (here g is generic) so that $\langle r, t_\alpha^\frown \langle x, F^{p^{**}}(x)\rangle, \bar F^{p^{**}} \rangle$ forces $\dot\eta = 0$,
- $\dot\eta_{\alpha,x} = 1$ if $\exists r \in$ ... forces $\dot\eta = \dot 1$,
- $\dot\eta_{\alpha,x} = 2$ if no such r decides $\dot\eta$.

This gives a name for a partition of such X. For $i = 0, 1, 2$, $\dot A^i_\alpha = \{X : \eta_{\alpha,X} = i\}$. Let A^*_α be a name forced to be equal to one of the A^i_α (the measure one set). Then take the diagonal intersection ΔA^*_α.[30] □

Exercise: Show that bounded subsets of κ come from finite products of collapses in the stem.

16. February 13

General framework for collapsing cardinals: Given a poset $\mathbb{P}$ and $\lambda < \mu$ both regular, if there is a sequence $\langle A_\alpha : \alpha < \lambda \rangle$ of antichains of size μ with enumerations $\langle p^i_\alpha : i < \mu \rangle$ such that $\forall i < \mu$, $\{p : (\exists \alpha)\, p \le p^i_\alpha\}$ is dense, then $\mathbb{P}$ adds a surjection from λ onto μ. Define $\dot f(\alpha)$ to be the unique i so that $p^i_\alpha \in \dot G$ if it exists. (Note, it may not exists, i.e. f may be a partial surjection.) So $\dot f$ names a (partial) surjection from λ onto μ.

In a Prikry-type setting, get some A_n for $n < \omega$ where $|A_n| = \kappa$ (say, maybe have κ-many choices for nth Prikry point) with the property above.

Diagonal Prikry Forcing

Let $\langle \kappa_n : n < \omega \rangle$ be an increasing sequence of measurable cardinals and $\langle U_n : n < \omega \rangle$ a sequence of ultrafilters (not necessarily normal) such that U_n is a κ_n-complete ultrafilter on κ_n. Let $\kappa := \sup_n \kappa_n$.

We define a poset $\mathbb{P} = \mathbb{P}_{\vec U}$. Conditions are $\langle \alpha_0, \alpha_1, \dots, \alpha_{n-1}, A_n, A_{n+1}, \dots \rangle$ such that for all $i < n$, $\alpha_i \in (\kappa_{i-1}, \kappa_i)$ (and set $\kappa_{-1} = \omega$) and for $i \ge n$, $A_i \in U_i$. For $p \in \mathbb{P}$, we write $p = \langle \alpha^p_0, \alpha^p_1, \dots, \alpha^p_{n-1}, A^p_n, A^p_{n+1}, \dots \rangle$, where $l(p) = n$ (the *length* of p) and we set $p \le q$ if

- $l(p) \ge l(q)$
- if $i < l(q)$, then $\alpha^p_i = \alpha^q_i$
- if $i \in [l(q), l(p))$, then $\alpha^p_i \in A^q_i$
- and if $i \ge l(p)$, then $A^p_i \subseteq A^q_i$.

Claim: $\mathbb{P}$ satisfies the strong Prikry Lemma.

Proof. (Sketch) For all stems s, get $\vec A_s$ (measure one sets $A_{l(s)}, A_{l(s)+1}, \dots$) so that $(s, \bar A_s) \in D$ if possible. For $n < \omega$, get

$$\vec A^n = \bigcap_{l(s)=n} \vec A_s \text{ (intersecting pointwise)}$$

and

$$\vec A^* = \bigcap \vec A^n \text{ (intersecting “almost” pointwise).}$$

[30] We're being sloppy here, probably want to deal with stems of each length separately.

Then define inductively

$$Y_0 := \left\{s : (s, \vec{A}^*) \in D\right\} \text{ and } Y_{n+1} = \left\{s : (\exists A \in U_{l(s)})\ (\forall \alpha \in A)\ s^\frown\langle\alpha\rangle \in Y_n\right\}.$$

We have enough closure to intersect measure one sets witnessing membership (and non-membership) of each s in each Y_n to get $\vec{A}^{**}$. If $q \leq (\varnothing, \bar{A}^{**})$ with $q \in D$, all $l(q)$-step extensions of $(\varnothing, \bar{A}^{**})$ are in D. □

Corollary: $\mathbb{P}$ doesn't add any bounded subsets of $\kappa := \sup_n \kappa_n$.

Claim: $\mathbb{P}$ has the κ^+-c.c.

Proof. Note that there are κ-many stems, and two conditions with the same stem are compatible. □

Let $U, \bar{U}$ be ultrafilters on κ; we define the Rudin-Keisler ordering: $\bar{U} \leq_{\mathrm{RK}} U$ if there exists $f : \kappa \longrightarrow \kappa$ such that $A \in \bar{U}$ iff $f^{-1}(A) \in U$. In particular, $\bar{U} \leq_{\mathrm{RK}} U$ implies:

- For all $B \in \bar{U}$, $\{\alpha : f(\alpha) \in B\} \in U$ (just definition of inverse image).
- For all $A \in U$, $f''A \in \bar{U}$.

Thus, in some sense, U is stronger than $\bar{U}$.

Let $\langle\kappa_n : n < \omega\rangle$ be as before. Let $\langle U_n : n < \omega\rangle$ and $\langle\bar{U}_n : n < \omega\rangle$ be sequences of ultrafilters such that each $U_n, \bar{U}_n$ is on κ_n. Now if $U_n \geq_{\mathrm{RK}} \bar{U}_n$ for all $n < \omega$ as witnessed by $f_n : \kappa_n \longrightarrow \kappa_n$, and if $\langle\alpha_n : n < \omega\rangle$ is $\mathbb{P}_{\langle U_n : n<\omega\rangle}$-generic, then $\langle f_n(\alpha_n) : n < \omega\rangle$ is $\mathbb{P}_{\langle\bar{U}_n : n<\omega\rangle}$-generic. (Need a new characterization of genericity here: for any ω-sequence of measure one sets coming from the κ_n, the Pirkry sequence is in a tail of this sequence.)

17. February 18

Assume GCH.

Goal: Coherently add many ω-sequences to a singular cardinal using ideas from projections between diagonal Prikry forcings.

Recall we had $\bar{U}_n \leq_{\mathrm{RK}} U_n$ with $\mathbb{P}_{\langle U_n : n<\omega\rangle}$ projecting onto $\mathbb{P}_{\langle\bar{U}_n : n<\omega\rangle}$. Need a coherent collection of measures, but that's what an extender is!

Setup: $\langle\kappa_n : n < \omega\rangle$ increasing sequence of regular cardinals with $\sup_n \kappa_n = \kappa$. For the first construction each κ_n was measurable. This time we develop "extender-based forcing with long extenders." Fix $\lambda > \kappa^+$.[31] We assume for each $n < \omega$ there is $j_n : V \longrightarrow M_n$ with $\mathrm{crit}(j_n) = \kappa_n$ and ${}^{\kappa_n}M_n \subseteq M_n$, $V_{\lambda+1} \subseteq M_n$ and $j_n(\kappa_n) > \lambda$.[32] Define for $\alpha < \lambda$

$$E_{n\alpha} = \{X \subseteq \kappa_n : \alpha \in j_n(X)\}$$

a measure on κ_n. Notice that $E_{n\kappa_n}$ is the usual normal measure on κ_n, and all $E_{n\alpha}$ are non-principle and κ_n-complete.

Definition: We say $\alpha \leq_n \beta$ if there is $f : \kappa_n \longrightarrow \kappa_n$ so that $j_n(f)(\beta) = \alpha$.

We claim that $\alpha \leq_n \beta$ implies $E_{n\alpha} \leq_{\mathrm{RK}} E_{n\beta}$ as witnessed by f. Fix $X \subseteq \kappa_n$. Then $X \in E_{n\alpha}$ iff $\alpha \in j_n(X)$ iff $j_n(f)(\beta) \in j_n(X)$ iff $\beta \in j_n(f)^{-1}(j_n(X))$ (inverse

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[31] λ is our target number of ω-sequences to add.

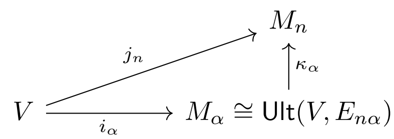
[32] Probably want each κ_n to be $\lambda + 1$-strong.

image) iff $\beta \in j_n(f^{-1}X)$ iff $f^{-1}X \in E_{n\beta}$.

Exercise: Determine a sense in which the direct limit of $\mathsf{Ult}(V, E_{n\alpha})$ for $\alpha < \lambda$ captures the properties of M_n.

Claim $\leq_n$ is κ_n-directed.[33]

Proof. Enumerate $\kappa_n^{<\kappa_n}$ in ordertype κ_n, say $\langle a_\alpha : \alpha < \kappa_n \rangle$, so that for all regular $\delta < \kappa_n$ and all $X \subseteq \delta$ of size $< \delta$, X is enumerated unboundedly often below δ (Using GCH). So $j_n(\langle a_\alpha : \alpha < \kappa_n \rangle) \restriction \lambda$ has the above property with "$\delta = \lambda$". Call this sequence $\langle a_\alpha : \alpha < \lambda \rangle$. (Note there is no confusion because of the critical point.) Fix $a \subseteq \lambda$ with $|a| < \kappa_n$ and $\alpha < \lambda$ so that $a = a_\alpha$. Notice that we can choose α as large as we please. We claim that $\forall \gamma \in a$, $\gamma \leq_n \alpha$. Consider the diagram



with $k_\alpha([f]_{E_{n\alpha}}) = j(f)(\alpha)$. So $a_\alpha = j_n(\langle a_\alpha : \alpha < \kappa_n \rangle)(\alpha) = k_\alpha(-)$. Since $|a_\alpha| < \kappa_n$, there is $b \in M_\alpha$ so that $k_\alpha(b) = k_\alpha'' b = a_\alpha$ (as $\mathrm{crit}(k_\alpha) \geq \kappa_n$). Choose γ^* so that $k_\alpha(\gamma^*) = \gamma$. Pick f so that $[f]_{E_{n\alpha}} = \gamma^*$. We check that $j_n(f)(\alpha) = \gamma$: $j_n(f)(\alpha) = k_\alpha([f]_{E_{n\alpha}}) = k_\alpha(\gamma^*) = \gamma$. □

Note: There were unboundedly many α's that we could have chosen. To we can witness directedness with an ordinal as large as we please.

Some notation: for $\alpha \leq_n \beta$, fix $\pi_{\beta\alpha} = \pi^n_{\beta\alpha}$ witnessing this (the "n" will be clear from context).

Claim A: If $\alpha < \beta$ and $\alpha, \beta \leq_n \gamma$, then $\{\nu < \kappa_n : \pi_{\gamma\alpha}(\nu) < \pi_{\gamma\beta}(\nu)\} \in E_{n\gamma}$.

Proof. We check that $j_n(\pi_{\gamma\alpha})(\gamma) < j_n(\pi_{\gamma\beta})(\gamma)$. But the first is α and the second is just β. □

Claim B: If $\alpha \leq_n \beta \leq_n \gamma$, then $\{\nu < \kappa_n : \pi_{\beta\alpha}(\pi_{\gamma\beta}(\nu)) = \pi_{\gamma\alpha}(\nu)\} \in E_{n\gamma}$.

Proof. We check that $j_n(\pi_{\beta\alpha}(\pi_{\gamma\beta}))(\gamma) = j(\pi_{\gamma\alpha})(\gamma)$. The RHS is just α. The LHS is $j_n(\pi_{\beta\alpha})(j_n(\pi_{\gamma\beta})(\gamma)) = j_n(\pi_{\beta\alpha})(\beta) = \alpha$. □

We'll try to add λ-many ω-sequences. Some of them will be controlled by $\langle E_{n\alpha} : n < \omega \rangle$, but we're not going to control all possible α's.

18. February 20

Recall: have a sequence $\langle \kappa_n : n < \omega \rangle$ increasing and $\sup_n \kappa_n = \kappa$. For each $n < \omega$, we have $j_n : V \longrightarrow M_n$ with $\mathrm{crit}(j_n) = \kappa_n$ and $j_n(\kappa_n) > \lambda$, ${}^{\kappa_n}M_n \subseteq M_n$ and $V_{\lambda+1} \subseteq M_n$ (where λ is target number of λ-sequences that we wish to add). We defined

$$E_{n\alpha} := \{X \subseteq \kappa_n : \alpha \in j_n(X)\}.$$

This is still a κ_n-complete, non-principal ultrafilter.

[33] More is true but this is enough.

We're trying to add many diagonal Prikry sequences corresponding to $\langle E_{n\alpha} : n < \omega\rangle$.

Definitions: (see Gitik)

(1) $\mathbb{Q}_{n1} := \{f : |f| \leq \kappa \wedge f \text{ is a partial function from } \lambda \text{ to } \kappa_n\}$, ordered by extension. This is a Cohen poset (adding λ-many subsets of κ^+, instead of the usual "2," we have "κ_n").

(2) $\mathbb{Q}_{n0}$: conditions are triples (a, A, f) such that
- $f \in \mathbb{Q}_{n1}$;
- $a \subseteq \lambda$, $|a| < \kappa_n$, and $a \cap \mathrm{dom}(f) = \varnothing$;
- a has a $\leq_n$-maximal element, $\mathrm{mc}(a)$;
- $A \in E_{n,\mathrm{mc}(a)}$;
- $\forall \alpha, \beta \in a$ if $\alpha \leq_n \beta$, then for all $\nu \in A$[34]

$$\pi_{\mathrm{mc}(a),\alpha}(\nu) = \pi_{\beta\alpha}\big(\pi_{\mathrm{mc}(a),\beta}(\nu)\big).$$

  (This is possible by Claim B.)
- For every $\alpha < \beta$ with $\alpha, \beta \in a$ and for all $\nu \in A$, we have

$$\pi_{\mathrm{mc}(a),\beta}(\nu) > \pi_{\mathrm{mc}(a),\alpha}(\nu)$$

  (This is possible by Claim A.)
- We declare $(a, A, f) \leq (b, B, g)$ if
  - $a \supseteq b$;
  - $f \leq g$;
  - $\pi''_{\mathrm{mc}(a),\mathrm{mc}(b)} A \subseteq B$.

(3) Our diagonal Prikry forcing will be denoted $\mathbb{P}$. Conditions are of the form

$$p = \langle f_0, f_1, \dots, f_{n-1}, \langle a_n, A_n, f_n\rangle, \langle a_{n+1}, A_{n+1}, f_{n+1}\rangle, \dots\rangle$$

such that
(a) $f_i \in \mathbb{Q}_{n1}$ for $i < n$;
(b) $(a_i, A_i, f_i) \in \mathbb{Q}_{n0}$ for $i \geq n$;
(c) For $i > j \geq n$, then $a_i \supseteq a_j$.

As usual, $n = l(p)$ and we will write $p = \langle f_0^p, f_1^p, \dots \langle a_n^p, A_n^p, f_n^p\rangle \dots\rangle$. We define $p \leq q$ if
- For $i < l(q)$, $f_i^p \leq f_i^q$;
- For $i \in [l(q), l(p))$, $f_i^p \leq f_i^q$, $f_i^p(\mathrm{mc}(a_i^q)) \in A_i^q$, and $\forall \gamma \in a_i^q$, $f_i^p(\gamma) = \pi_{\mathrm{mc}(a_i^q),\gamma}(f_i^p(\mathrm{mc}(a_i^q)))$.
- For $i \geq l(p)$, $(a_i^p, A_i^p, f_i^p) \leq (a_i^q, A_i^q, f_i^q)$

A helpful definition: if $p \in \mathbb{P}$ and $\nu \in A^p_{l(p)}$, then we say $p^\frown\nu$ is the weakest condition p^* of length $l(p)+1$ where

$$f^{p^*}_{l(p)} = f^p_{l(p)} \cup \Big\{(\gamma, \pi_{\mathrm{mc}(a^p_{l(p)}),\gamma}(\nu)) : \gamma \in a^p_{l(p)}\Big\}$$

(recall $a^p_{l(p)} \cap \mathrm{dom}(f^p_{l(p)}) = \varnothing$) and everything else is fixed. We similarly define $p^\frown\vec{\nu}$ for $\vec{\nu} = \langle \nu_0, \dots, \nu_k\rangle$.

As usual, $p \leq^* q$ iff $p \leq q$ and $l(p) = l(q)$.

[34]Think of ν as a reflection of the maximal coordinate.

Claim: $\mathbb{P}$ satisfies the strong Prikry Lemma.

Other things to think about for next time: generically, for each $n < \omega$, we'll have a $F_n : \lambda \longrightarrow \kappa_n$: define $t_\alpha(n) = F_n(\alpha)$. If $\alpha \in \mathrm{dom}(f_i^p)$ for all i, then $t_\alpha \in V$. On the other hand, if $\alpha \in a_i^p$ for some i (so in larger ones too), then $t_\alpha \notin V$.

19. February 23

We'll start by trying to get intuition about what's going on. In the background, we have $\langle \kappa_n : n < \omega \rangle$ with $\sup \kappa_n = \kappa$. We are planning to add λ-many cofinal ω-sequences. We'll add functions $F_n : \lambda \longrightarrow \kappa_n$ and define $t_\alpha : \omega \longrightarrow \kappa$ by $t_\alpha(n) = F_n(\alpha)$ (think of this as a $\lambda \times \omega$-matrix). Note $t_\alpha(n) < \kappa_n$, so each $t_\alpha \in \prod_n \kappa_n$. Some of these sequences will be new, and some will be old. The point of the forcing: to carefully control which are new and which are old (i.e., in V).

The definition of $\mathbb{P}$ is to make some t_α's new and some old. The new t_α are going to be Prikry sequences in $\langle E_{n\alpha} : n < \omega \rangle$ (diagonal Prikry sequences, using more-and-more complete measures). When is t_α new? Consider $(a, A, f) \in \mathbb{Q}_{n0}$. t_α is new when $\alpha \in a$ for some generic condition. Recall that if $i < j$, then $a_i \subseteq a_j$ by definition of being a condition. So if you're controlling α at some coordinate, you're controlling it at all future conditions, hence adding a Prikry sequence.

Consider a condition $\langle f_0, f_1, \ldots, f_{n-1}, \langle a_n, A_n, f_n \rangle, \ldots \rangle$; the question remains: how do we take a (a_n, A_n, f_n) and output some function f_n'?

Easy Fact: $\forall p, q \in \mathbb{P}$, if $p \leq q$, then $\exists! \vec{\nu}$ such that $p \leq^* q^\frown \vec{\nu}$.

Claim: $\mathbb{P}$ satisfies the strong Prikry Lemma.

Claim 1: Let $D \subseteq \mathbb{P}$ be dense open, and $p \in \mathbb{P}$. Then there is a $p_0 \leq^* p$ so that for all $q \leq p_0$ with $q \in D$, $q \restriction l(q)^\frown p_0 \restriction [l(q), \omega) \in D$.

Proof. (Sketch) By induction on $n < \omega$. Construct a $\leq^*$-decreasing sequence $\langle q_n : n < \omega \rangle$ with $q_0 = p$, and where q_{n+1} diagonalizes over possible $(n+1)$-step extensions of q_n. Suppose we're given q_n for some $n < \omega$. Enumerate $(n+1)$-step extensions $\langle \vec{\nu}_\alpha : \alpha < \kappa_{l(p)+n} \rangle$. We construct $\langle q_{n,\alpha} : \alpha < \kappa_{l(p)+n} \rangle$ which are $\leq^*$-decreasing. $q_{0,n} := q_n$. At stage $\alpha + 1$, ask: is there a direct extension q of $q_{n,\alpha}^\frown \vec{\nu}_\alpha$ such that $q \in D$? We get $q_{n,\alpha+1}$ by strengthening $q_{n,\alpha}$ as follows:

- It has f_i^q for $i < l(p) = l(q_{n,\alpha})$;
- $f_i^q \restriction (\lambda \backslash a_i^{q_{n,\alpha}})$ for $i \in [l(q), l(q) + n + 1)$;
- (a_i^q, A_i^q, f_i^q) for $i \geq l(q) + n + 1$.

This completes the successor step. At limits, we have enough closure to take lower bounds. The f-parts are κ^+-closed, and $\mathbb{Q}_{i0}$ is κ_i-closed.

Now let q_{n+1} be a $\leq^*$-lower bound for $\langle q_{n\alpha} : \alpha < \kappa_{l(p)+n} \rangle$. Finally, let p_0 be a $\leq^*$-lower bound for $\langle q_n : n < \omega \rangle$. □

20. February 25

Recall we were on our way to proving the strong Prikry Lemma last time.

Claim 2:[35] Let D be dense open, and let p_0 be as in the Claim 1 from last time. There are $\langle p_m : m < \omega \rangle$ and sets $\langle Y_m : m < \omega \rangle$ so that

$$Y_0 = \{q \restriction l(q) : q \restriction l(q)^\frown p_0 \restriction [l(q), \omega) \in D\},$$

and Y_{m+1} is the set of initial segments of conditions such that there is a measure one set of trivial, one-step extensions that get into Y_m. Note the witnessing measure one set above depends implicitly on p_m. The above are such that $\forall m \geq 1$, if $q \leq p_m$ and $q \restriction l(q) \in Y_{m-1}$, then $q \restriction (l(q) - 1)^\frown(\text{trivial one-step extension}) \in Y_{m-1}$.

Proof. (Sketch) By induction on $m < \omega$. By definition p_0 is the p_0 from Claim 1. At stage $m + 1$, to construct p_{m+1}, we construct a sequence $\langle p^n_m : n < \omega \rangle$ by induction. p^{n+1}_m diagonalizes over all n-step extensions of p^n_m. Note: everything here is $\leq^*$-decreasing.

Enumerate these as $\langle \vec{\nu}_\alpha : \alpha < \kappa_{l(p)+n-1} \rangle$. Induct over α. $\langle p^{n,\alpha}_m : \alpha < \kappa_{l(p)+n-1} \rangle$. At stage $\alpha + 1$, consider $(p^{n,\alpha}_m {}^\frown \vec{\nu}_\alpha) \restriction (l(p_0) + n - 1)$ (restrict to length). Ask: is there an extension of of the above condition which is in Y_m? If yes, strengthen the f-parts of $p^{n,\alpha}_m$ by this extension to get $p^{n,\alpha+1}_m$. Note: closure is no problem since each of the f-parts are κ^+-closed. This is enough to finish the claim. □

Let p_ω be a $\leq^*$-lower bound for $\langle p_m : m < \omega \rangle$. Intersect the measure one sets witnessing that $p_\omega {}^\frown \vec{\nu} \restriction$(its length)$\in Y_m$ (or not) for each m. Let $p_{\omega+1}$ be p_ω restricted to measure one sets. If $q \leq p_{\omega+1}$ and $q \in D$, then every $n := (l(q) - l(p_{\omega+1}))$-step extension of $q \restriction l(p_{\omega+1})^\frown p_{\omega+1} \restriction [l(p_{\omega+1}), \omega)$ is in D. Argue along the way that $q \restriction l(q) \in Y_n$. This proves the strong Prikiry Lemma.

Now we show that κ and κ^+ are preserved.

Corollary 20.1. *No bounded subsets of κ are added. Hence κ is preserved.*

Corollary 20.2. *κ^+ is preserved.*

Proof. Assume for a contradiction that $\dot{f} : \mu \longrightarrow \kappa^+$ is a name for a function which is cofinal, with $\mu < \kappa$. Choose p so that $\kappa_{l(p)} > \mu$. Apply the strong Prikry Lemma for each $\alpha < \mu$ to $D_\alpha := \left\{ q \in \mathbb{P} : q \text{ decides } \dot{f}(\alpha) \right\}$. Then get a p^* which works for each such α, and an n_α such that every n_α-step extension of p^* decides $\dot{f}(\alpha)$. Note that there are $\kappa_{l(p)+n_\alpha}$ sequences $\vec{\nu}$ of length n_α that can be added. This implies that the range of $\dot{f}$ is bounded. □

Exercise: Characterize genericity for $\mathbb{P}$.

Claim: $\mathbb{P}$ has the κ^{++}-c.c.[36]

Proof. Fix $\langle p_\alpha : \alpha < \kappa^{++} \rangle$. Assume they all have the same length. Form a Δ-system out of the domains

$$\left\{ \bigcup_{i<\omega} \text{dom}(f^{p_\alpha}_i) \cup \bigcup_{j \geq l(p_\alpha)} a^{p_\alpha}_j : \alpha < \kappa^{++} \right\}.$$

[35]This is a derivative process of sorts, similar to the one used for Magidor's forcing.
[36]$\mathbb{P}$ is in fact Knaster.

Fix the values of the $f_i^{p_\alpha} \restriction$(root of the Δ-system). Note we are using GCH to do both of these. Now take α, α'. For each i, $f_i^{p_\alpha} \cup f_i^{p_{\alpha'}}$ is a condition, and for each $j \geq l(p_\alpha) = l(p_{\alpha'})$, we can take γ_j which is $\geq_j$-greater than $a_j^{p_\alpha}, a_j^{p_{\alpha'}}$ and which is not in the domain of $f_j^{p_{\alpha'}}$. Let's define $q \leq p_\alpha, p_{\alpha'}$ by $f_i^q = f_i^{p_\alpha} \cup f_i^{p_{\alpha'}}$, $a_j^q = \{\gamma_j\} \cup a_j^{p_\alpha} \cup a_j^{p_{\alpha'}}$, and A_j^q is in $E_{j\gamma_j}$ such that $\pi_{\gamma_j, \mathrm{mc}(a_j^{p_\alpha})}{}''A_j^q \subseteq A_j^{p_\alpha}$ and similarly for α'. □

21. February 27

Finishing up the extender-based forcing. Recall conditions looked like

$$\langle f_0, f_1, \ldots, f_{n-1}, \langle a_n, A_n, f_n\rangle, \langle a_{n+1}, A_{n+1}, f_{n+1}\rangle, \ldots\rangle.$$

Going to approximate a sequence $F_n : \lambda \longrightarrow \kappa_n$ for $n < \omega$. Each f_i is a κ-sized approximation to this F_i.

Recall, $t_\alpha : \omega \longrightarrow \kappa$, and $t_\alpha(n) = F_n(\alpha)$ so $t_\alpha \in \prod_n \kappa_n$[37]. For this forcing to be interesting, we take $\lambda \geq \kappa^{++}$.

Claim: $\mathbb{P}$ adds λ-many cofinal ω-sequences in κ.

Proof. We prove that $\forall \alpha < \lambda$, $\exists \beta > \alpha$ such that $\forall \gamma < \beta$, $t_\gamma <^* t_\beta$ (i.e,. unboundedly many are Prikry generic). Fix a condition $p \in \mathbb{P}$ and $\alpha < \lambda$. Choose $\beta > \sup \mathrm{dom}(f_n^p), \sup a_n^p$. Form a condition $p^* \leq^* p$ with $\beta \in a_i^{p^*}$ for $i \geq l(p^*)$. We claim that this β works. Fix $\gamma < \beta$. There are two possibilities for this γ : either for some $q \leq p^*$, $\gamma \in \mathrm{dom}(f_i^q)$ for all large enough i, or for some $q \leq p^*$, $\gamma \in a_i^q$ for all large enough i.

Case 1: $\gamma \in \mathrm{dom}(f_i^q)$ for all large enough i. This implies that $t_\gamma \in V$, since it's values are determined on a tail end by the f's. For large enough i, we can choose $A_i \in E_{n,\mathrm{mc}(a_i^q)}$ so that $\forall \nu \in A_i$, $\pi_{\mathrm{mc}(a_i^q),\beta}(\nu) > t_\gamma(i)$. Then $q \Vdash$ "for all large $i t_{\mathrm{mc}(a_i^q)}(i) \in A_i$". So it forces that $t_\beta(i) > t_\gamma(i)$ for all large enough i.

Case 2: $\gamma \in a_i^q$ for all large enough i. Similarly, for all large enough i, we can choose $A_i \in E_{n,\mathrm{mc}(a_i^q)}$ so that $\forall \nu \in A_i$, $\pi_{\mathrm{mc}(a_i^q),\beta}(\nu) > \pi_{\mathrm{mc}(a_i^q),\gamma}(\nu)$ (this is a clause in the definition of $\mathbb{P}$). Use the same argument as in Case 1 to finish. □

Exercise: Modify $\mathbb{P}$ so that $|f_n| \leq \kappa_n$. Prove that this collapses κ to be countable.

22. March 2

Question: Do extender-based forcings (like the one we talked about) force strong weak-square principles?[38]

Question: (Woodin, 1980's) Is it consistent that SCH fails at some κ and κ^+ has the tree property? In particular, does this hold for $\aleph_\omega$?

Theorem 22.1. (Gitik-Sharon) *If κ is supercompact, then there is a forcing extension in which κ is singular, strong limit, $2^\kappa = \kappa^{++}$, and $\kappa^+ \notin I[\kappa^+]$. In particular, there are no special κ^+-trees in the extension.*

Theorem 22.2. (Cummings-Foreman) *In the Gitik-Sharon model, there is a bad scale of length κ^+, which implies that $\kappa^+ \notin I[\kappa^+]$.*

[37]Although these do not form a scale, generic ones "look much like" a scale

[38]S. thinks answer is "not always" though maybe the one we talked about does.

Exercise: Learn the following definitions and prove:

$$\square^*_\mu \implies \mu^+ \in I[\mu^+] \implies \text{“All scales of length } \mu^+ \text{ are good”}.$$

Theorem 22.3. (Neeman) *From ω-supercompact cardinals, we can improve the Gitik-Sharon result to get the tree property at κ^+.*

So this answer's Woodin's question for some random singular cardinal. Collapsing κ to $\aleph_{\omega^2}$ was done in the original Gitik-Sharon model.

Exercise: Prove there is a bad scale at $\aleph_{\omega^2}$ in the collapsed Gitik-Sharon model.[39]

Getting $\kappa = \aleph_{\omega^2}$ with the tree property at κ^+ is a result of Sinapova.

Gitik and Sharon's Poset

Let κ be supercompact and U a normal measure on $\mathcal{P}_\kappa(\kappa^{+\omega+1})$. Define U_n to be the projection of U to $\mathcal{P}_\kappa(\kappa^{+n})$. Note: the completeness of each of the U_n is κ. Thus we'll need to use normality to take lots of diagonal intersections.

Conditions in $\mathbb{P}$ look like $\langle \chi_0, \chi_1, \ldots, \chi_{n-1}, A_n, A_{n+1}, \ldots \rangle$. Let

$$Z_i = \left\{ X \in \mathcal{P}_\kappa(\kappa^{+i}) : X \cap \kappa \in \kappa \text{ and } \operatorname{otp}(X) = (X \cap \kappa)^{+i} \right\} \in U_i.$$

Conditions must satisfy:

(1) For $i < n$, $Y_i \in Z_i$;
(2) $\vec{X}$ is $\prec$-increasing where $X \prec Y$ if $|X| < \kappa_Y$ and $X \subseteq Y$;
(3) $A_i \in U_i \cap \mathcal{P}(Z_i)$ for $i \geq n$.

Usual conventions: $l(p) = n$, and we write $p = \langle X^p_0, X^p_1, \ldots, X^p_{n-1}, A_n, A_{n+1}, \ldots \rangle$. $p \leq q$ if

- $l(p) \geq l(q)$;
- $\forall i < l(q)$, $X^p_i = X^q_i$;
- $\forall i \in [l(q), l(p))$, $X^p_i \in A^q_i$;
- $\forall i \geq l(p)$, $A^p_i \subseteq A^q_i$.

Exercise: $\mathbb{P}$ satisfies the (strong) Prikry Lemma. Also find a characterization of genericity.

Therefore, no bounded subsets of κ are added.

Claim: $\kappa^{+\omega}$ is collapsed to be size κ.

Proof. $\bigcup X_n = \kappa^{+\omega}$ by genericity (so $\kappa^{+\omega}$ is a countable union of sets of size $\leq \kappa$). □

Note: if we had started with $2^\kappa = \kappa^{+\omega+2}$, then we get $\neg\mathsf{SCH}$ in the extension since $2^\kappa = \kappa^{++}$ in the extension.

Claim: $\mathbb{P}$ has the $\kappa^{+\omega+1}$-c.c.

Proof. There are just $\kappa^{+\omega}$-many stems. □

[39] This exercise will make more sense later.

Claim: There is a bad scale of length κ^+ in $\prod_n \kappa_{X_n}^{+n+1}$. Hence, $\kappa^+ \notin I[\kappa^+]$ and there are no special Aronszajn trees.

Recall some definitions: if $f, g \in \prod_n \mu_n$ where $\langle \mu_n : n < \omega \rangle$ is increasing sequence of regular cardinals with $\mu := \sup_n \mu_n$, we write $f <^* g$ to mean that for all large enough n, $f(n) < g(n)$. A sequence $\langle f_\alpha : \alpha < \mu^+ \rangle$ is a *scale* if it is increasing and cofinal in $(\prod_n \mu_n, <^*)$.

Theorem 22.4. (Shelah) *Scales exist.*

A point $\gamma < \mu^+$ is *good* for the scale $\vec{f}$ if there are $n < \omega$ and $A \subseteq \gamma$ unbounded s.t. $(\forall n \geq N)\ \langle f_\alpha(n) : \alpha \in A \rangle$ is strictly increasing.

Exercise: γ is good iff there is a sequence $\langle H_i : i < \mathrm{cf}(\gamma) \rangle$ which are pointwise increasing so that

- $(\forall \alpha < \gamma)\ (\exists i)\ \big[f_\alpha <^* H_i\big]$;
- $(\forall i < \mathrm{cf}(\gamma))\ (\exists \alpha < \gamma)\ \big[H_i <^* f_\alpha\big]$.

"$\vec{H}$ is cofinally interweaved with $\vec{f} \restriction \gamma$."

A *good scale* is a scale with club-many good points. (Note: every point of cofinality ω is good.) A scale is *bad* if it is not good, i.e., there is a stationary set of non-good points.

Theorem 22.5. (Shelah) *If κ is $\kappa^{+\omega+1}$-supercompact, then any scale in $\prod_n \kappa^{+n}$ of length $\kappa^{+\omega+1}$ is bad.*

Proof. Fix a scale $\vec{f}$ of length $\kappa^{+\omega+1}$ in $\prod_n \kappa^{+n}$ and $j : V \longrightarrow M$ witnessing that κ is $\kappa^{+\omega+1}$-supercompact. In M, let $\gamma = \sup j''\kappa^{+\omega+1} < j(\kappa^{+\omega+1})$. We show that γ is not good for $j(\vec{f})$ in M. Standard reflection arguments then give a stationary set of bad points below $\kappa^{+\omega+1}$. Let H be $n \mapsto \sup j''\kappa^{+n} \leq j(\kappa^{+n})$. Note that $j(\vec{f}) \restriction \gamma$ is cofinal in $\prod_{n<\omega} H(n)$. H is what is called an exact upper bound (eub) of non-uniform cofinality. This then precludes γ from being good. So γ is bad, completing the proof. $\square$

23. March 4

Today we'll work towards proving the following claim:

Claim: There is a bad scale in the extension.

Fix a scale $\vec{f}$ in $\prod_n \kappa^{+n+1}$ of length $\kappa^{+\omega+1}$. Reflect it to a scale in $\prod \kappa_{X_n}^{+n+1}$. Fix F_n^γ for $\gamma < \kappa^{+n+1}$ so that $[F_n^\gamma]_{U_n} = \gamma$. In the extension by some $\mathbb{P}$-generic sequence $\langle X_n : n < \omega \rangle$, we define $g_\alpha(n) = F_n^{f_\alpha(n)}(X_n)$.

We first claim $\vec{g}$ is a scale. We show $\vec{g}$ is (i) $<^*$-increasing and (ii) cofinal.

For (i), fix $\alpha < \beta < \kappa^{+\omega+1} = \kappa^+ =: \mu$. Then for all large enough n, $f_\alpha(n) < f_\beta(n)$ so for all large enough n, $[F_n^{f_\alpha(n)}]_{U_n} < [F_n^{f_\beta(n)}]_{U_n}$. So for all large enough n, $\exists A_n \in U_n, \forall x \in A_n, F_n^{f_\alpha(n)}(x) < F_n^{f_\beta(n)}(x)$ and $X_n \in A_n$, (for large enough n), hence we're done.

Next we show (ii), that $\vec{g}$ is cofinal. But first we need a lemma:

Lemma: (Bounding) $\forall g \in \prod \kappa_{X_n}^{+n+1}$, there is a sequence $\langle H_n : n < \omega \rangle$ (where $H_n : \mathcal{P}_\kappa(\kappa^{+n}) \longrightarrow \mathrm{On}$) from V so that for all large n, $g(n) < H_n(X_n)$, and for all x, $H_n(x) < \kappa_X^{+n+1}$ (equivalently, $[H_n] < \kappa^{+n+1}$).

Proof. Given a stem $\vec{X}$ of length $n+1$, then there is a sequence of measure one sets $\bar{A}_{\vec{X}}$ so that $(\vec{X}, \bar{A}_{\vec{X}})$ decides $\dot{g}(n)$. This is because (roughly)

$$\vec{X} \Vdash \dot{g}(n) < \kappa_{X_n}^{+n+1}$$

where $\vec{X} = \langle X_0, X_1, \dots, X_n \rangle$. Lets call this value $\gamma_{\vec{X}}$. Now we define

$$H_n(X) = \sup \left\{ \gamma_{\vec{X}} : \vec{X} \text{ ends with } X \right\} < \kappa_X^{+n+1}.$$

Now we capture the measure one sets. Take some diagonal intersections to get $\bar{A}^*$. Then for all large n, $X_n \in A_n^* \implies$ for all large n, $\dot{g}(n) < H_n(X_n)$. □

Fix a name $\dot{g}$ for an element of $\prod \kappa_{X_n}^{+n+1}$. We then get $\langle H_n : n < \omega \rangle$ as in the lemma. Now choose α s.t. $(n \mapsto [H_n]_{U_n}) <^* f_\alpha$. Then it is easy to check that this works, i.e., that $\dot{g} <^* (n \mapsto H_n(X_n)) <^* g_\alpha$. Hence $\vec{g}$ is cofinal, and thus a scale.

We now want to show that $\vec{g}$ is bad. Let $S = \left\{ \gamma < \mu : \gamma \text{ is bad for } \vec{f} \right\}$. Since κ is supercompact, S is stationary. By chain conditions on our poset[40], S is stationary in $V[\vec{X}]$. Therefore, it is enough to show that if γ is good for $\vec{g}$ in $V[\vec{X}]$, then γ is good for $\vec{f}$ in V.

We need another lemma:

Lemma: In $V[\vec{X}]$, $\forall \gamma\ \omega < \mathrm{cf}(\gamma) < \kappa$, $\forall A \subseteq \gamma$ unbounded, there is $B \subseteq A$ unbounded with $B \in V$.

Proof. In the extension, write

$$A = \bigcup_{n<\omega} \left\{ \alpha : (\exists p \in G_{\vec{X}}) [\text{length n } \wedge p \vdash \alpha \in \dot{A} \right\}.$$

One of these sets is unbounded in the extension, since $\kappa > \mathrm{cf}(\gamma) > \omega$. Fix such an $n < \omega$. We can then work in V to make a condition of length n forcing an unbounded set (from V) into A. (Note $\omega < \mathrm{cf}^V(\gamma) < \kappa$.) □

We can now show if γ is good for $\vec{g}$ in $V[\vec{X}]$, then γ is good for $\vec{f}$ in V. Let γ be good for $\vec{g}$ witnessed by A, N. By the lemma we get $B \subseteq A$ unbounded with $B \in V$. Take a condition p forcing this. We can assume $N \leq l(p)$. Then $\forall \alpha \in B \cap \beta$ and all $n \geq l(p)$,

$$\left\{ X : F_n^{f_\alpha(n)}(X) > F_n^{f_\beta(n)}(X) \right\} \supseteq A_n^p.$$

This implies that $[F_n^{f_\alpha(n)}]_{U_n} < [F_n^{f_\beta(n)}]_{U_n}$ as witnessed by the measure one set above. So $f_\alpha(n) < f_\beta(n)$ for all $\alpha < \beta$ from B and $n \geq l(p)$. But this is the definition of γ being good for $\vec{f}$.

[40] $\mathbb{P}$ is μ-c.c.

Hence we have shown that there is a bad scale in the extension as claimed.

Remark: Suppose we had $H_\alpha : \kappa \longrightarrow \kappa$ so that $j_0(H_\alpha)(\kappa) = \alpha$ for $\alpha < \kappa^{+\omega+1}$. (Recall j_0 is the ultrapower embedding via the normal measure U_0, i.e., the projection onto $\mathcal{P}_\kappa(\kappa)$.) Given a scale $\vec{f}$ as before, we can now define $g_\alpha(n) = H_{f_\alpha(n)}(\kappa_{X_n})$. This works as before, giving a bad scale.

24. March 6

"Guiding principle"[41]: If we have a Prikry sequence $\langle \kappa_n : n < \omega \rangle$ and the forcing collapses $\kappa_n^{+\alpha}$ for some α and all large n, then the forcing poset collapses $\kappa^{+\alpha}$.

How do we add collapses to the Gitik-Sharon poset? The natural thing to do using the guiding principle is to preserve $\kappa_n^{\omega+1}$ for all $n < \omega$.

Question: Why might the guiding principle be true? Have $F : \mathcal{P}_\kappa(\kappa^{+k}) \longrightarrow$ Collapses which constrains collapses in the stem. The Prikry forcing typically incorporates

$$(\left\{[F]_U : F : \mathcal{P}_\kappa(\kappa^{+k}) \longrightarrow \text{Collapses}\right\}, \leq_U).$$

If we are working in a closed enough ultrapower, if

$$\left\{X : F(X) \text{ is in a poset that collapses } \kappa_X^{+\alpha}\right\} \in U,$$

then the forcing poset of classes $[F]$ collapses $\kappa^{+\alpha}$.

A vague description of Gitik-Sharon forcing with collapses: conditions look like

$$\langle X_0, f_0, X_1, f_1, \ldots, X_{n-1}, f_{n-1}, F_n, F_{n+1}, \ldots \rangle$$

where $\langle \vec{X}, \langle \mathrm{dom}(F_i) : i \geq n \rangle \rangle$ is a Gitik-Sharon condition. Furthermore, each $f_i \in \mathrm{Coll}(\kappa_{X_i}^{+\omega+2}, < \kappa_{X_{i+1}})$ for $i < n-1$ and $f_{n-1} \in \mathrm{Coll}(\kappa_{X_{n-1}}^{+\omega+2}, < \kappa)$. Moreover, $F_i(X) \in \mathrm{Coll}(\kappa_X^{+\omega+2}, < \kappa)$.

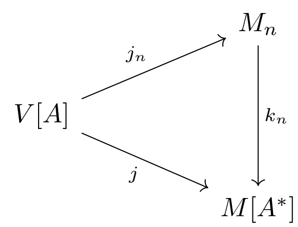
Exercise: Try to show that this poset preserves cardinals.

We can force this poset to have chain condition by taking $[F_i]$ in some generic filter for the ith poset of classes. As with the "improvement" of Magidor's poset, we need a lemma to construct these generics.

Claim: Starting in V, with GCH and κ supercompact, if we iterate $\mathrm{Add}(\alpha, \alpha^{+\omega+2})$ for $\alpha \leq \kappa$ (with Easton support), then we can lift an embedding $j : V \longrightarrow M$ witnessing $\kappa^{+\omega+1}$-supercompactness then for a generic A for the iteration, we can put, in $V[A]$, $j : V[A] \longrightarrow M[A^*]$ so that $(\forall \alpha < j(\kappa))$ $(\exists f : \kappa \longrightarrow \kappa)$ with $j(f)(\kappa) =$

[41]For adding collapses.

α. In particular this implies (for the factor maps k_n below) $\text{crit}(k_n) \geq j(\kappa)$.



Over $M[A^*]$ we can build a generic H for $\text{Coll}^{M[A^*]}(\kappa^{+\omega+2}, < j(\kappa))$ (because the ultrapower is sufficiently closed, so is that forcing, i.e. that this poset is $\kappa^{+\omega+2}$-closed in $V[A^*]$). The high critical point of k_n means that H "pulls back" to H_n, a generic for the version of the above collapse in M_n. Taking $[F_i] \in H_i$ gives the Gitik-Sharon with collapses $\kappa^{+\omega+1}$-c.c. (As now conditions with the same stem are compatible.)

(Note, for each of the following, we are working in $V[A]$ after we've done the iteration.)

Try 1: Gitik-Sharon for $\aleph_\omega$. Use $\text{Coll}(\kappa_{X_n}^{+n+2}, < \kappa_{X_{n+1}})$. These collapses have too much information, and will collapse $\kappa^{+\omega+1}$.

Try 2: Use $\text{Coll}^V(\kappa_{X_n}^{+n+2}, < \kappa_{X_{n+1}})$. Has the Prikry property. What are the obstacles? These forcings are no longer closed, but are still distributive.[42] This forcing is still bad, because of the guiding principle at $\kappa^{+\omega+2}$ implies it will collapse $\kappa^{+\omega+2}$.

Exercise: Figure out how to do the Prikry property argument using distributivity, rather than closure.

Try 3: Use $\text{Coll}^V(\kappa_{X_n}^{+n+1}, \kappa^{+\omega+1}) \times \text{Coll}^V(\kappa_{X_n}^{+\omega+2}, < \kappa_{X_{n+1}})$.

Theorem 24.1. (Spencer, Dima) *Try 3 works to give* $\neg\mathsf{SCH}$ *at* $\aleph_\omega$.

25. March 9

Start with V where κ is supercompact and GCH holds. Fix $j : V \longrightarrow M$ witnessing $\kappa^{+\omega+1}$-supercompactness. Iterate $\text{Add}(\alpha, \alpha^{+\omega+2})$ for $\alpha \leq \kappa$. Let A be generic for the iteration. In $V[A]$, there is a generic A^* over M for j(iteration) such that we get $j : V[A] \longrightarrow M[A^*]$ which witnesses $\kappa^{+\omega+1}$-supercompactness of κ in $V[A]$ and $\forall \alpha < j(\kappa)\ \exists f : \kappa \longrightarrow \kappa$ s.t. $j(f)(\kappa) = \alpha$.

Remark: (1) We did a similar argument when we revised Magidor's poset to be κ^+-c.c. (2) In this setup, we can build generics for $\text{Coll}^{M[A^*]}(\kappa^{+\omega+2}, < j(\kappa))$. This is what you need to get Gitik-Sharon down to $\aleph_{\omega^2}$ with a $\kappa^{+\omega+1}$-c.c. poset. This $\kappa^{+\omega+1}$-c.c. is key in preservation of the stationary set of bad points.

[42] The proof of this involves breaking up the iteration in just the right way in order to apply Easton's Lemma.

To get Gitik-Sharon down to $\aleph_\omega$, we use a product of collapses $\mathrm{Coll}^V(\kappa_{X_n}^{+n+2}, \kappa_{X_n}^{+\omega+1}) \times \mathrm{Coll}^V(\kappa_{X_n}^{+\omega+2}, < \kappa_{X_{n+1}})$ between successive Prikry points. (It (hopefully) is clear what the definition of the forcing should be.) Define a poset $\mathbb{P}$ using these collapses and measures derived from j. (Not using "guiding generics" like in Remark 2.)

In $\mathbb{P}$ we have functions $F_n : \mathcal{P}_\kappa(\kappa^{+n}) \longrightarrow$ Collapses, where

$$\forall X, F(X) \in \mathrm{Coll}^V(\kappa_X^{+n+2}, \kappa_X^{+\omega+1}) \times \mathrm{Coll}^V(\kappa_X^{+\omega+2}, < \kappa).$$

The classes $[F_n]_{U_n}$ (where U_n is measure on $\mathcal{P}_\kappa^{\kappa^{+n}}$) are in

$$\mathbb{C}_n^0 \times \mathbb{C}_n^1 := \mathrm{Coll}^{W_n}(\kappa^{+n+2}, \kappa^{+\omega+1}) \times \mathrm{Coll}^{W_n}(\kappa^{+\omega+2}, < j(\kappa))$$

(where this poset is in $M_n \cong \mathsf{Ult}(V[A], U_n)$ and W_n is the V-like inner model of M_n.)[43] The fact about functions from κ to κ and j (above) imply that $k_n : M_n \longrightarrow M[A^*]$ has a critical point $\geq j(\kappa)$. Now $k_n(\mathbb{C}_n^0 \times \mathbb{C}_n^1) = \mathrm{Coll}^M(\kappa^{+n+2}, \kappa^{+\omega+1}) \times \mathrm{Coll}^M(\kappa^{+\omega+2}, < j(\kappa))$ (in $M[A^*]$).

Note: $\mathrm{crit}(k_n) \geq j(\kappa)$ implies that $k_n(\mathbb{C}_n^0) = k_n''(\mathbb{C}_n^0) = \mathbb{C}_n^0$.

Define $\mathbb{D} = \prod_{n<\omega} \mathbb{C}_n^0 \times \mathbb{C}_n^1/\mathrm{fin}$.

Claim: $\mathbb{P}$ projects onto $\mathbb{D}$.

Proof. The map

$$\langle X_0, f_0, \ldots, X_{n-1}, f_{n-1}, F_n, F_{n+1}, \ldots\rangle \mapsto \langle \varnothing, \varnothing, \ldots, \varnothing, [F_n], [F_{n+1}], \ldots\rangle$$

is a projection. □

Claim 1: $\mathbb{D}$ preserves cardinals over $V[A]$.

Claim 2: In $V[\mathbb{D}]$, $\mathbb{P}/\mathbb{D}$ has $\kappa^{+\omega+1}$-c.c.

Claim 3: Over V, $\mathbb{D}_0 := \prod_{n<\omega} \mathbb{C}_n^0/\mathrm{fin}$ adds a $\square_{\kappa^{+\omega}}^*$-sequence.

Corollary: $\mathbb{D}_0$ destroys the stationarity of the set of bad points of any scale in $V[A]$.

26. March 11

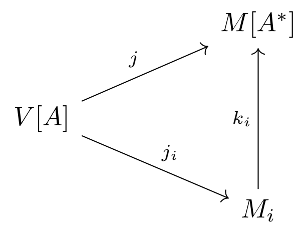
Recall we had two models $V, V[A]$ and $j : V[A] \longrightarrow M[A^*]$ witnessing $\kappa^{+\omega+1}$-supercompactness. In $V[A]$, $2^\kappa = \kappa^{+\omega+2}$. We defined a version of Gitik-Sharon where conditions look like

$$\langle X_0, f_0, \ldots, X_{n-1}, f_{n-1}, F_n, F_{n+1}, \ldots\rangle.$$

Each $f_i \in \mathrm{Coll}^V(\kappa_{X_i}^{i+2}, < \kappa_{X_i}^{+\omega+1}) \times \mathrm{Coll}^V(\kappa_{X_i}^{+\omega+2}, < \kappa_{X_{i+1}})$. Each F_i is defined appropriately. $[F_i] \in \mathbb{C}_i^0 \times \mathbb{C}_i^1 = \mathrm{Coll}^{W_i}(\kappa^{+i+2}, \kappa^{+\omega+1}) \times \mathrm{Coll}^{W_i}(\kappa^{+\omega+2}, < j_i(\kappa))$ in $M_i \cong \mathsf{Ult}(V[A], U_i)$.

[43] Have $\bar{U}$ in V on $\mathcal{P}_\kappa(\kappa^{+\omega+1})$ and extends to U in $V[A]$. Can lift both. On the other hand, can project $\bar{U}$ to $\bar{U}_n$ on $\mathcal{P}_\kappa(\kappa^{+n})$ and similarly for U projecting to U_n. Now $\mathsf{Ult}(V, \bar{U}_n)$ embeds into $\mathsf{Ult}(V[A], U_n)$; on the other hand, $M = \mathsf{Ult}(V, \bar{U}) \subseteq \mathsf{Ult}(V[A], U) = M[A^*]$.

Properties of j :



$\text{crit}(k_i) \geq j(\kappa)$, so $k_i(\mathbb{C}_i^0) = k_i''(\mathbb{C}_i^0) = \mathbb{C}_i^0$, i.e. $k_i \restriction \mathbb{C}_i^0$ is the identity. $k_i(\mathbb{C}_i^0) = \text{Coll}^M(\kappa^{+i+2}, \kappa^{+\omega+1}) = \text{Coll}^V(\kappa^{+i+2}, \kappa^{+\omega+1})$ (since we have a ultrapower by a highly closed measure).

We have the following claims from last time.

Claim -1: $\mathbb{P}$ satisfies the strong Prikry Lemma.

Claim 0: $\mathbb{P}$ induces a generic for $\mathbb{D} := \prod \mathbb{C}_n^0 \times \mathbb{C}_n^1/\text{fin}$.

Claim 1: $\mathbb{D}$ preserves cardinals.

Proof. Exercise. The fact the poset is defined mod finite will allow a strategic closure argument. □

Claim 2: If H is $\mathbb{D}$-generic, then in $V[A][H]$, $\mathbb{P}/H$ has $\kappa^{+\omega+1}$-c.c.

Proof. First, the strong Prikry Lemma implies that $\mathbb{P}$ preserves $\kappa^{+\omega+1} =: \mu$.[44] Fix a $\mathbb{P}$-generic G which projects to H. Also fix $\langle p_i : i < \mu \rangle$ in $\mathbb{P}/H$. We can think of G as $\langle \vec{X}, \vec{C} \rangle$ where $\vec{X}$ is a Prikry sequence and C_n is a generic for the nth collapse.[45] $\forall p \in \mathbb{P}/H$, $\{q : \forall \text{ large enough } n,\ [F_n^q] \leq [F_n^p]\}$ is dense in $\mathbb{P}/H$. This means that $\forall i < \mu$, $\exists k_i < \omega$, $\forall n \geq k_i$, $X_n \in A_n^{p_i}$ and $F_n^{p_i}(X_n) \in C_n$ (in $V[G]$).

Fix the choice of k_i to k on a set of size μ in $V[A][G]$. Extend each p_i to q_i to have length k by only picking new Prikry points. Further fix the stem of q_i on a large set. Choose q_i, q_j forced to be in the good set of size μ. We claim that q_i, q_j are compatible. Let $s = \text{stem}(q_i) = \text{stem}(q_j)$. There is N so that $\forall n \geq N$, $[F_n^{q_i}] \parallel [F_n^{q_j}]$ since q_i, q_j are in H. For $n \in [l(s), N)$, we use $(X_n, F_n^{q_i}(X_n) \cup F_n^{q_j}(X_n))$ for the nth-piece. Note $X_n \in A_n^{q_i} \cap A_n^{q_j}$, and the union on the right is in C_n (as q_i, p_i give same information here). The witnessing condition is $\langle s^\frown \langle (X_n, F_n^{q_i}(X_n) \cup F_n^{q_j}(X_n) : n \in [l(s), N) \rangle, \langle [F_n^{q_i}] \wedge [F_n^{q_j}] : n \geq N \rangle \rangle$. □

Lemma:[46] $\mathbb{P}$ is κ-c.c. implies $\forall \langle p_\alpha : \alpha < \kappa \rangle$ $(\exists \alpha < \kappa)$, $p_\alpha \Vdash$ "$\left\{ \beta < \kappa : p_\beta \in \dot{G} \right\}$ is unbounded".

Claim 3: $\mathbb{D}_0 := \prod_n \mathbb{C}_n^0/\text{fin} = \prod_n \text{Coll}^V(\kappa^{+i+2}, \kappa^{+\omega+1})/\text{fin}$ adds a $\square^*_{\kappa^{+\omega}}$-sequence.

[44] We've done this type of argument before, see for example the long extenders poset.

[45] We're using a characterization of genericity somewhere in here.

[46] Not relevant here, but its proof is similar to the previous claim and it is "Something all good people should know." -S.

27. March 13

Recall we had $V \subseteq V[A]$. In $V[A]$, we have $\mathbb{P} \in V[A]$, a version of Gitik-Sharon for $\aleph_\omega$. We saw that $\mathbb{P}$ induces a generic for a poset $\mathbb{D}_0 \in V$. We had $\mathbb{D}_0 = \prod_{n<\omega} \mathrm{Coll}^V(\kappa^{+n+2}, \kappa^{+\omega+1})/\mathrm{fin}$.

<u>Claim</u>: Forcing with $\mathbb{D}_0$ over V adds a $\square^*_{\kappa^{+\omega}}$-sequence.

Note that in V, $\kappa^{+\omega}$ is strong limit.

Some extra from last time: If $2^\omega > \omega_1$ and we force with $\mathrm{Add}(\omega_1, 1)$, then 2^ω is collapsed. If $2^\kappa > \kappa^{+\omega+1}$ and we force with $\prod_{n<\omega} \mathrm{Coll}(\kappa^{+n+2}, \kappa^{+\omega+1})/\mathrm{fin}$, then 2^κ is collapsed. Note this gives some idea why we need the collapses to be from some inner model.

Proof. (of claim) Let $\dot{C}_n$ be a $\mathrm{Coll}^V(\kappa^{+n+2}, \kappa^{+\omega+1})$ be a name for a club in $\kappa^{+\omega+1}$ of ordertype κ^{+n+2}. First notice that, if we set

$$\dot{X} := \left\{\gamma < \kappa^{+\omega+1} : \exists d \in G_{\mathbb{D}_0} \text{ for all large n, } d(n) \Vdash \text{``}\dot{C}_n \cap \gamma \text{ club in } \gamma\text{''}\right\}$$

then

$$\Vdash_{\mathbb{D}_0} \text{``}\dot{X} \text{ is } > \omega \text{ club''}.$$

For $> \omega$-closed, fix increasing $\langle \gamma_i : i < \mu\rangle$ with $\mu = \mathrm{cf}(\mu) > \omega$. Notice that $\mu = \mathrm{cf}(\mu) < \kappa^{+n}$ for some n. Fix witnessing $\langle d_i : i < \mu\rangle$ s.t. $d_i \Vdash \gamma_i \in \dot{X}$. Using the fact below, $G_{\mathbb{D}_0}$ is κ^+-directed closed, so can get $d \in G_{\mathbb{D}_0}$ which is a lower bound for all of the d_i. It is not hard to see that d witnesses $\sup_i \gamma_i := \gamma \in \dot{X}$. Unbounded is similar, using the fact that for all γ, the set

$$\left\{d : \text{ for some } \gamma' > \gamma,\ d \Vdash \dot{\gamma}' \in \dot{X}\right\}$$

is dense.

Now we want to get the $\square^*_{\kappa^{+\omega}}$-sequence. Work in the extension $V[G_{\mathbb{D}_0}]$. Let $d_\gamma \in G_{\mathbb{D}_0}$ witness that $\gamma \in \dot{X}$ if possible. Also let C^γ_n be club in γ so that $d_\gamma(n) \Vdash \dot{C}_n \cap \gamma = C^\gamma_n$. This works for all large enough n. Note that if $\gamma < \gamma'$ from X, then for all large n, $C^{\gamma'}_n = C^\gamma_n \cap \gamma'$. Define for $\gamma \in \dot{X}$,

$$\mathcal{C}_\gamma = \left\{C \subseteq \gamma : C \text{ club } C \subseteq \bigcap_{n \geq k} C^\gamma_n \text{ for some } k\right\}.$$

Also, if $\gamma \notin \dot{X}$, then $\mathrm{cf}(\gamma) = \omega$, so we just set $\mathcal{C}_\gamma = \{\text{some cofinal } \omega\text{-sequence}\}$. Let $\gamma \in \lim C$ for $C \in \mathcal{C}$. For all large n, $C \cap \gamma' \subseteq C^\gamma_n \cap \gamma' = C^{\gamma'}_n$. $\square$

<u>Fact</u>: If a poset $\mathbb{Q}$ is $\kappa + 1$-strategically closed and H is $\mathbb{Q}$-generic, then H is κ^+-directed closed.

<u>Parting Thoughts</u>

We'll end with some general talk about the form of diagonal Prikry forcing.

Conditions are usually of the form $\langle p_n : n < \omega\rangle$. We also have some (length) function $l : \mathbb{P} \longrightarrow \omega$. We will typically say $p \leq^* q$ if $p \leq q$ and $l(p) = l(q)$.

Now let

$$\mathbb{P}^0 = \{p : l(p) = 0\} \text{ and } \mathbb{D}^* = (\mathbb{P}^0, \leq^*).$$

Let further

$$\mathbb{P}^n = \left\{p \restriction [n, \omega) : p \in \mathbb{P}^0\right\} \text{ and } \mathbb{D} := \bigcup_{n<\omega} \mathbb{P}^n,$$

where $p \restriction [n, \omega) \leq q \restriction [m, \omega)$ if $\exists k \geq \max\{m, n\}$ so that $p \restriction [k, \omega) \leq q \restriction [k, \omega)$.

Observation 1: If $\mathbb{D}^*$ collapses "the successor of the singular," over a model where its predecessor (the singular) is strong limit, then $\mathbb{D}$ adds weak square.[47]

Observation 2: If $\mathbb{D}^*$ preserves the successor of the singular and some supercompactness and $\mathbb{P}/\mathbb{D}$ has chain condition, then $\mathbb{P}$ forces the failure of weak square.[48]

[47]This also applies to the extender-based forcing from earlier.

[48]Gitik-Sharon falls in this case.