FORCING EXERCISES
DAY 7

Problem 1. *Express the axioms for a dense linear order without endpoints as sentences in the language of a single binary relation* $<$.

Problem 2 (The Tarski-Vaught test)**.** *Show that* $\mathcal{M} \prec \mathcal{N}$ *if and only if for all formulas* $\phi(u, \vec{v})$ *and all* $\vec{a} \in M^n$, $\mathcal{N} \vDash \exists x \phi(x, \vec{a})$ *if and only if there is* $b \in M$, $\mathcal{N} \vDash \phi(b, \vec{a})$. *The following list outlines the statements that you need to prove.*

(1) *Prove the forward direction of the theorem. Use the definition of satisfaction at the key point.*
(2) *For the reverse direction, assume that the right hand side of the 'if and only if' statement holds and use the technique of* induction on formulas *outlined below. Formulas were constructed inductively using a certain process. We specified certain* basic *or* atomic *things as formulas. For example,* $t_1 = t_2$ *where* t_1 *and* t_2 *are terms. Then we specified how new formulas are built out of previously defined formulas. For example if* ϕ *and* ψ *are formulas, then* $\phi \wedge \psi$ *is also a formula. In the following order prove*
   (a) *If* $\phi(\vec{v})$ *is an atomic formula and* $\vec{a} \in M^n$, *then* $\mathcal{M} \vDash \phi(\vec{a})$ *if and only if* $\mathcal{N} \models \phi(\vec{a})$.
   (b) *For each of the methods of constructing a new formula from an existing formula, show that if elementarity (*$\mathcal{M} \vDash \phi$ *if and only if* $\mathcal{N} \vDash \phi$*) holds for the old formula(s), then if holds for the new formula. Hints for this step: The definition of satisfaction of formulas is key. The only place the assumption is used is for formulas* $\exists x \phi(x, \vec{a})$.

Problem 3 (Lowenheim-Skolem theorem)**.** *Show that for every structure* $\mathcal{N}$ *and every subset* $N_0 \subseteq N$, *there is an elementary submodel* $\mathcal{M} \prec \mathcal{N}$, *such that* $N_0 \subseteq M$ *and* $|M| \leq |N_0| \cdot \aleph_0 \cdot |\tau|$. *Hints: Use the Tarski-Vaught test. Use the axiom of choice to define a function for each existential formula, which given a sequence of parameters returns a witness to the formula if it exists.*

Problem 4. *Can the* completeness *of the real line be expressed as a first order sentence about the structure* $(\mathbb{R}, +\cdot, 0, 1, <)$*? Give a proof of your answer.*

Theorem 1 (Compactness Theorem)**.** *Let* T *be a theory. If every finite subset* T' *of* T *has a model, then* T *has a model.*

Problem 5. *Use the Completeness theorem to prove the Compactness theorem.*

Problem 6. *Use the compactness theorem to construct a non-Archimedean field. Recall that a field is Archimedean if there is no* $x > 0$ *such that for all* $n \geq 1$, $x < \frac{1}{n}$.

Problem 7. *Use the compactness theorem to prove that there is a non-standard model of Peano Arithmetic. A non-standard model of Peano Arithmetic is not isomorphic to the standard model.*

Problem 8 (Upward Lowenheim-Skolem theorem)**.** *Suppose that T is a theory in a signature τ with an infinite model. Show that T has models of any cardinality κ where $\kappa \geq |\tau| \cdot \aleph_0$. Hint: Create a signature τ' which contains τ and also has many constant symbols. Show there is a κ sized model of a suitable τ'-theory.*