FORCING EXERCISES
DAY 10

Problem 1. *Revisit problem 6 from Day 9 and write down a clean copy of the solution (as though you were turning it in).*

Problem 2. *Read a proof of the Lowenheim-Skolem theorem. Be ready to present the proof in your problem session.*

Problem 3. *Suppose $p \Vdash \phi$ and ZFC $\vdash (\phi \to \psi)$. Then $p \Vdash \psi$. (You can assume $M[G]$ is a model of ZFC).*

Problem 4. *Do the following:*

(1) *Show that for any formulas ϕ and ψ and $\tau_1, \dots, \tau_n \in M^{\mathbb{P}}$ and $\sigma_1, \dots, \sigma_m \in M^{\mathbb{P}}$ we have that $p \Vdash \phi(\tau_1, \dots, \tau_n) \wedge \psi(\sigma_1, \dots, \sigma_m)$ if and only if both $p \Vdash \phi(\tau_1, \dots, \tau_n)$ and $p \Vdash \psi(\tau_1, \dots, \tau_m)$.*

(2) *Show that for any formula ϕ and $\tau_1, \dots, \tau_n \in M^{\mathbb{P}}$, $p \Vdash \neg\phi(\tau_1, \dots, \tau_n)$ if and only if there is no $q \leq p$ such that $q \Vdash \phi$.*

Problem 5. *Read the definition of ultraproducts and the proof of Los' Theorem starting on page 158 of Jech. If you don't have a copy, then search google for a reputable source.*

Problem 6 (The Compactness Theorem)**.** *Suppose that T is a theory and every finite subset of T has a model. Let $I = \{T_0 \mid T_0 \subseteq T \text{ is finite}\}$. Show the following:*

(1) *Define $A_\phi \subseteq I$ for $\phi \in T$ by $A_\phi = \{T_0 \in I \mid \phi \in T_0\}$. Show that $\{A_\phi \mid \phi \in T\}$ has the finite intersection property.*

(2) *Define an ultrafilter on I. For each $i \in I$ let M_i be a model of i. Recall i is a finite subset of T. Define the ultraproduct* Ult *of $\langle M_i \mid i \in I \rangle$ by your ultrafilter. Show that* Ult $\vDash T$ *by Los' Theorem.*