THE TREE PROPERTY AT THE TWO IMMEDIATE SUCCESSORS OF A SINGULAR CARDINAL

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Abstract. We present an alternative proof that from large cardinals, we can force the tree property at κ^+ and κ^{++} simultaneously for a singular strong limit cardinal κ. The advantage of our method is that the proof of the tree property at the double successor is simpler than in the existing literature. This new approach also works to establish the result for $\kappa = \aleph_{\omega^2}$.

1. Introduction

A regular uncountable cardinal κ has the *tree property* if every κ-tree has a cofinal branch, or equivalently there are no κ-Aronszajn trees. The tree property belongs to a family of *compactness properties* which are of great interest in combinatorial set theory. Compactness is the phenomenon where if some property holds for every strictly smaller substructure of an object, it holds for the entire object. Like other compactness properties such as stationary reflection, the tree property is a property of "large cardinal type" which can consistently hold at certain small regular cardinals.

An old question due to Magidor asks whether the tree property can consistently hold simultaneously at all regular $\kappa > \aleph_1$. There are several obstacles to obtaining a positive consistency result for this problem:

- Specker showed that if $\tau^{<\tau} = \tau$, then there is a special τ^+-Aronszajn tree. So a positive result requires a model where there are no strongly inaccessible cardinals and the GCH fails everywhere. In particular the SCH must fail at every singular strong limit cardinal.
- Jensen showed that the weak square principle $\square^*_\tau$ is equivalent to the existence of a special τ^+-Aronszajn tree. Results from inner model theory show that the failure of $\square^*_\tau$ for any singular τ requires very substantial large cardinal hypotheses.

A test question which exposes some of the main difficulties asks whether there can exist a singular strong limit cardinal κ such that both κ^+ and κ^{++}

Cummings was partially supported by the National Science Foundation, DMS-1500790.
Hayut was partially supported by FWF, M 2650 Meitner-Programm.
Neeman was partially supported by the National Science Foundation, DMS-1764029.
Sinapova was partially supported by the National Science Foundation, Career-1454945.
Unger was partially supported by the National Science Foundation, DMS-1700425.

have the tree property. This question was resolved positively by Sinapova [5], and in subsequent work Sinapova and Unger [6] showed that the singular cardinal κ can consistently be $\aleph_{\omega^2}$.

In [5], the main forcing was diagonal Prikry forcing interleaved into a Mitchell style poset. The proof of the tree property at κ^{++} generalized an argument of Cummings and Foreman [1]; this form of argument also appears in work of Unger [7]. The hardest point in the argument of [5] was to show the tree property at κ^+, which required a complicated argument. One disadvantage to interleaving the Prikry forcing inside the Mitchell poset is that the construction is not amenable to including collapses to make κ small; in particular this approach is not suited to make κ into $\aleph_{\omega^2}$.

That motivated the construction in [6], where the main forcing is a two step iteration, with a Mitchell's forcing poset followed by diagonal Prikry forcing. In that case one can incorporate collapses between the Prikry points. However, the argument for the tree property at the double successor of the singular cardinal became much more complicated. The hardest technical issue was "branch preservation" with respect to a quotient forcing of the form $j(\text{Mitchell} * \text{Prikry})/\text{Mitchell} * \text{Prikry}$, where j is an appropriate elementary embedding.

Here we present a simpler construction for the results of [5] and [6]. More precisely, we give an alternative proof that forcing with certain iterations of the form Mitchell $*$ Prikry yields the tree property at κ^{++}. Our argument avoids dealing with the above-mentioned quotient poset, and so bypasses all the technical issues connected with that poset.

2. The forcing posets

In this section, we describe the two main posets in our construction: a Mitchell style forcing and a diagonal Prikry forcing.

We will use the following version of the Mitchell forcing from [3]. Given regular cardinals κ, μ, λ where $\kappa < \mu < \lambda$, $\kappa^{<\kappa} = \kappa$ and λ is strongly inaccessible, we define a forcing poset $\mathbb{M} = \mathbb{M}(\kappa, \mu, \lambda)$. Conditions are pairs (p, q) such that:

(1) $p \in \text{Add}(\kappa, \lambda)$.
(2) q is a partial function on λ such that $|q| < \mu$.
(3) For all $\alpha \in \text{dom}(q)$, $q(\alpha)$ is an $\text{Add}(\kappa, \alpha)$-name for a condition in $\text{Add}(\mu, 1)$.

The ordering is given by $(p_1, q_1) \leq (p_0, q_0)$ iff $p_1 \leq p_0$, $\text{dom}(q_0) \subseteq \text{dom}(q_1)$, and $p \upharpoonright \alpha \Vdash q_1(\alpha) \leq q_0(\alpha)$ for all $\alpha \in \text{dom}(q_0)$.

We refer the reader to [1] for a detailed account of the properties of $\mathbb{M}$. We will use the following facts:

- $\mathbb{M}$ is κ-directed closed and λ-cc.
- $\mathbb{M}$ is the projection of $\text{Add}(\kappa, \lambda) \times \mathbb{R}$, where $\mathbb{R}$ is μ-closed[1].

[1]This means decreasing $< \mu$ sequences have lower bounds.

- All $< \mu$-sequences of ordinals in the generic extension by $\mathbb{M}$ lie in the subextension by $\mathrm{Add}(\kappa, \lambda)$.
- $\mathbb{M}$ preserves all cardinals except those in the interval (μ, λ), which are collapsed to μ.
- $\mathbb{M}$ forces $2^\kappa = \lambda = \mu^+$.

If λ is measurable[2] then λ still has the tree property in the extension by $\mathbb{M}$. The key points are that if $j : V \to M$ is an embedding with critical point λ into a model M such that ${}^\lambda M \subseteq M$, then:

- In M, $\mathbb{M}$ is an initial segment of $j(\mathbb{M})$.
- $j \restriction \mathbb{M}$ is a complete embedding of $\mathbb{M}$ into $j(\mathbb{M})$.
- If G is $\mathbb{M}$-generic then in $M[G]$ the quotient forcing $j(\mathbb{M})/G$ is the projection of a product $\mathbb{A} \times \mathbb{Q}$, where $\mathbb{A} = \mathrm{Add}(\kappa, j(\lambda) - \lambda)$.
- In $V[G]$, $\mathbb{A}$ is κ^+-Knaster and $\mathbb{Q}$ is μ-closed.

Next, we describe the diagonal supercompact Prikry forcing $\mathbb{P}$. This forcing was first defined by Gitik and Sharon [2] to prove the consistency of failure of SCH at κ with failure of $\Box^*_\kappa$, and then modified by Neeman [4] to prove the consistency of failure of SCH at κ with the tree property holding at κ^+.

Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of regular cardinals. Let $\kappa = \kappa_0$ and assume that κ is a supercompact cardinal. Let $\mu = (\sup_n \kappa_n)^+$. Let U be a normal measure on $\mathcal{P}_\kappa(\mu)$, and for every n let U_n be the projection of U to $\mathcal{P}_\kappa(\kappa_n)$. The measure U_n concentrates on x such that $x \cap \kappa$ is an inaccessible cardinal: we only consider $x \in P_\kappa \kappa_n$ with this property, and write κ_x for $x \cap \kappa$.

Conditions in $\mathbb{P}$ are of the form $p = \langle x_0, ..., x_{n-1}, A_n, A_{n+1} \ldots \rangle$, where each $x_i \in \mathcal{P}_\kappa(\kappa_i)$, $x_i \prec x_{i+1}$ (i.e. $x_i \subseteq x_{i+1}$ and $|x_i| < \kappa_{x_{i+1}}$), and each $A_k \in U_k$. We say that $\mathrm{lh}(p) = n$ and the stem of p is $\mathrm{s}(p) = \langle x_0, ..., x_{n-1} \rangle$.

The ordering is given by $q \leq p$ iff:

- $\mathrm{lh}(q) \geq \mathrm{lh}(p)$.
- For all i with $i < \mathrm{lh}(p)$, $x^q_i = x^p_i$.
- For all i with $\mathrm{lh}(p) \leq i < \mathrm{lh}(q)$, $x^q_i \in A^p_i$.
- For all i with $\mathrm{lh}(q) \leq i$, $A^q_i \subseteq A^p_i$.

We say that q is a *direct extension* of p, and write $q \leq^* p$, if $q \leq p$ and they have the same length,

The following are standard facts about $\mathbb{P}$:

1. $\mathbb{P}$ has the *Prikry property*: for any statement in the forcing language ϕ and any $p \in \mathbb{P}$, there is $q \leq^* p$ deciding ϕ. In particular $\mathbb{P}$ adds no bounded subsets of κ.
2. Conditions with the same stem are compatible, and so $\mathbb{P}$ has the μ-chain condition.
3. $\mathbb{P}$ forces that $\mathrm{cf}(\kappa_n) = \omega$ for all $n \geq 0$.
4. $\mathbb{P}$ preserves κ and μ, and forces that $\mu = \kappa^+$

[2]With a bit more work, weak compactness suffices here.

Given a formula ϕ and a stem h, we will say that $h \Vdash^* \phi$ if there is a condition p with stem h which forces ϕ. Note that by the Prikry property, for every ϕ, h either we have $h \Vdash^* \phi$ or $h \Vdash^* \neg\phi$.

3. The main construction

Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of indestructible supercompact cardinals with limit κ_ω. Let $\mu = \kappa_\omega^+$ and $\kappa = \kappa_0$. Suppose also that λ is a measurable cardinal above μ. Let $\mathbb{M}$ be the Mitchell forcing $\mathbb{M}(\kappa, \mu, \lambda)$ from the last section, and G be $\mathbb{M}$-generic. Since $\mathbb{M}$ is κ-directed closed, κ is still supercompact in $V[G]$.

In $V[G]$ let $\mathbb{P}$ be the diagonal Prikry forcing described in the preceding section, built using normal measures U_n on $\mathcal{P}_\kappa(\kappa_n)$ for $n < \omega$. Let H be $\mathbb{P}$-generic over $V[G]$. In $V[G][H]$, κ is preserved, $\mathrm{cf}(\kappa) = \omega$, $\kappa^+ = \mu$, $\kappa^{++} = \lambda = 2^\kappa$.

Theorem 3.1. *The tree property holds at λ in $V[G][H]$.*

Proof. In $V[G]$ let $\dot{T}$ be $\mathbb{P}$-name for a λ-tree that is forced to be a counterexample. As usual assume that for $\alpha < \lambda$, the levels $\dot{T}_\alpha$ are simply $\{\alpha\} \times \mu$.

Let $j : V \to M$ be the ultrapower map by a normal measure on λ. Lift it to $j : V[G] \to M[G^*]$ in $V[G][K \times A]$, where K is generic for a μ-closed forcing $\mathbb{Q}$ and A is generic for a κ^+-Knaster forcing $\mathbb{A}$. Of course we could have lifted j working in $V[G^*]$, but $\mathbb{Q} \times \mathbb{A}$ is more tractable than $j(\mathbb{M})/G$.

Lemma 3.2. *In $V[G]$ there exist an unbounded set $J \subseteq \lambda$, a stem h, and a function $f : J \to \mu$, such that for all $\alpha < \beta$ both in J,*

$$h \Vdash^* \langle \alpha, f(\alpha) \rangle <_{\dot{T}} \langle \beta, f(\beta) \rangle.$$

Proof. Work in $V[G][K \times A]$. Let $u = \langle \lambda, 0 \rangle$, so that u is a node on the λ-th level of $j(\dot{T})$. Then for all $\alpha < \lambda$ there exist $p_\alpha \in j(\mathbb{P})$ and $\xi_\alpha < \mu$ such that $p_\alpha \Vdash \langle \alpha, \xi_\alpha \rangle <_{j(\dot{T})} u$. Let h_α be the stem of p_α, and note that h_α is a stem in the original forcing $\mathbb{P}$.

Since there are only κ_ω-many stems and the cofinality of λ is μ in $M[G^*]$, there exist an unbounded $J \subseteq \lambda$ and a stem $\bar{h}$, such that for all $\alpha \in J$ we have $\bar{h} = h_\alpha$. For all $\alpha < \lambda$, if $\alpha \in J$ then $\bar{h} \Vdash^* \langle \alpha, \xi_\alpha \rangle <_{j(\dot{T})} u$. Increasing J if needed we may assume that the converse also holds.

Define $f : J \to \mu$ by $f(\alpha) = \xi_\alpha$. Then if $\beta \in J$, we have that for all $\alpha < \beta$, $\alpha \in J$ if and only f there is some $\xi < \mu$ such that $\bar{h} \Vdash^* \langle \alpha, \xi \rangle <_{\dot{T}} \langle \beta, f(\beta) \rangle$, and in this case ξ_α is the unique ξ with this property. This implies that for all $\gamma < \lambda$ both $J \cap \gamma$ and $f \restriction \gamma$ are in $V[G]$.

Next we want to find such a J and f in $V[G]$. Since $\mathbb{A}^2$ has the κ^+-approximation property in $V[G][K]$, it is easy to see that the versions of J and f which we just constructed lie in $V[G][K]$. In $V[G][K \times A]$, for every stem h extending $\bar{h}$, let $J_h = \{\alpha \in J \mid \exists \xi (h \Vdash^* \langle \alpha, \xi \rangle <_{j(\dot{T})} u)\}$ and define $f_h : J_h \to \mu$ by setting $f_h(\alpha)$ to be the unique ξ witnessing that $\alpha \in J_h$.

As above, if J_h is unbounded then J_h and f_h are in $V[G][K]$. Let $\bar{\alpha} < \lambda$ be forced by $\mathbb{A}$ to be a bound for all J_h which are bounded in λ.

For each h, let $\dot{J}_h, \dot{f}_h \in V[G][K]$ be $\mathbb{A}$-names for J_h, f_h. In $V[G][K]$ let $\mathcal{C}_h = \{C \subseteq \lambda \mid C \setminus \bar{\alpha} \neq \emptyset, \exists b \in \mathbb{A}(b \Vdash C = \dot{J}_h)\}$. Then by the above remark and since $\mathbb{A}$ is κ^+-Knaster, if J_h is unbounded in λ, we have that $1 \leq |\mathcal{C}_h| \leq \kappa$. Enumerate $\mathcal{C}_h$ as $\langle C_{h,\eta} \mid \eta < \kappa\rangle$ (possibly with repetitions) when it is not empty. For every $\eta < \kappa$ pick some $b_\eta \in \mathbb{A}$ forcing that $C_{h,\eta} = \dot{J}_h$, and let $f_{h,\eta} : C_{h,\eta} \to \mu$ be defined by setting $f_{h,\eta}(\alpha)$ equal to the unique ξ witnessing $\alpha \in C_{h,\eta}$ as forced by b_η. That is to say, $b_\eta \Vdash^{V[G][K]}_{\mathbb{A}} (h \Vdash^* \langle \alpha, \xi\rangle <_{j(\dot{T})} u)$.

Working in $V[G]$, for each h such that $\dot{\mathcal{C}}_h$ is not the empty set and for each $\eta < \kappa$, fix $\mathbb{Q}$-names $\dot{C}_{h,\eta}$ and $\dot{f}_{h,\eta}$. We want to show that for some h and η the pair $(\dot{C}_{h,\eta}, \dot{f}_{h,\eta})$ can be forced to be in $V[G]$. Towards a contradiction, suppose that for all h and η we have $1_{\mathbb{Q}} \Vdash (\dot{C}_{h,\eta}, \dot{f}_{h,\eta}) \notin V[G]$.

We say that q_0, q_1 in $\mathbb{Q}$ *force contradictory information about* $\dot{f}_{h,\eta}(\alpha)$ if q_0, q_1 both decide "$\alpha \in \dot{C}_{h,\eta}$" with at least one of them forcing a positive decision, and $(q_0, q_1) \Vdash_{\mathbb{Q}\times\mathbb{Q}} \dot{f}_{h,\eta}[\dot{G}_L](\alpha) \neq \dot{f}_{h,\eta}[\dot{G}_R](\alpha)$.

Since we have assumed that each $\dot{f}_{h,\eta}$ is forced to be new, we have the following in $V[G]$:

Claim 3.3. *Suppose that $q_0, q_1 \in \mathbb{Q}$, h is a stem extending $\bar{h}$, $\eta < \kappa$, and $\alpha < \lambda$. Then there are $\alpha' > \alpha$ and $q_0' \leq q_0, q_1' \leq q_1$, forcing contradictory information about $\dot{f}_{h,\eta}(\alpha')$.*

Working in $V[G]$ we build a binary tree of conditions in $\mathbb{Q}$, forcing contradictory information about $f_{h,\eta}$ for every pair (h,η) at every splitting. More precisely we build $\langle q_\sigma, \alpha_{\sigma,h,\eta} \mid \sigma \in 2^{<\kappa}, h \text{ a stem }, h \sqsupseteq \bar{h}, \eta < \kappa\rangle$, such that for all (σ, h, η) the conditions $q_{\sigma ^\frown 0}$ and $q_{\sigma ^\frown 1}$ force contradictory information about $f_{h,\eta}(\alpha_{\sigma,h,\eta})$. We build this tree by induction on $|\sigma|$, and at every stage we apply Claim 3.3 repeatedly for each h and η. We use that $\mathbb{Q}$ is μ-closed and the number of pairs (h,η) is κ_ω.

Let $\alpha^* = \sup_{\sigma,h,\eta} \alpha_{\sigma,h,\eta}$. For each $i \in 2^\kappa$, let $q_i \leq q_{i\restriction\eta}$ for all $\eta < \kappa$. Now let $q_i' \leq q_i$ be such that there are h_i, $\eta_i < \kappa$, $\xi_i < \mu$ and $b \in \mathbb{A}$ such that:

- $(q_i', b) \Vdash_{\mathbb{Q}\times\mathbb{A}}$ "$h_i \Vdash^* \langle \alpha^*, \xi_i\rangle <_{j(\dot{T})} u$",
- $(q_i', b) \Vdash_{\mathbb{Q}\times\mathbb{A}} \dot{C}_{h_i,\eta_i} = \dot{J}_{h_i}, \dot{f}_{h_i} = \dot{f}_{h_i,\eta}$ [3] (and so $q_i' \Vdash \alpha^* \in \dot{C}_{h,\eta_i}$)

Note that for all $\alpha < \alpha^*$, $q_i' \Vdash \dot{f}_{h_i,\eta_i}(\alpha) = \delta$ iff $h_i \Vdash^* \langle \alpha, \delta\rangle <_{\dot{T}} \langle \alpha^*, \xi_i\rangle$.

Since $2^\kappa = \lambda > \mu$, there exist distinct $i, j \in 2^\kappa$ and (h, η, ξ) such that $h_i = h_j = h$, $\eta_i = \eta_j = \eta$ and $\xi_i = \xi_j = \xi$. Let σ be the node where i and j split. By construction the conditions q_i and q_j cannot force contradictory information about $\dot{f}_{h,\eta}(\alpha_{h,\sigma,\eta})$. This is a contradiction, so in $V[G]$ we may find a stem h, set J and function f as required. □

Let J, f, h be given by Lemma 3.2, and let $n = |h|$. As above, let $\xi_\alpha = f(\alpha)$.

[3]Here we identify $\dot{J}_h, \dot{f}_h$ and $\dot{C}_{h_i,\eta_i}, \dot{f}_{h_i,\eta}$ with their natural corresponding $\mathbb{Q}\times\mathbb{A}$-names.

Lemma 3.4. *There are $\rho < \lambda$ and U_n-measure one sets $\langle A_\alpha \mid \alpha \in J \setminus \rho \rangle$ in $V[G]$, such that for all $x \in A_\alpha \cap A_\beta$, $h^\frown x \Vdash^* \langle \alpha, \xi_\alpha \rangle <_{\dot{T}} \langle \beta, \xi_\beta \rangle$.*

Proof. By the same ideas as in [4], such measure one sets exist in $V[G][K]$. We go over the proof for completeness:

Claim 3.5. *There are $\langle A^*_\alpha \mid \alpha \in J \setminus \rho \rangle$ in $V[G][K]$, satisfying the conclusion of Lemma 3.4.*

Proof. For every $x \in \mathcal{P}_\kappa(\kappa_n)$ let $J_x = \{\alpha \in J \mid h^\frown x \Vdash^* \langle \alpha, \xi_\alpha \rangle <_{j(\dot{T})} u\}$. Let $\bar{\alpha} < \lambda$ be such that if J'_x is bounded in λ, then $J'_x \subseteq \bar{\alpha}$. Here we use that the number of x's is less than $\mu = \mathrm{cf}^{M[G^*]} \lambda$.

Redefine $J_x = \{\alpha \in J \setminus \bar{\alpha} \mid h^\frown x \Vdash^* \langle \alpha, \xi_\alpha \rangle <_{j(\dot{T})} u\}$ Then each J_x is either empty or unbounded. In particular each $J_x \in V[G][K]$, since $\mathbb{A}$ has the κ^+-approximation property. In $V[G][K]$, define $\mathcal{C}_x$ to be the set of possible values for J_x. Then it is routine to show:

- $|\mathcal{C}_x| \leq \kappa$.
- For all $C \in \mathcal{C}_x$ and for all $\alpha < \beta$ both in C, $h^\frown x \Vdash^* \langle \alpha, \xi_\alpha \rangle <_{\dot{T}} \langle \beta, \xi_\beta \rangle$.
- Distinct elements C and C' of $\mathcal{C}_x$ are disjoint on a final segment of λ.

Fix $\rho < \lambda$ such that $\bar{\alpha} < \rho$, and $C \cap C' \subseteq \rho$ for all x and all pairs of distinct elements C and C' in $\mathcal{C}_x$. Let $\alpha_0 = \min(J \setminus \rho)$, and define $f(x, \alpha)$ to be the unique $C \in \mathcal{C}_x$ such that $\alpha \in C$, if such exists. Then let $A^*_\alpha = \{x \mid f(x, \alpha) = f(x, \alpha_0)\}$, where the equality means that the values $f(x, \alpha)$ and $f(x, \alpha_0)$ are defined and equal. It is clear that $A^*_\alpha \in U_n$. □

Fix $\langle A^*_\alpha \mid \alpha \in J \setminus \rho \rangle$ as in Claim 3.5. By construction, we have that for all $\alpha < \beta$ both in $J \setminus \rho$ and for all $x \in A^*_\beta$, $x \in A^*_\alpha$ if and only if $h^\frown x \Vdash^* \langle \alpha, \xi_\alpha \rangle <_{\dot{T}} \langle \beta, \xi_\beta \rangle$.

Working in $V[G][K]$, for each x we let $b_x = \{\alpha \in J \setminus \rho \mid x \in A^*_\alpha\}$. By increasing ρ, we may assume that each b_x is unbounded in λ or empty. We say that $\dagger_x$ holds if b_x is unbounded and $b_x \in V[G]$. Let $A = \{x \in \mathcal{P}_\kappa(\kappa_n) \mid \dagger_x \text{ holds } \}$. Then $A \in V[G]$ since $\mathbb{Q}$ is μ-closed.

Claim 3.6. *$A \in U_n$.*

Proof. Suppose otherwise. Then $A^c = \mathcal{P}_\kappa(\kappa_n) \setminus A \in U_n$. Find q_0 and q_1 in $\mathbb{Q}$ with the following property: for all $x \in A^c$ such that $\dot{b}_x$ is not forced to be empty, there is $\alpha_x < \lambda$ such that q_0 and q_1 decide "$\alpha_x \in \dot{b}_x$" in opposite ways. Let $\beta \geq \sup_x \alpha_x$ with $\beta \in J$.

Extend each condition q_i to q'_i such that for each $x \in A^c$, $q'_i \parallel x \in \dot{A}^*_\beta$. This is possible by the closure of $\mathbb{Q}$. Let $A_i = \{x \in A^c \mid q'_i \Vdash x \in \dot{A}^*_\beta\}$, then $A_i \in U_n$ because $A^c \in U_n$.

Take $x \in A_0 \cap A_1$ and let $\alpha = \alpha_x$. Then both q'_0, q'_1 force that $x \in \dot{A}^*_\beta$. Then $\dot{b}_x$ is forced to be nonempty, so q'_0 and q'_1 decide "$\alpha \in \dot{b}_x$" in opposite ways. Without loss of generality:

(1) $q'_0 \Vdash \alpha \in \dot{b}_x$.
(2) $q'_1 \Vdash \alpha \notin \dot{b}_x$.

The first item implies that $q'_0 \Vdash x \in \dot{A}^*_\alpha \cap \dot{A}^*_\beta$, and so $h^\frown x \Vdash^* \langle \alpha, \xi_\alpha \rangle <_{\dot{T}} \langle \beta, \xi_\beta \rangle$. But then since $q'_1 \Vdash x \in \dot{A}^*_\beta$, we get that $q'_1 \Vdash \alpha \in b_x$. Contradiction. □

So, we have $A \in U_n$. By the closure of $\mathbb{Q}$, the sequence $\langle b_x \mid x \in A \rangle$ is also in $V[G]$. Let $A_\alpha = \{x \in A \mid \alpha \in b_x\}$. Since each $A_\alpha = A \cap A^*_\alpha$ and $A^*_\alpha \in j(U_n)$, we get that $A_\alpha \in U_n$. This completes the lemma. □

We can now apply this lemma inductively as in [4] to get conditions $\langle p_\alpha \mid \alpha \in J \setminus \rho \rangle$ so that for all $\alpha < \beta$, $p_\alpha \wedge p_\beta \Vdash \langle \alpha, \xi_\alpha \rangle <_{\dot{T}} \langle \beta, \xi_\beta \rangle$. By the μ chain condition, there are unboundedly many α's such that p_α is forced into a generic for $\mathbb{P}$. This gives a branch. □

Remark 3.7. The above arguments can also be carried out by assuming that λ is weakly compact.

The arguments of [6, Section 3] can be used to show that μ has the tree property in $V[G][H]$. The setting here is rather simpler and we just outline the argument. Let $\dot{T} \in V[G]$ be a $\mathbb{P}$-name for a μ-tree. Recalling that $\mathbb{M}$ can be written as a projection of $\mathrm{Add}(\kappa, \lambda) \times \mathbb{R}$ for a suitable term forcing $\mathbb{R}$, we force to obtain $V[G] \subseteq V[A \times B]$ where $A \times B$ is $\mathrm{Add}(\kappa, \lambda) \times \mathbb{R}$-generic. The Prikry poset $\mathbb{P}$ has the same definition in $V[A \times B]$ as in $V[G]$, and in this model the cardinals κ_n are generically supercompact via reasonably nice forcing. We can use this to argue as in [4] that T has a branch in $V[A \times B][H]$, and then use a suitable "branch lemma" to pull back and get a branch of T in $V[G][H]$.

We have therefore given a new proof of the main theorem of [5].

Theorem 3.8. *From ω many supercompact cardinals and a weakly compact λ above them, it is consistent to have the tree property at both κ^+ and κ^{++} for a singular strong limit cardinal κ.*

Next, we use the above arguments to give another proof of the main theorem of [6].

Theorem 3.9. *From ω many supercompact cardinals and a weakly compact λ above them, it is consistent to have the tree property at both $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$, where $\aleph_{\omega^2}$ is strong limit.*

Proof. We use the same forcing as in [6]. Namely, we prepare the ground model V, so that:

(1) $\kappa_n = \kappa^{+n}$, and κ_n is generically supercompact.
(2) After we force with the Mitchell forcing $\mathbb{M} \times \mathrm{Add}(\kappa, \lambda^+ \setminus \lambda)$, for each n we have normal measures U_n on $\mathcal{P}_\kappa(\kappa_n)$, and "guiding generics" K_n, for $Col(\kappa^{+\omega+3}, < j_{U_n}(\kappa))$ over the ultrapower by U_n.

(3) The guiding generics above are for a $\lambda^+ = \mu^{++}$-closed forcing since, they are pulled back from a genering for a $Col(\kappa^{+\omega+3}, < j^*(\kappa))$ where j^* is a λ-supercompact embedding with critical point κ, and there the ultrapower is closed under λ-sequences. For more details, see Section 2 of [6].

Let G be $\mathbb{M} \times \text{Add}(\kappa, \lambda^+ \setminus \lambda)$-generic. In $V[G]$ define the diagonal Prikry forcing $\mathbb{P}$ with interleaved collapses to make $\kappa = \aleph_{\omega^2}$ as follows. Conditions are of the form $p = \langle d, x_0, c_o, ..., x_{n-1}, c_{n-1}, A_n, C_n, ...\rangle$, where,

(1) $\langle x_0, ..., x_{n-1}, A_n ...\rangle$ is in the diagonal Prikry forcing defined from the measures $\langle U_i \mid i < \omega\rangle$, as defined in section 2.
(2) $d \in Col(\omega, \kappa_{x_0}^{+\omega})$,
(3) for $i < n-1$, $c_i \in Col(\kappa_{x_i}^{+\omega+3}, < \kappa_{x_{i+1}})$, $c_{n-1} \in Col(\kappa_{x_{n-1}}^{+\omega+3}, < \kappa)$,
(4) for $i \geq n$, $\text{dom}(C_i) = A_i$, each $C_i(x) \in Col(\kappa_x^{+\omega+3}, < \kappa)$, $[C_i]_{U_i} \in K_i$.

Let H be $\mathbb{P}$-generic over $V[G]$. In $V[G][H]$, $\kappa = \aleph_{\omega^2}$, $\mu = \aleph_{\omega^2+1}, \lambda = \aleph_{\omega^2+2}, 2^\kappa = \aleph_{\omega^2+3}$. By the arguments of [6, Section 3], which we already sketched above in the setting of Theorem 3.1, we have the tree property at $\mu = \aleph_{\omega^2+1}$.

To prove the tree property at $\aleph_{\omega^2+2}$, we use the the same approach as in Theorem 3.1. Note that we still have that the number of stems is κ_ω. Also, any two conditions with the same stem are compatible, as witnessed by a common lower bound with that same stem. So $\mathbb{P}$ has the μ-chain condition, and we can define as before the notion of $h \Vdash^* \phi$.

Let $j : V \to M$ be an elementary embedding with critical point λ. As in Theorem 3.1, we lift j to $j : V[G] \to M[G^*]$ in $V[G][K \times A]$, where K is generic for a μ-closed forcing $\mathbb{Q}$ and A is generic for a κ^+-Knaster forcing. The critical point is still above any fixed condition $p \in \mathbb{P}$, that is to say $j(p) = p$. So we have the analogue of Lemma 3.2: in $V[G]$, there is an unbounded $J \subseteq \lambda$, a stem h, and a function $f : J \to \mu$, such that for all $\alpha < \beta$ both in J,

$$h \Vdash^* \langle \alpha, f(\alpha)\rangle <_{\dot{T}} \langle \beta, f(\beta)\rangle.$$

Next we prove the following analogue of Lemma 3.4.

Lemma 3.10. *There are $\rho < \lambda$ and a sequence $\langle A_\alpha, C_\alpha \mid \alpha \in J \setminus \rho\rangle$ in $V[G]$, such that:*

- *$A_\alpha \in U_n$, $\text{dom}(C_\alpha) = A_\alpha$, $C_\alpha(x) \in Col(\kappa_x^{+\omega+3}, < \kappa)$ for all $x \in A_\alpha$, and $[C_\alpha]$ is in the guiding generic K_n.*
- *For all $x \in A_\alpha \cap A_\beta$, if $C_\alpha(x)$ and $C_\beta(x)$ are compatible, then*

$$h^\frown\langle x, C_\alpha(x) \cup C_\beta(x)\rangle \Vdash^* \langle \alpha, \xi_\alpha\rangle <_{\dot{T}} \langle \beta, \xi_\beta\rangle.$$

Proof. For every $\alpha < \beta$ both in J, let $r_{\alpha,\beta} \in \mathbb{P}$ be a condition with stem h, such that $r_{\alpha,\beta} \Vdash \langle \alpha, \xi_\alpha\rangle <_{\dot{T}} \langle \beta, \xi_\beta\rangle$. Denote $r_{\alpha,\beta} = h^\frown(A_{\alpha,\beta}, C_{\alpha,\beta})^\frown r_{\alpha,\beta} \restriction [n+1, \omega)$. Since each $[C_{\alpha,\beta}]$ belongs to a μ^{++}-closed guiding generic K_n, there is a lower bound $[C]$ in K_n for all of them. So we may assume that $\text{dom}(C) = \mathcal{P}_\kappa(\kappa_n)$ and that $C_{\alpha,\beta} = C \restriction A_{\alpha,\beta}$ for all α and β, by shrinking measure one sets and extending values of $C_{\alpha,\beta}(x)$ as needed. It follows that for every $\alpha < \beta$ both in J, for U_n-many x,

$$h^\frown\langle x, C(x)\rangle \Vdash^* \langle \alpha, \xi_\alpha\rangle <_{\dot{T}} \langle \beta, \xi_\beta\rangle.$$

The rest of the proof is exactly as in Lemma 3.4, only we replace instances of $h^\frown x$ with $h^\frown\langle x, C(x)\rangle$. □

Now we can apply Lemma 3.10 inductively, exactly as we applied Lemma 3.4 in the proof of Theorem 3.1. We construct conditions $\langle p_\alpha \mid \alpha \in J \setminus \rho\rangle$ so that for all $\alpha < \beta$, $p_\alpha \wedge p_\beta \Vdash \langle \alpha, \xi_\alpha\rangle <_{\dot{T}} \langle \beta, \xi_\beta\rangle$. By the μ chain condition, there are unboundedly many α's such that p_α is forced into a generic for $\mathbb{P}$. This gives a branch. □

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