FORCING EXERCISES
DAY 11

Problem 1. *Let $\mathbb{P} \in M$ be a poset, and let $p \in \mathbb{P}$. We say that a set D is* dense below p *if for every $q \leq p$ there is an $r \leq q$ with $r \in D$. Do the following.*

(1) *Suppose that G is $\mathbb{P}$-generic, and that $p \in G$. Show that G intersects D.*
(2) *Let $\mathbb{P} \in M$ and $p \in \mathbb{P}$. Let ϕ be a formula of the forcing language. Suppose that the set $D = \{q \in \mathbb{P} : q \Vdash \phi\}$ is dense below p. Show that $p \in D$.*

Recall that a filter G is $\mathbb{P}$-generic over M if and only if it intersects every maximal antichain of $\mathbb{P}$ that belongs to M. The following fact is very important!

Problem 2 (Maximality of the Forcing Language)**.** *Show that if $p \Vdash (\exists x)\phi(x, \tau_1, \ldots, \tau_n)$ then there is a name $\sigma \in M^{\mathbb{P}}$ such that $p \Vdash \phi(\sigma, \tau_1, \ldots, \tau_n)$.*

Problem 3. *The forcing $\mathbb{C}$ of finite partial functions from ω to ω ordered by extension is commonly referred to as* Cohen forcing. *A* Cohen real *over M is a function $c : \omega \to \omega$ such that the corresponding filter $G = \{p \in \mathbb{C} : p \subseteq c\}$ is $\mathbb{C}$-generic over M.*

(1) *Show that if G is a $\mathbb{C}$-generic filter over M then the function $g = \bigcup G$ is a Cohen real over M. (So Cohen generic filters and Cohen generic reals are essentially the same kind of object).*
(2) *Suppose that c is a Cohen real over M. Show that the function mapping n to $c(2n)$ is also a Cohen real over M.*
(3) *More generally suppose that $A \subseteq \omega$ is any infinite subset* belonging to M. *Let e_A enumerate A in increasing order. Show that the function mapping n to $c(e_A(n))$ is a Cohen real over M.*
(4) *Give an example of an infinite $A \subseteq \omega$ belonging to $M[c]$ such that the function mapping n to $c(e_A(n))$ is not a Cohen real over M.*

Problem 4. *Let $\mathbb{B}$ be the poset of finite partial functions from ω to 2. Do the following.*

(1) *Let G be $\mathbb{C}$-generic over M. Show that $M[G]$ contains a filter H which is $\mathbb{B}$-generic over M. (Hint: it may be easier to think about it in terms of reals instead of in terms of filters).*
(2) *Let G be $\mathbb{B}$-generic over M. Show that $M[G]$ contains a Cohen real over M.*