SUCCESSIVE FAILURES OF APPROACHABILITY

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Abstract. Motivated by showing that in ZFC we cannot construct a special Aronszajn tree on some cardinal greater than $\aleph_1$, we produce a model in which the approachability property fails (hence there are no special Aronszajn trees) at all regular cardinals in the interval $[\aleph_2, \aleph_{\omega^2+3}]$ and $\aleph_{\omega^2}$ is strong limit.

1. Introduction

In the 1920's, König [11] proved that every tree of height ω with finite levels has a cofinal branch. In the 1930's Aronszajn [12] showed that the analogous theorem for ω_1 fails. In particular he constructed a tree of height ω_1 whose levels are countable which has no cofinal branch. Such trees have come to be known as Aronszajn trees. The first Aronszajn tree is *special* in the sense that there is a function $f : T \to \omega$ such that $f(s) \neq f(t)$ whenever s is below t in T. This function f witnesses that T has no cofinal branch.

These two theorems provide a strong contrast between the combinatorial properties of ω and ω_1. König's theorem shows that ω has a certain compactness property, while an Aronszajn tree is a canonical example of a noncompact object of size ω_1. These properties admit straightforward generalizations to higher cardinals. We say that a regular cardinal λ has the *tree property* if it satisfies a higher analog of König's theorem. If λ does not have the tree property, then we call a counter example a λ-Aronszajn tree. The natural question is: Which cardinals carry Aronszajn trees?

A full answer to this question is connected to the phenomena of independence in set theory and large cardinals. The first evidence of this comes from a theorem of Specker [26] which shows that Aronszajn's construction can be generalized in the context of an instance of the generalized continuum hypothesis. In particular if $\kappa^{<\kappa} = \kappa$, then there is a special κ^+-Aronszajn tree (for the appropriate generalization of the notion of special). On the other hand, the tree property has a strong connection with the existence of large cardinals. We say that an uncountable cardinal is *weakly compact* if it satisfies a higher analog of infinite Ramsey's theorem. By theorems of Tarski and Erdös [6] and Monk and Scott [15], an uncountable cardinal λ is weakly compact if and only if it is inaccessible and has the tree property.

The invention of forcing provided a method for proving the consistency of the tree property at accessible cardinals. An early theorem of Mitchell and Silver [14], shows that the tree property at ω_2 is equiconsistent with the existence of a weakly compact cardinal. So if the existence of a weakly compact cardinal is consistent, then it is impossible to construct an ω_2-Aronszajn tree from the usual axioms of set theory. Moreover, the assumption that a weakly compact cardinal is consistent is necessary. This result gives an approach to resolving which cardinals have Aronszajn trees.

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The conjecture is that if the existence of enough large cardinals is consistent, then we cannot prove the statement that for some λ there is λ-Aronszajn tree. A weaker goal which captures many of the difficulties of this conjecture is to show that in ZFC one cannot prove that there is a cardinal λ which carries a *special* λ-Aronszajn tree.

It is this weaker goal that we address in this paper. We prove

Theorem 1.1. *Under suitable large cardinal hypotheses it is consistent that $\aleph_{\omega^2}$ is strong limit and the approachability property fails for every regular cardinal in the interval $[\aleph_2, \aleph_{\omega^2+3}]$.*

As we will mention below, the failure of the approachability property at a cardinal λ is a strengthening of the nonexistence of special λ-Aronszajn trees. So the theorem represents partial progress towards the construction of a model with no special Aronszajn trees on any regular cardinal greater than $\aleph_1$ and hence the weaker goal above.

Our theorem combines two approaches to the problem, which have until now seemed incompatible. The first is a ground up approach where one constructs models where longer and longer initial segments of the regular cardinals carry no special trees (or even have the tree property). The major advances in this approach are due to Abraham [1], Cummings and Foreman [5], Magidor and Shelah [13], Sinapova [21], Neeman [17] and the author [29] for the tree property, and Mitchell [14] and the author [28] for the nonexistence of special trees. This approach cannot continue through the first strong limit cardinal without some changes suggested by the second approach.

The second is an approach for dealing with the successors of singular strong limit cardinals. By Specker's theorem if ν is singular strong limit and there are no special ν^{++}-Aronszajn trees, then $2^\nu > \nu^+$. So the *singular cardinals hypothesis* fails at ν. The singular cardinals hypothesis is an important property in the study of the continuum function on singular cardinals and obtaining a model where it fails requires the existence of large cardinals. Any model for the nonexistence of special trees above $\aleph_1$ must be a model where GCH fails everywhere. Such a model was first constructed by Foreman and Woodin [7] using a complex Radin forcing construction.

For some time, a major problem for the second approach was whether it is consistent to have the failure of SCH at ν and the nonexistence of special Aronszajn trees at ν^+. This was resolved by Gitik and Sharon [9] and their result was later improved by Neeman [16] to give the tree property. Note that by our remarks above such models are required to get the nonexistence of special trees (or the tree property) at ν^+ and ν^{++} where ν is singular strong limit. Further advances in this area are due to Cummings and Foreman [5], the author [27], Sinapova [21, 20, 22] and the author and Sinapova [25].

The main forcing in this paper combines the two approaches outlined above. In particular it combines the ground up approach in [28] with a version of Gitik and Sharon's [9] Prikry type forcing. In the jargon, collapses which enforce the nonexistence of special trees are *interleaved* with the Prikry points. The main difficulty of the paper comes from controlling new collapses as Prikry points are added. This is typically done by leaving some gaps between the Prikry points and the associated collapses. In the current work, we have no such luxury since we wish to control the combinatorics of every regular cardinal below the cardinal which becomes singular.

The paper is organized as follows. In Section 2, we make some preliminary definitions most of which are standard in the study of either singular cardinal combinatorics or compactness properties at double successors. In Section 3, we describe the preparation forcing and derive the measures that we need for the main forcing. In Section 4, we describe the main forcing and prove that it gives the desired cardinal structure. In Section 5, we give a schematic view of an argument for the failure of approachability at a double successor cardinal. In Section 6, we prove that the extension by the main forcing gives the desired failure of approachability. For the double successor cardinals we apply the scheme from the previous section and for the successor of each singular cardinal we apply arguments from singular cardinal combinatorics.

2. Preliminaries

In this section we define the combinatorial notions and forcing posets which are at the heart of the paper. By a theorem of Jensen [10], the existence of a special σ^+-Aronszajn tree is equivalent to the existence of a *weak square sequence* at σ.

Definition 2.1. *A weak square sequence at σ is a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \sigma^+\rangle$ such that*

1. *For all $\alpha < \sigma^+$, $1 \leq |\mathcal{C}_\alpha| \leq \sigma$,*
2. *For all $\alpha < \sigma^+$ and all $C \in \mathcal{C}_\alpha$, C is a club subset of α of ordertype at most σ and*
3. *For all $\alpha < \sigma^+$, all $C \in \mathcal{C}_\alpha$ and all $\beta \in \lim(C)$, $C \cap \beta \in \mathcal{C}_\beta$.*

If there is such a sequence, then we say that *weak square holds* at σ.

In this paper we are interested in the weaker *approachability property* isolated by Shelah [19, 18]. For a cardinal τ and a sequence $\langle a_\alpha \mid \alpha < \tau\rangle$ of bounded subsets of τ, we say that an ordinal $\gamma < \tau$ is *approachable* with respect to $\vec{a}$ if there is an unbounded $A \subseteq \gamma$ such that $\mathrm{otp}(A) = \mathrm{cf}(\gamma)$ and for all $\beta < \gamma$ there is $\alpha < \gamma$ such that $A \cap \beta = a_\alpha$. Using this we can define *the approachability ideal $I[\tau]$*.

Definition 2.2. *$S \in I[\tau]$ if and only if there are a sequence $\langle a_\alpha \mid \alpha < \tau\rangle$ and a club $C \subseteq \tau$ such that every $\gamma \in S \cap C$ is approachable with respect to $\vec{a}$.*

It is easy to see that $I[\tau]$ contains the nonstationary ideal. We say that *the approachability property* holds at τ if $\tau \in I[\tau]$. We note that weak square at σ implies the approachability property at σ^+ and refer the interested reader to [3] for a proof.

In the case where σ is a singular cardinal, the approachability property at σ^+ is connected with the notion of good points in Shelah's PCF theory. We give a brief description of this in the case when $\mathrm{cf}(\sigma) = \omega$, since we will make use of it in the final argument below.

Suppose that $\langle \sigma_n \mid n < \omega\rangle$ is an increasing sequence of regular cardinals cofinal in σ. A sequence of functions $\langle f_\alpha \mid \alpha < \sigma^+\rangle$ is a *scale* of length σ^+ in $\prod_{n<\omega} \sigma_n$ if it is increasing and cofinal in $\prod_{n<\omega} \sigma_n$ under the eventual domination ordering. A remarkable theorem of Shelah is that there are sequences $\langle \sigma_n \mid n < \omega\rangle$ for which scales of length σ^+ exist.

If $\vec{f}$ is a scale of length σ^+ in $\prod_{n<\omega} \sigma_n$, then we say that γ is *good* for $\vec{f}$ if there are an unbounded $A \subseteq \gamma$ and an $N < \omega$ such that for all $n \geq N$, $\langle f_\alpha(n) \mid \alpha \in A\rangle$ is strictly increasing. A scale $\vec{f}$ is *good* if there is a club $C \subseteq \sigma^+$ such that every $\gamma \in C$ with $\mathrm{cf}(\gamma) > \omega$ is good for $\vec{f}$. A scale is *bad* if it is not good. We note that approachability at σ^+ implies that all scales of length σ^+ are good and refer the reader to [3] for a proof.

Each of the principles described above can be thought of as an instance of incompactness. In this paper we will be interested in the negation of these properties. We summarize the implications discussed above, but in terms of the negations.

(1) For all cardinals σ, the failure of approachability at σ^+ implies the failure of weak square at σ.
(2) For all singular cardinals σ, there is a bad scale of length σ^+ implies the failure of approachability at σ^+.

In the arguments below we will either argue for the existence of a bad scale or directly for the failure of approachability. In particular we will have the failure of weak square on a long initial segment of the cardinals or equivalently the nonexistence of special Aronszajn trees on that interval.

We now define some of the forcing posets which will form the collapses between the Prikry points in our main forcing. The poset is essentially due to Mitchell [14] from his original argument for nonexistence of special $\aleph_2$-Aronszajn trees. We use a more flexible notation for this poset developed by Neeman [17].

Definition 2.3. *Let $\rho < \sigma < \tau \leq \eta$ be regular cardinals and let $\mathbb{P} = \mathrm{Add}(\rho, \eta)$. Let $\mathbb{C}(\mathbb{P}, \sigma, \tau)$ be the collection of partial functions f of size less than σ whose domain is a set of successor ordinals contained in the interval (σ, τ) such that for all $\alpha \in \mathrm{dom}(f)$, $f(\alpha)$ is a $\mathbb{P} \restriction \alpha$-name for an element of $\mathrm{Add}(\sigma, 1)_{V[\mathbb{P}\restriction\alpha]}$. We order the poset by $f_1 \leq f_2$ if and only if $\mathrm{dom}(f_1) \supseteq \mathrm{dom}(f_2)$ and for all $\alpha \in \mathrm{dom}(f_2)$, $\Vdash_{\mathbb{P}\restriction\alpha} f_1(\alpha) \leq f_2(\alpha)$.*

Note that such posets are easily 'enriched' in the sense of Neeman to give Mitchell like collapses. The *enrichment* of $\mathbb{C}(\mathbb{P}, \sigma, \tau)$ by $\mathbb{P}$ is the poset defined in the generic extension by $\mathbb{P}$ with the same underlying set as $\mathbb{C}(\mathbb{P}, \sigma, \tau)$, but with the order $f_1 \leq f_2$ if and only if $\mathrm{dom}(f_1) \supseteq \mathrm{dom}(f_2)$ and there is p in the generic for $\mathbb{P}$ such that for all $\alpha \in \mathrm{dom}(f_2)$, $p \restriction \alpha \Vdash f_1(\alpha) \leq f_2(\alpha)$. If P is $\mathbb{P}$-generic we will write $\mathbb{C}^{+P}$ for the enrichment of $\mathbb{C}$ by P. We will drop the P and just write $\mathbb{C}^+$ when it is clear from context which P is required. Note that the poset $\mathbb{C}(\mathbb{P}, \sigma, \tau)$ is σ-closed, τ-cc if τ is inaccessible and collapses every regular cardinal in the interval (σ, τ) to have size σ. Hence it makes τ into σ^+.

For notational convenience we make the following definition.

Definition 2.4. *Let $\rho < \sigma < \tau$ be cardinals. If $\mathbb{P} = \mathrm{Add}(\rho, \tau)$ and $\mathbb{C} = \mathbb{C}(\mathbb{P}, \sigma, \tau)$, then we write $\mathbb{M}(\rho, \sigma, \tau)$ for $\mathbb{P} * \mathbb{C}^+$.*

We note that $\mathbb{M}(\rho, \sigma, \tau)$ essentially gives the family of Mitchell-like posets defined by Abraham [1].

We also need two facts about term forcing. For more complete presentation of term forcing we refer the interested reader to [4]. For completeness we sketch the relevant definitions. Suppose that $\mathbb{P}$ is a poset and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset. Let $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$ be the set of $\mathbb{P}$-names $\dot{q}$ which are forced to be elements of $\dot{\mathbb{Q}}$ with the order $\dot{q}_1 \leq \dot{q}_2$ if $\Vdash_{\mathbb{P}}$ "$\dot{q}_1 \leq \dot{q}_2$ in $\dot{\mathbb{Q}}$".

We note that $\mathbb{P} \times \mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$ induces a generic for $\mathbb{P} * \dot{\mathbb{Q}}$ by taking the upwards closure in the order on $\mathbb{P} * \dot{\mathbb{Q}}$. This means that the extension by any two step iteration has an outer model which is the extension by a product. The $\mathbb{C}$ posets above can be seen as a kind of term forcing and $\mathbb{P} * \mathbb{C}^+$ can be seen as a kind of two step iteration. We will use this idea extensively in the proof.

In certain cases, the term poset can be realized as a nice poset in V.

Fact 2.5. *If $\kappa^{<\kappa} = \kappa$, $\lambda \geq \kappa$ and $\mathbb{P}$ is a κ-cc poset, then $\mathcal{A}(\mathbb{P}, \mathrm{Add}^{V^{\mathbb{P}}}(\kappa, \lambda))$ is isomorphic to $\mathrm{Add}(\kappa, \lambda)$.*

If $\mathbb{A}$ is an iteration, then we can define a term ordering on $\mathbb{A}$ by $a \leq a'$ if the support of a contains the support of a' and for every α in the support of a', it is forced by $\mathbb{A} \restriction \alpha$ that $a(\alpha) \leq a'(\alpha)$. We call this poset $\mathcal{A}(\mathbb{A})$. It is straightforward to see that $\mathcal{A}(\mathbb{A})$ induces a generic for $\mathbb{A}$.

It is straightforward to see that the poset $\mathcal{A}(\mathbb{A})$ has many of the same properties at the iteration. We will need the following fact.

Fact 2.6. *Let κ be a 2-Mahlo cardinal. Suppose that $\mathbb{A}$ is an Easton support iteration of length κ where the set of nontrivial coordinates in the iteration is a stationary set S. If for every $\alpha \in S$, α is Mahlo and it is forced by $\mathbb{A} \restriction \alpha$ that $\mathbb{A}(\alpha)$ is α-closed and a member of V_κ, then $\mathcal{A}(\mathbb{A})$ is κ-cc and preserves the Mahloness of κ.*

3. The preparation forcing

We work in a model V and let κ be a supercompact cardinal. We assume that there is an increasing sequence $\langle \kappa_i \mid i \leq \omega + 3 \rangle$ such that $\kappa_0 = \kappa$, $\kappa_\omega = \sup \kappa_i$, $\kappa_{\omega+1} = \kappa_\omega^+$ and for all i (except ω) κ_{i+1} is the least Mahlo cardinal above κ_i. For simplicity we set $\theta = \kappa_{\omega+3}$. For inaccessible α, we define $\langle \alpha_i \mid i \leq \omega + 3 \rangle$ and $\alpha_{\omega+3} = \theta_\alpha$ as we did for κ.

We fix a supercompactness measure U on $\mathcal{P}_\kappa(\theta)$ and let $j : V \to M$ be the ultrapower map. It is straightforward to show that there is a set $Z \subseteq \kappa$ such that $\kappa \in j(Z)$ and for every $\gamma \in Z$, γ is $\gamma_{\omega+1}$-supercompact and closed under the function $\alpha \mapsto \theta_\alpha$. In particular we have $j(\alpha \mapsto \alpha_i)(\kappa) = \kappa_i$ for all $i \leq \omega + 3$.

We define an iteration with Easton support where we do nontrivial forcing at each $\alpha \in Z$. Suppose that we have defined $\mathbb{A} \restriction \alpha$ for some $\alpha \in Z$. At stage α we force with $(\mathbb{P}_0(\alpha) * \mathbb{C}_0^+(\alpha)) \times (\mathbb{P}_1(\alpha) * \prod_{0<i\leq\omega+1} \mathbb{C}_i^+(\alpha)) \times \mathrm{Add}^{V[\mathbb{A}\restriction\alpha]}(\alpha_1, \theta_\alpha^+ \setminus \theta_\alpha)$ where

1. $\mathbb{P}_0(\alpha) = \mathrm{Add}(\alpha_0, \alpha_2)$ as computed in $V[\mathbb{A} \restriction \alpha]$,
2. $\mathbb{C}_0(\alpha) = \mathbb{C}(\mathbb{P}_0(\alpha), \alpha_1, \alpha_2)$,
3. $\mathbb{P}_1(\alpha) = \mathrm{Add}(\alpha_1, \theta_\alpha)$ as computed in $V[\mathbb{A} \restriction \alpha]$ and
4. for all i with $0 < i \leq \omega + 1$, $\mathbb{C}_i(\alpha) = \mathbb{C}(\mathbb{P}_1(\alpha), \alpha_{i+1}, \alpha_{i+2})$.

We take the product $\prod_{0<i\leq\omega+1} \mathbb{C}_i^+(\alpha)$ with full support.

For ease of notation we let $\mathbb{A} = \mathbb{A} \restriction \kappa$, $\mathbb{P} = \mathbb{P}_0(\kappa) \times \mathbb{P}_1(\kappa)$ and in the extension by $\mathbb{P}$ we let $\mathbb{C}^+ = \prod_{i\leq\omega+1} \mathbb{C}_i^+(\kappa)$. We will now lift j preserving its large cardinal properties precisely with one further addition. Let G be $\mathbb{A}$-generic and let $H = (H_0 * H_1) \times H_2$ be $(\mathbb{P} * \mathbb{C}^+) \times \mathrm{Add}(\kappa_1, \theta^+ \setminus \theta)$-generic over $V[G]$.

Lemma 3.1. *In $V[G * H]$, there is a generic $G^* * H^*$ for $j(\mathbb{A} * (\mathbb{P} * \mathbb{C}^+) \times \mathrm{Add}^{V[\mathbb{A}]}(\kappa_1, \theta^+ \setminus \theta))$ such that j extends to $j : V[G * H] \to M[G^* * H^*]$ witnessing* *that κ is θ-supercompact and for all $\gamma < j(\kappa_1)$, there is a function $f : \kappa_1 \to \kappa_1$ such that $j(f)(\sup j``\kappa_1) = \gamma$.*

Proof. Much of the proof is standard, so we sketch some parts and give details where important. To construct G^* we note

(1) $j(\mathbb{A} \restriction \kappa) \restriction \kappa + 1 = \mathbb{A} \restriction \kappa * ((\mathbb{P} * \mathbb{C}^+) \times \mathrm{Add}(\kappa_1, \theta^+ \setminus \theta))$ and
(2) the poset $j(\mathbb{A})/G * H$ is θ^+-closed in $V[G * H]$ and has just θ^+ antichains in $M[G * H]$.

Standard arguments allow us to build a generic for the tail forcing and thus form G^* which is generic for $j(\mathbb{A})$ over M and such that j lifts to $j : V[G] \to M[G^*]$.

By closure properties of M, $j``(H_0 * H_1) \in M[G^*]$ and is a directed set there of cardinality $\theta$. Moreover, $j(\mathbb{P} * \mathbb{C}^+)$ is $j(\kappa)$-directed closed and hence we can find a master condition for $j``H_0 * H_1$. Another routine counting of antichains allows us to build a generic $H_0^* * H_1^*$ for $j(\mathbb{P} * \mathbb{C}^+)$ over $M[G^*]$ containing our master condition. At this point we can lift j to $j : V[G * (H_0 * H_1)] \to M[G^* * (H_0^* * H_1^*)]$.

The final difficulty is to lift to the extension by $\mathrm{Add}(\kappa_1, \theta^+ \setminus \theta)$. It is here that we will control the values of $j(f)(\sup j``\kappa_1)$ for generic functions f from κ_1 to κ_1. In preparation for this step we let $\langle \eta_\gamma \mid \gamma \in \theta^+ \setminus \theta\rangle$ be an enumeration of $j(\kappa_1)$.

By the usual argument counting antichains we can construct a generic I for $j(\mathrm{Add}(\kappa_1, \theta^+ \setminus \theta))$ over $M[G^* * H_0^* * H_1^*]$. For each γ, let I^γ be the restriction of I to coordinates below $j(\gamma)$. Let H^γ be the natural modification of I^γ which contains the condition $p_\gamma = \bigcup_{p \in j``H_2} p \restriction j(\gamma) \cup \{((j(\beta), \sup j``\kappa_1), \eta_\beta) \mid \beta < \gamma\}$. Since we have only changed a small number of coordinates, H^γ remains generic.

Note that j is continuous at θ^+, so that $j(\mathrm{Add}(\kappa_1, \theta^+ \setminus \theta)) = \bigcup_{\gamma < \theta^+} \mathrm{Add}(\kappa_1, j(\gamma) \setminus j(\theta))$. Moreover the sequence of H^γ is increasing and hence $\bigcup_\gamma H^\gamma$ is generic for $j(\mathrm{Add}(\kappa_1, \theta^+ \setminus \theta))$ over $M[G^* * H_0^* * H_1^*]$ and compatible with j. This is enough to finish the proof. □

For each $n < \omega$ we can derive a supercompactness measure U_n on $\mathcal{P}_\kappa(\kappa_n)$ from j in $V[G * H]$. We let $j_n : V[G * H] \to M_n$ be the ultrapower map and let $k_n : M_n \to M[G^* * H^*]$ be the factor map. By the way that we lifted j, for $n \geq 1$ we have $j(\kappa_1) + 1 \subseteq \mathrm{ran}(k_n)$ and hence $\mathrm{crit}(k_n) > j(\kappa_1)$. This property is essential in the arguments below.

We end the section with a proposition which will help us prove that the relevant cardinals are preserved by the final forcing.

Proposition 3.2. *For every $\alpha \in Z \cup \{\kappa\}$ and every $n \leq \omega + 3$ where n is not ω or $\omega + 1$, there are an outer model W of $V[G * H]$ and posets $\bar{\mathbb{Y}}$ and $\hat{\mathbb{Y}}$ such that*

(1) *W is an extension of V by $\bar{\mathbb{Y}} \times \hat{\mathbb{Y}}$,*
(2) *$\bar{\mathbb{Y}}$ is α_n-Knaster,*
(3) *$\hat{\mathbb{Y}}$ is α_n-closed and*
(4) *there is a generic for $\bar{\mathbb{Y}}$ in $V[G * H]$.*

Proof. Let α and n be as in the proposition. We work through a few cases. First suppose that $n = 0$. It is straightforward to see that the iteration can be broken up as $\mathbb{A} \restriction \alpha$ which is α-cc followed by the rest of the iteration which is forced to be α-closed. So we can set $\bar{\mathbb{Y}}$ to be $\mathbb{A} \restriction \alpha$ and $\hat{\mathbb{Y}}$ to be the poset of $\mathbb{A} \restriction \alpha$-names for elements of the rest of the iteration.

Suppose that $n = 1$. Then there is an outer model of $V[G * H]$ where we have a generic for $\mathcal{A}(\mathbb{A} \restriction \alpha, \mathbb{P}_1(\alpha) \times \prod_{i \leq \omega+1} \mathbb{C}_i) \times \mathcal{A}(\mathbb{A} \restriction \alpha+1, \dot{\mathbb{A}} \restriction [\alpha+1, \kappa+1))$. Note that this poset is α_1-closed in V. So we take it to be $\hat{\mathbb{Y}}$ and set $\bar{\mathbb{Y}}$ to be $\mathbb{A} \restriction \alpha * \mathbb{P}_0(\alpha)$.

Suppose that $n \geq 2$. Then there is an outer model of $V[G * H]$ where we have a generic for $\mathcal{A}(\mathbb{A} \restriction \alpha, \prod_{n-1 \leq i \leq \omega+1} \mathbb{C}_i) \times \mathcal{A}(\mathbb{A} \restriction \alpha + 1, \dot{\mathbb{A}} \restriction [\alpha + 1, \kappa + 1))$. Note that this poset is α_n-closed in V. So we can take it to be $\hat{\mathbb{Y}}$ and set $\bar{\mathbb{Y}}$ to be $\mathbb{A} \restriction \alpha * \mathbb{P}_0(\alpha) * \mathbb{C}_0^+(\alpha) \times \mathbb{P}_1(\alpha) * \prod_{1 \leq i < n-1} \mathbb{C}_i^+$. □

It is immediate that each such α_n is preserved and for all $n \geq 1$, $\alpha_{n+1} = \alpha_n^+$ in $V[G * H]$. Further standard arguments using the above proposition show that α_ω and $\alpha_{\omega+1}$ are preserved for all $\alpha \in Z \cup \{\kappa\}$.

4. The main forcing

In order to define a diagonal Prikry forcing we define a collection of Mitchell-like collapses which will go between the Prikry points. Let $\mathbb{Q}(\alpha, \beta) = \mathbb{Q}^0(\alpha, \beta) \times \mathbb{Q}^1(\alpha, \beta)$ where

$$\begin{aligned}\mathbb{Q}^0(\alpha, \beta) &= \mathrm{Add}(\alpha_{\omega+2}, \beta) * \mathbb{C}^+(\mathrm{Add}(\alpha_{\omega+2}, \beta), \alpha_{\omega+3}, \beta)\\ \mathbb{Q}^1(\alpha, \beta) &= \mathrm{Add}(\alpha_{\omega+3}, \beta_1) * \mathbb{C}^+(\mathrm{Add}(\alpha_{\omega+3}, \beta_1), \beta, \beta_1)\end{aligned}$$

Extremely important to the construction is that we take all these posets as defined in V. The intention is to force β to become $\alpha_{\omega+3}^+$ and β_1 to be β^+. We will be sloppy and write $\mathbb{Q}(x, y)$ for $\mathbb{Q}(\kappa_x, \kappa_y)$.

We are now ready to define the diagonal Prikry poset. Let Z_n be the set of x in $\mathcal{P}_\kappa(\kappa_n)$ such that $\kappa_x = x \cap \kappa \in Z$ and $\mathrm{otp}(x)$ is α_n where $\alpha = \kappa_x$. Clearly $Z_n \in U_n$. For $n < m$, $x \in Z_n$ and $y \in Z_m$, we write $x \prec y$ for $x \subseteq y$ and $|x| < \kappa_y$.

Definition 4.1. *Let $\mathbb{R}$ be a poset where conditions are of the form*

$$\langle q_0, x_0, q_1, x_1, \dots q_{n-1}, x_{n-1}, f_n, F_{n+1}, F_{n+2}, \dots \rangle$$

such that

1. *$n \neq 0$ implies $q_0 \in \mathrm{Coll}(\omega, \alpha_\omega)$ where $\alpha = \kappa_{x_0}$.*
2. *for all $i < n$, $x_i \in Z_i$ and $\vec{x}$ is $\prec$-increasing,*
3. *for all $i \in [1, n)$, $q_i \in \mathbb{Q}(x_{i-1}, x_i)$,*
4. *There is a sequence of measure one sets $\langle A_i \mid i \geq n \rangle$ such that $\mathrm{dom}(f_n) = A_n$ and for all $i \geq n + 1$, $\mathrm{dom}(F_i) = A_{i-1} \times A_i$,*
5. *$n = 0$ implies for all $x \in A_n$, $f_n(x) \in \mathrm{Coll}(\omega, \alpha_\omega)$ where $\alpha = \kappa_x$ and otherwise for all $x \in A_n$, $f_n(x) \in \mathbb{Q}(x_{n-1}, x)$*
6. *for all $i \geq n + 1$ and $(x, y) \in \mathrm{dom}(F_i)$, $F_i(x, y) \in \mathbb{Q}(x, y)$.*

For a condition $p \in \mathbb{P}$ we adorn each part of p with a superscript to indicate its connection with p. For example q_0^p, x_0^p etc. We also let $\ell(p) = n$ denote the length of the condition p. For $p, r \in \mathbb{P}$ we say that $p \leq^* r$ if $\ell(p) = \ell(r) = n$ and

1. *$\vec{x}^p = \vec{x}^r$,*
2. *$\vec{q}^p \leq \vec{q}^r$,*
3. *for all $i \geq n$, $A_i^p \subseteq A_i^r$,*
4. *f_n^p is below f_n^r on A_n^p,*
5. *for all $i \geq n + 1$, $F_i^p \leq F_i^r$ on $A_{i-1} \times A_i$.*

In this case we say that p is a direct extension of r.

For $x \in A^r_{\ell(r)}$, we define the one point extension of r by x, $r \frown x$, to be the condition p of length $\ell(r)+1$ given by

(1) $\vec{x}^p = \vec{x}^r \frown x$,
(2) $\vec{q}^p = \vec{q}^r \frown f^r_{\ell(r)}(x)$,
(3) *for all* $i \geq \ell(p)$, $A^p_i = A^r_i$,
(4) *for all* $i \geq \ell(p)+1$, $F^p_i = F^r_i$ *and*
(5) $f^p_{\ell(p)} = F^r_{\ell(r)+1}(x, -)$.

We let $p \leq r$ if p is obtained from r by a sequence of one point extensions and direct extensions.

It is straightforward to see that every extension $p \leq r$ can be written as a single direct extension of finitely many one point extensions of r. We fix some terminology:

(1) We call the lower part or stem of a condition any sequence of the form $\langle q_0, x_0, q_1, \dots q_{n-1}, x_{n-1}\rangle$ which is an intial segment of some condition. Note that we have not included f_n.
(2) If r is a condition, then we write stem(r) to denote the stem of r.
(3) If $s = \langle q_0, x_0, q_1, \dots q_{n-1}, x_{n-1}\rangle$ is a stem, then we let top$(s) = x_{n-1}$.
(4) We call the one variable functions f^p_n the f-part of p.
(5) We call the sequence of two variable functions $\vec{F}^p$ the upper part or constraint of p.

Remark 4.2. *For all $n < \omega$, there are κ_{n-1} stems of length n. There are at most κ_1 equivalence classes of functions f_n which are the f-part of some condition.*

For each $i > 1$, the class of a function F_i as above modulo $U_{i-1} \times U_i$ is a member of $\mathbb{Q}_i = \mathbb{Q}(\kappa, j_{i-1}(\kappa))$ as computed in the (external) ultrapower of V by $U_{i-1} \times U_i$. This forcing is a product of Mitchell-like collapses below $j_{i-1}(\kappa_1)$ which is the least (formerly) Mahlo cardinal above $j_{i-1}(\kappa)$.

We fix some notation. For $i > 1$, $M^i_{i-1} = \mathrm{Ult}(V[G * H], U_{i-1} \times U_i))$ and $j_{i-1}(M[G^* * H^*)) = \mathrm{Ult}(M_{i-1}, j_{i-1}(U))$. By standard arguments there are elementary embeddings, $k : M^i_{i-1} \to j_{i-1}(M[G^* * H^*])$ and $\hat{k} : j_{i-1}(M[G^* * H^*]) \to \mathrm{Ult}(V[G * H], U \times U)$.

Claim 4.3. $\mathrm{crit}(\hat{k} \circ k) > j(\kappa_1)$

Proof. Note that k can be defined in M_{i-1} as the factor embedding between the ultrapower by $j_{i-1}(U_i)$ and the ultrapower by $j_{i-1}(U)$, $j_{i-1}(k_i)$. By the elementarity of j_{i-1}, k has critical point above $j_{i-1}(j(\kappa_1)) \geq j(\kappa_1)$ by the way that we extended the embedding j.

Note that k_{i-1} maps M_{i-1} to $M[G^* * H^*]$ with critical point above $j_{i-1}(\kappa_1)$. Inside M_{i-1}, $j_{i-1}(M[G^* * H^*])$ is the ultrapower by $j_{i-1}(U)$. Let π be the ultrapower map as defined in M_{i-1}. It follows that in $M[G^* * H^*]$, $k_{i-1}(\pi)$ is the definable ultrapower map in to $\mathrm{Ult}(M[G^* * H^*], j(U)) \simeq \mathrm{Ult}(V[G * H], U \times U)$. Moreover if a is the generator of $j_{i-1}(U)$ in M_{i-1} then the map $\hat{k}$ is given by $\pi(f)(a) \mapsto k_{i-1}(\pi)(k_{i-1}(f))(k_{i-1}(a))$. We note that this map is exactly k_{i-1} applied on $j_{i-1}(M[G^* * H^*])$ and hence the critical point of $\hat{k}$ is at least the critical point of k_{i-1} which is at least $j(\kappa_1)$ by the way that we lifted the embedding. It follows that the composition $\hat{k} \circ k$ has critical point above $j(\kappa_1)$. $\square$

Corollary 4.4. *For $i > 1$, $\mathbb{Q}_i$ is $\mathbb{Q}(\kappa, j(\kappa))$ as computed in M.*

Proof. Recall that $\mathbb{Q}_i$ is $\mathbb{Q}(\kappa, j_{i-1}(\kappa))$ as computed in the external ultrapower of V by $U_{i-1} \times U_i$. Of course using the high critical point of k_{i-1}, $j_{i-1}(\kappa) = j(\kappa)$. Moreover, by the previous lemma, $(\hat{k} \circ k)(\mathbb{Q}_i) = (\hat{k} \circ k)``\mathbb{Q}_i$ and $(\hat{k} \circ k) \restriction \mathbb{Q}_i$ is the identity map. Here we use that $j_{i-1}(\kappa_1) = j(\kappa_1)$ is inaccessible in the inner model where we compute $\mathbb{Q}_i$ and hence $k \circ \hat{k}$ fixes a bijection between $j(\kappa_1)$ and $V_{j(\kappa_1)}$ as computed in this model. It follows that $\mathbb{Q}_i$ is $\mathbb{Q}(\kappa, j(\kappa))$ as computed in $\mathrm{Ult}(V, U \times U)$. However this ultrapower is highly closed inside $M = \mathrm{Ult}(V, U)$ and the conclusion follows. $\square$

So we have shown that each $\mathbb{Q}_i$ for $i \geq 1$ is in fact the same forcing and we drop the dependence on i and just write $\mathbb{Q}$. In the course of the proof of the Prikry lemma below the forcing $\mathbb{Q}$ will be represented in different ways in different ultrapowers. For clarity we record the following remark.

Remark 4.5. *For each $i > 1$, $\mathbb{Q}$ is isomorphic to the set $\{f \in M_{i-1} \mid \mathrm{dom}(f) \in j_{i-1}(U_i)$ and for all $x \in \mathrm{dom}(f)$, $f(x) \in \mathbb{Q}(\kappa, \kappa_x)$ as computed the external ultrapower of V by $U_{i-1}\}$ with the natural ordering. These functions f are the ones that represent elements of $\mathbb{Q}_i$ in the ultrapower by $j_{i-1}(U_i)$. We note that they can also be represented as elements of M_{i-1} using functions from $V[G * H]$ as $j_{i-1}(F)(j_{i-1}``\kappa_{i-1})$ where $F = F_i^p$ for some condition p.*

Next we work towards showing that forcings $\mathbb{Q}$ and $\mathbb{Q}(\alpha, \beta)$ are well-behaved.

Claim 4.6. *The full support power $\mathbb{Q}^\omega$ is $< \kappa_{\omega+2}$-distributive in $V[G * H]$.*

Proof. $\mathbb{Q}$ is defined in M (hence V) and is $\kappa_{\omega+2}$-closed in V by the closure of M. Hence $\mathbb{Q}^\omega$ is $\kappa_{\omega+2}$ closed in V. Let W, $\bar{\mathbb{Y}}$ and $\hat{\mathbb{Y}}$ be the outer model and posets from Proposition 3.2 applied to $\kappa_{\omega+2}$. By Easton's lemma, every $< \kappa_{\omega+2}$-sequence from $W^{\mathbb{Q}^\omega}$ is in $V^{\bar{\mathbb{Y}}}$, but $V[G * H]$ contains a generic for $\bar{\mathbb{Y}}$. So we are done. $\square$

A similar argument establishes the following claim.

Claim 4.7. *For all $\alpha < \beta \leq \kappa$, $\mathbb{Q}(\alpha, \beta)$ is $< \alpha_{\omega+2}$-distributive in $V[G * H]^{\mathbb{Q}^\omega}$.*

Remark 4.8. *It is clear from the proof above that the conclusion of the previous claim holds in any forcing extension by a poset from V of size less than $\alpha_{\omega+1}$. The small poset can be included in $\bar{\mathbb{Y}}$.*

We pause here to prove that the posets $\mathbb{Q}(\alpha, \beta)$ have the desired effect on the cardinals between α and β.

Proposition 4.9. *In any extension of $V[G * H]$ by a poset of size less than $\alpha_{\omega+1}$ from V, $\mathbb{Q}(\alpha, \beta)$ is β_1-cc, preserves the cardinals $\alpha_{\omega+3}$ and β and forces $\beta = \alpha^+_{\omega+3}$ and $\beta_1 = \beta^+$.*

Proof. Let $\mathbb{Y}$ be a poset of size less than $\alpha_{\omega+1}$ in V. First we consider the outer model W and posets $\bar{\mathbb{Y}}$ and $\hat{\mathbb{Y}}$ from Proposition 3.2 applied to β_1. In V, the product $\mathbb{Y} \times \bar{\mathbb{Y}} \times \mathbb{Q}(\alpha, \beta)$ is β_1-Knaster. By Easton's lemma, it follows that $\mathbb{Q}(\alpha, \beta)$ is β_1-cc in $W^{\mathbb{Y}}$ and hence in $V[G * H]^{\mathbb{Y}}$.

Second we consider the outer model W and posets $\bar{\mathbb{Y}}$ and $\hat{\mathbb{Y}}$ from Proposition 3.2 applied to β. Using the definition of the $\mathbb{C}$ posets, there is an outer model of $V^{\mathbb{Q}(\alpha,\beta)}$ which is an extension by the partial order $\mathbb{Q}^0(\alpha, \beta) \times \mathrm{Add}(\alpha_{\omega+3}, \beta_1) \times$

$\mathbb{C}(\mathrm{Add}(\alpha_{\omega+3},\beta_1),\beta,\beta_1)$. Hence $W^{\mathbb{Q}(\alpha,\beta)}$ is contained in a generic extension of V by the product of $\mathbb{Y}\times\bar{\mathbb{Y}}\times\mathbb{Q}^0(\alpha,\beta)\times\mathrm{Add}(\alpha_{\omega+3},\beta_1)$ and $\hat{\mathbb{Y}}\times\mathbb{C}(\mathrm{Add}(\alpha_{\omega+3},\beta_1),\beta,\beta_1)$. The first piece is β-cc and the second is β-closed. Hence β is preserved in $V[G*H]^{\mathbb{Y}\times\mathbb{Q}(\alpha,\beta)}$.

The argument that $\alpha_{\omega+3}$ is preserved is similar to the argument that β is preserved, but we split up $\mathbb{Q}^0(\alpha,\beta)$ instead of $\mathbb{Q}^1(\alpha,\beta)$ and incorporate $\mathbb{Q}^1(\alpha,\beta)$ into the closed part.

The argument that cardinals in the intervals $(\alpha_{\omega+3},\beta)$ and (β,β_1) are collapsed is standard for Mitchell type posets. $\square$

Claim 4.10. *Forcing with $\mathbb{Q}^\omega$ over $V[G*H]$ preserves cardinals up to $\kappa_{\omega+3}$.*

Proof. By Claim 4.6, it is enough to show that $\kappa_{\omega+3}$ is preserved. Recall that $\mathbb{Q}$ is computed in an ultrapower of V which is closed under $\theta=\kappa_{\omega+3}$-sequences. In particular, it can be written as the projection of a product $\mathrm{Add}(\kappa_{\omega+2},j(\kappa))$ which is $\kappa_{\omega+3}$-cc in V and $\mathbb{C}(\mathrm{Add}(\kappa_{\omega+2},j(\kappa)),\kappa_{\omega+3},j(\kappa))\times\mathbb{Q}^1(\kappa,j(\kappa))$ which is $\kappa_{\omega+3}$-closed in V. Since the iteration to add $G*H$ is $\kappa_{\omega+3}$-cc, we have that $\kappa_{\omega+3}$ is preserved when we force with $\mathbb{Q}^\omega$. $\square$

We are now ready to prove the Prikry lemma. The main elements come from a combination of [2] and [23]. Suppose that we force with $\mathbb{Q}^\omega$ to obtain $\vec{K}=\langle K_n \mid n>1\rangle$. We let $\bar{\mathbb{R}}$ be the set of conditions r such that $\langle [F^r_n] \mid n\geq\max(\ell(p),2)\rangle\in\prod_{n\geq\max(\ell(p),2)}K_n$ ordered as a suborder of $\mathbb{R}$. The following claims are straightforward.

Claim 4.11. *In $V[G*H]^{\mathbb{Q}^\omega}$, $\bar{\mathbb{R}}$ is κ_ω-centered below any condition of length at least one.*

Note that in $\bar{\mathbb{R}}$, conditions of length at least one with the same stem and (equivalence class of) f-part are compatible and there are at most κ_ω such pairs.

Claim 4.12. *$\mathbb{Q}^\omega*\bar{\mathbb{R}}$ projects to $\mathbb{R}$.*

The definition of the map is clear and the fact that the map preserves the length of the condition in $\mathbb{R}$ allows us to prove the Prikry lemma for $\bar{\mathbb{R}}$ in place of $\mathbb{R}$.

Lemma 4.13. *Work in $V[G*H][\vec{K}]$. Let $r\in\bar{\mathbb{R}}$ be a condition of length at least 1 and φ be a statement in the forcing language for $\bar{\mathbb{R}}$. There is an $r^*\leq^* r$ which decides φ.*

It is immediate from the lemma that $\mathbb{R}$ satisfies the Prikry property. The following claims all take place in $V[G*H][\vec{K}]$. We note that the distributivity of $\vec{K}$ implies that the measures U_n remain measures and hence their ultrapower maps lift trivially to this extension. For ease of notation, we denote this slightly larger ultrapower by M_n and note that $j_n(\bar{\mathbb{R}})$ can be defined there.

In the following claims, we work with conditions in $\bar{\mathbb{R}}$ and its images in various ultrapowers.

Claim 4.14. *There is an $r_0\leq^* r$ such that for all $p\leq r_0$ if p is at least a one point extension of r_0 and it decides φ, then there is an upper part $\vec{F}$ such that $\mathrm{stem}(p)\frown F^{r_0}_{\ell(p)}(x^p_{\ell(p)-1})\frown\vec{F}$ decides φ.*

Proof. We go by induction on the length of extensions of r. Let $r^{\ell(r)} = r$. Assume that we have constructed r^n for some $n < \omega$. Work in the ultrapower M_n and consider conditions of length $n+1$ of the form

$$s \frown (q, j_n``\kappa_n) \frown (f^+, \vec{F}^+)$$

which are below $j_n(r^n)$.

Note that the collection of possible stems s here is exactly the collection of j_n pointwise images of stems of length n from $\mathbb{R}$, since $\mathrm{top}(s) \prec j_n``\kappa_n$ implies that $\mathrm{top}(s) = j_n``x$ for some x in $\mathcal{P}_\kappa(\kappa_{n-1})$. Further, $q \in \mathbb{Q}(\kappa_x, \kappa)$ where x is as above, which has size κ_1. It follows that there are at most κ_{n-1} many such pairs s, q.

Note that for $f^+ \leq j_n(F^{r^n}_{n+1})(j_n``\kappa_n)$ in $M_n$, there is a function $F^*$ such that $f^+ = j_n(F^*)(j_n``\kappa_n)$.

For each s and q, the set $D_{s,q}$ defined as

$$\{f^+ \mid \text{if } \exists \vec{F}^+, \exists f^{++} \leq f^+ \text{ such that } s \frown (q, j_n``\kappa_n) \frown (f^{++}, \vec{F}^+) \text{ decides } j_n(\varphi)$$
$$\text{then } \exists \vec{F}^{++} s \frown (q, j_n``\kappa_n) \frown (f^+, \vec{F}^{++}) \text{ decides } j_n(\varphi)\}$$

is dense open in $\mathbb{Q}_{n+1}$, which is just $\mathbb{Q}$. Here we use Remark 4.5 to see that the natural forcing defined from functions f^+ as above is isomorphic to $\mathbb{Q}_{n+1} = \mathbb{Q}$. In the definition of $D_{s,q}$, since we only consider extension of conditions of length $n+1$ in $j_n(\bar{\mathbb{R}})$, we only need a tail of $\vec{K}$ to define this restricted poset. Hence $D_{s,q}$ can be defined in the model $V[G * H][\langle K_i \mid i > n+1\rangle]$ where $\mathbb{Q}$ is $< \kappa_{\omega+2}$-distributive. So the set $D = \bigcap_{s,q} D_{s,q}$ is dense open in $\mathbb{Q}$. Since D can be defined in $V[G * H][\langle K_i \mid i > n+1\rangle]$ and by the product lemma K_{n+1} is generic for $\mathbb{Q}$ over this model, we can take a function F_{n+1} so that $j_n(F_{n+1})(j_n``\kappa_n) \in K_{n+1} \cap D$. We can assume that $F_{n+1} \leq F^{r^n}_{n+1}$ on a $U_n \times U_{n+1}$ large set.

By Los' theorem the set A_n given by

$$\{x \mid \forall s \text{ if } \mathrm{top}(s) = x, \exists \vec{F}^+ \exists f^{++} \leq F_{n+1}(x),\ s \frown (f^{++}, \vec{F}^+) \text{ decides } \varphi$$
$$\text{then } \exists \vec{F}^{++},\ s \frown (F_n(x), \vec{F}^{++}) \text{ decides } \varphi\}$$

is U_n measure one. We define r^{n+1} by refining $F^{r^n}_{n+1}$ to $F_{n+1} \restriction A$ and leaving the rest of r^n unchanged.

We let r_0 be a lower bound for $\langle r^n \mid n \geq \ell(r)\rangle$. It is straightforward to check that r_0 satisfies the conditions of the claim. $\square$

Claim 4.15. *There is an $r_1 \leq^* r_0$ such that if $p \leq r_1$ is at least a one point extension and it decides φ, then $\mathrm{stem}(p) \frown F^{r_1}_{\ell(p)}(x^p_{\ell(p)-1}) \frown \vec{F}^{r_1}$ decides φ.*

Proof. We collect witnesses to the previous claim. Let s be a stem of a condition which is at least a one point extension of r_0. If possible we select an upper part $\vec{G}^s$ witnessing that the condition with stem s from the previous claim decides φ. Using the distributivity of $\mathbb{Q}^\omega$ (in particular that the sequence of generics $\vec{K}$ is closed) and the fact that for each n there are just κ_n many stems of length n, for each $n \geq \ell(r_0)+1$ we can find $\vec{G}^n$ such that for all $k \geq n$ and all s of length n, $[G^n_k] \leq [G^s_k]$. It is straight forward to find a sequence $\langle G_k \mid k \geq \ell(r_0)+1\rangle$ such that for all $n \geq \ell(r_0)+1$, $[G_k] \leq [G^n_k]$.

For each stem s there is a sequence of measure one sets $\vec{A}^s$ on which $\vec{G} \restriction [\ell(s), \omega)$ is below $\vec{G}^s$. We can assume that the sequence $\vec{A}^s$ is contained (pointwise) in the sets which form the domains of $\vec{G}^s$. By a standard diagonal intersection argument there is a sequence of measure one sets $\langle A_n \mid n \geq \ell(r_0)+1\rangle$ such that for all $x \in A_n$ if s is a stem with $s \prec x$, then $x \in A^s_n$.

We obtain r_1 by for all $n \geq \ell(r_0)+1$ restricting $F^{r_0}_n$ to $G_n \restriction A_n$. It is straightforward to check that this condition satisfies the claim. $\square$

Claim 4.16. *There is a condition $r_2 \leq^* r_1$ such that if $p \leq r_2$ is at least a two point extension and it decides φ, then*

$$\mathrm{stem}(p) \restriction \ell(p)-1 \frown (F^{r_2}_{\ell(p)-1}(x^p_{\ell(p)-2}, x^p_{\ell(p)-1}), x_{\ell(p)-1}) \frown (F^{r_2}_{\ell(p))}(x^q_{\ell(p)-1}), \vec{F}^{r_2})$$

decides φ.

Proof. We work by induction on the lengths of possible extensions of r_1 of at least two points. Let $r_1 = r^{\ell(r)}$ and assume that we have constructed r^n for some n. We work in M^{n+1}_n and consider conditions of length $n+2$ of the form:

$$s \frown (x^*, q, y^*)) \frown (j^{n+1}_n(F^{r^n}_{n+2})(y^*), j^{n+1}_n(\vec{F}^{r^n} \restriction [n+3,\infty)))$$

where $x^* = j^{n+1}_n``\kappa_n$, $y^* = j_n(j_{n+1})``j_n(\kappa_{n+1})$, $q \in \mathbb{Q}$ and s is a stem of length n from $\mathbb{R}$ adjoined with an element of $\mathbb{Q}(\kappa_{\mathrm{top}(s)}, \kappa)$. Recall that there are just κ_{n-1} stems of length n in $\mathbb{R}$ and there are just κ_1 possible top collapsing conditions. We denote this condition above $r_{s,q}$.

For each such s, the set D_s

$$\{q \mid r_{s,q} \text{ decides } j^{n+1}_n(\varphi) \text{ or for no extension } q' \text{ of } q \text{ does } r_{s,q'} \text{ decide } j^{n+1}_n(\varphi)\}$$

is dense in $\mathbb{Q}$ and defined in $V[G*H][\langle K_m \mid m > n+1\rangle]$. Using the distributivity of $\mathbb{Q}$ in this model, the set $D = \bigcap_{\ell(s)=n} D_s$ is dense in $\mathbb{Q}$. So we can find a function F_{n+1} such that $[F_{n+1}] \in D \cap K_{n+1}$ with $F_{n+1} \leq F^{r^n}_{n+1}$.

Reflecting the configuration from M^{n+1}_n, we let $r_{s,x,q,y}$ be the natural condition with stem given by $s \frown (x,q,y)$ and constraints given by those from r^n.

By Los' theorem the set

$$\{(x,y) \mid r_{s,x,F_{n+1}(x,y),y} \text{ decides } \varphi \text{ or there is no } q \leq F_{n+1}(x,y) \text{ for which } r_{s,x,q,y} \text{ decides } \varphi\}$$

is measure one for $U_n \times U_{n+1}$. We fix measure one sets A^s_n, A^s_{n+1} so that every $(x,y) \in A^s_n \times A^s_{n+1}$ with $x \prec y$ is in the above set. We let A_n, A_{n+1} be appropriate diagonal intersections and refine r^n to r^{n+1} by replacing $F^{r^n}_{n+1}$ with $F_{n+1} \restriction A_n \times A_{n+1}$.

At the end of the construction we let r_2 be a lower bound for r^n for $n \geq \ell(r_1)$. It is straightforward to check that r_2 has the desired property. $\square$

For a stem s of length $n \geq \ell(r_2)+1$, we partition $A^{r_2}_n$ in to three sets

$$A^0_s = \{x \mid s \frown (F^{r_2}_n(\mathrm{top}(s),x),x) \frown (F^{r_2}_{n+1}(x), \vec{F}^{r_2}) \Vdash \varphi\}$$
$$A^1_s = \{x \mid s \frown (F^{r_2}_n(\mathrm{top}(s),x),x) \frown (F^{r_2}_{n+1}(x), \vec{F}^{r_2}) \Vdash \neg\varphi\}$$
$$A^2_s = \{x \mid s \frown (F^{r_2}_n(\mathrm{top}(s),x),x) \frown (F^{r_2}_{n+1}(x), \vec{F}^{r_2}) \text{ does not decide } \varphi\}$$

Let A_s be the unique set above which is U_n-measure one. Let r_3 be obtained by restricting the measure one sets of r_2 to the diagonal intersections of the A_s.

Let $p^* \leq r_3$ be an extension of minimal length deciding φ.

Claim 4.17. $\ell(p^*) \leq \ell(r_3)+1$

Proof. Suppose not. Then p^* is at least a two point extension of r_3. Let $n = \ell(p^*)-1$ and s be $\mathrm{stem}(p^*) \restriction n$ and let $x = x_n^{p^*}$. From our previous claims, the condition

$$s \frown (F_n^{r_3}(\mathrm{top}(s), x), x) \frown (F_{n+1}^{r_3}(x), \vec{F}^{r_2})$$

decides φ. Without loss of generality we assume that it forces φ. It follows that $A_s = A_s^0$ and hence $s \frown (F_n^{r_3}(\mathrm{top}(s)), \vec{F}^{r_3})$ forces φ, a contradiction to the minimality of the length of p^*. □

Claim 4.18. *There is a direct extension of* r_3 *which decides* φ.

Proof. For each $x \in A_{\ell(r_3)}^{r_3}$, we use the distributivity of $\mathbb{Q}(x_{\ell(r)-1}^r, x)$ to record a condition $q_x \leq f_{\ell(r_3)}^{r_3}(x)$ such that for all relevant stems s the condition

$$r(s, x) = s \frown (q_x, x) \frown (F_{\ell(r_3)}^{r_3}(x), \vec{F}^{r_3})$$

either decides φ or no extension of q_x in the above condition decides φ.

For each stem s, there is a measure one set A_s of x which all give the same decision above relative to s. Let A be the diagonal intersection of the A_s. Let r_4 be obtained from r_3 by restricting $f_{\ell(r_3)}^{r_3}$ to the function $x \mapsto q_x$ on A. Now by the previous lemma there is a one point extension p of r_4 which decides φ. Without loss of generality we assume it forces φ. Let s be $\mathrm{stem}(p) \restriction \ell(p) - 1$. By the above construction, $r(s, x)$ forces φ for all $x \in A_s$. It follows that the condition $s \frown (f_{\ell(r_4)}^{r_4}, \vec{F}^{r_3})$ is a direct extension of r_3 forcing φ. □

This finishes the proof of the Prikry lemma.

Corollary 4.19. *In the extension by* $\mathbb{R}$, $\kappa = \aleph_{\omega^2}$ *and if* $\lambda_n = x_n \cap \kappa$ *and* $\lambda_{n,i} = (\alpha \mapsto \alpha_i)(\lambda_n)$, *then* $\lambda_{n,i}$ *is preserved for all* $n < \omega$ *and* $i \leq \omega + 3$.

Proof. By Remark 4.8 and Proposition 4.9, it is enough to show that if $\dot{X}$ is an $\mathbb{R}$-name for a subset of some $\mu < \kappa$ and r is a condition with $\mu < \lambda = \kappa \cap \mathrm{top}(\mathrm{stem}(r))$, then r forces $\dot{X}$ is in the extension by $\prod_{i<\ell(r)} \mathbb{Q}(x_{i-1}^r, x_i^r)$.

By the Prikry Lemma, $\dot{X}$ can be interpreted by an $(\mathbb{R}, \leq^*)$-generic below the condition r. By an argument similar to Claim 4.7, the forcing $(\mathbb{R}, \leq^*)$ below r decomposes in to the product of $\prod_{i<\ell(r)} \mathbb{Q}(x_{i-1}^r, x_i^r)$ and a poset $\hat{\mathbb{Y}}$ where $\hat{\mathbb{Y}}$ is $< \lambda_{\omega+2}$-distributive in the extension by $\prod_{i<\ell(r)} \mathbb{Q}(x_{i-1}^r, x_i^r)$.

So we take a direct extension r^* of r which forces (in the sense of $(\mathbb{R}, \leq^*)$) that the interpretation of $\dot{X}$ is in the extension by $\prod_{i<\ell(r)} \mathbb{Q}(x_{i-1}^r, x_i^r)$. Hence r^* forces the same as a condition in $\mathbb{R}$. □

Recall that $\mathbb{Q}^\omega$ is the full support product. Clearly both $\mathbb{R}$ and $\mathbb{Q}^\omega$ project to $\mathbb{Q}^\omega/fin$. Let I be the $\mathbb{Q}^\omega/fin$-generic induced by $\vec{K}$.

Claim 4.20. $\mathbb{R}/I$ *has the* $\kappa_{\omega+1}$*-Knaster property.*

Proof. Work in $V[G * H][I]$ and let $\langle r_\alpha \mid \alpha < \kappa_{\omega+1} \rangle$ be a sequence of elements of $\mathbb{R}/I$. We can assume that each r_α has some fixed length l.

Let $\langle [F_i] \mid i < \omega \rangle / fin$ be a condition in I forcing this. By the distributivity of $\mathbb{Q}^\omega/fin$ we can assume that the F_i have the property that for each α there is an n_α such that for all $i \geq n_\alpha$, $[F_i] \leq [F_i^{r_\alpha}]$. By passing to an unbounded subset of the r_α, we can assume there is n^* such that $n^* = n_\alpha$ for all $\alpha < \kappa_{\omega+1}$. Further, extending each r_α if necessary we can assume that $l = \ell(r_\alpha) \geq n^*$. By passing to a further unbounded subset, we can assume that for all $i < l$, $x_i^{r_\alpha} = x_i^{r_\beta}$ for all α and β.

Now for each α, $r_\alpha \restriction l+1$ essentially comes from the poset $\prod_{i<l} \mathbb{Q}(x_{i-1}^r, x_i^r) \times \mathbb{Q}(x_{l-1}, j_l``\kappa_l)$ where the latter forcing is computed in (an inner model of) M_l. This forcing has cardinality less than κ_ω and hence we can find an unbounded set of α on which any r_α and r_β are compatible. □

Corollary 4.21. *$\mathbb{R}$ preserves the cardinals $\kappa_{\omega+1}, \kappa_{\omega+2}$ and $\kappa_{\omega+3}$.*

By Claim 4.10 all three cardinals are preserved in $V[I]$ which is an inner model of an extension by $\mathbb{Q}^\omega$ and by the previous claim they are preserved in the extension by $\mathbb{R}$.

5. A schematic view of arguments for the failure of approachability at double successors

In this section we give an abstract overview of arguments for the failure the approachability property at double successor cardinals. Before we begin, note that none of the cardinals and posets in this section bear any relation to those defined elsewhere in the paper.

We begin remarking that the approachability property is upwards absolute to models with the same cardinals. So to prove that it fails in some model, it is enough to show that it fails in an outer model with the same cardinals. We formalize this in the following remark.

Remark 5.1. *Suppose that $W \subseteq W'$ are models of set theory and λ is a regular cardinal in W'. If $\lambda \notin I[\lambda]$ in W', then $\lambda \notin I[\lambda]$ in W.*

We will also use a theorem of Gitik and Krueger [8] which allows us to preserve the failure of approachability in some outer models.

Theorem 5.2. *Suppose that $\lambda = \mu^{++}$ and $\mathbb{P}$ is μ-centered. If $\lambda \notin I[\lambda]$ in V, then it is forced by $\mathbb{P}$ that $\lambda \notin I[\lambda]$.*

The bulk of this section is devoted to giving an abstract view of the failure of approachability in the extension by the Mitchell posets as described in Section 2. In particular we need to show that these posets have approximation properties.

Definition 5.3 (Hamkins)**.** *Let $\mathbb{P}$ be a poset and κ be a cardinal. We say that $\mathbb{P}$ has the* κ-approximation property *if for every ordinal μ and every name $\dot{x}$ for a subset of μ, if for all $z \in \mathcal{P}_\kappa(\mu)$ $\Vdash_{\mathbb{P}} \dot{x} \cap z \in V$, then $\Vdash_{\mathbb{P}} \dot{x} \in V$.*

We consider the following general situation. Let $\rho < \sigma < \tau$ be regular cardinals with τ Mahlo and let $\mathbb{M} = \mathbb{M}(\rho, \sigma, \tau)$ as defined in Section 2. Let $\mathbb{X}$ be a poset such that for all $\alpha \leq \tau$, $\mathbb{X}$ is α-cc in the extension by $\mathbb{M} \restriction \alpha$. (Here $\mathbb{M} \restriction \tau = \mathbb{M}$.) Suppose that $\langle \dot{a}_\gamma \mid \gamma < \tau \rangle$ and $\dot{C}$ are $\mathbb{M} \times \mathbb{X}$-names for witnesses that $\tau \in I[\tau]$.

We assume that there are a club D and subforcings $\mathbb{X} \restriction \alpha$ of $\mathbb{X}$ for $\alpha \in D \cup \{\tau\}$ such that for all $\alpha \in D \cup \{\tau\}$, $\langle \dot{a}_\gamma \mid \gamma < \alpha \rangle$ is in the extension by $\mathbb{M} \restriction \alpha \times \mathbb{X} \restriction \alpha$. By the τ-cc of $\mathbb{M} \times \mathbb{X}$ we can assume that $\dot{C}$ is in V, so we rename it C. Since τ is Mahlo, there is an inaccessible α in $D \cap C$.

It follows that $\alpha = \sigma^+$ in the extension by $\mathbb{M} \restriction \alpha \times \mathbb{X} \restriction \alpha$ and $\alpha \in I[\alpha]$ as witnessed by $\langle a_\gamma \mid \gamma < \alpha \rangle$ and $C \cap \alpha$. Since α is approachable in the extension by $\mathbb{M} \times \mathbb{X}$, there is an $\mathbb{M}/\mathbb{M} \restriction \alpha \times \mathbb{X}/\mathbb{X} \restriction \alpha$-name $\dot{A}$ for a subset of α of ordertype σ such that for all $\delta < \alpha$, $\dot{A} \cap \delta = a_\gamma$ for some $\gamma < \alpha$.

Since forcing with $\mathbb{X}$ over the extension by $\mathbb{M} \restriction \alpha$ preserves α, for the failure of approachability at τ it is enough to show that $\mathbb{M}/\mathbb{M} \restriction \alpha$ has the λ-approximation property in the extension by $\mathbb{X} \times \mathbb{M} \restriction \alpha$ for some $\lambda \leq \sigma$.

In the arguments for the failure of approachability below the choice of $\mathbb{X}$ will vary, but we can prove a single lemma which captures all of the different choices. We say that a poset $\mathbb{X}$ preserves the λ-cc of a poset $\mathbb{P}$ if $\mathbb{P}$ is λ-cc in the extension by $\mathbb{X}$.

Lemma 5.4. *Let $\rho < \sigma < \tau$ be cardinals and let $\mathbb{M} = \mathbb{M}(\rho, \sigma, \tau)$. Let $\lambda \leq \sigma$ be a cardinal and let $\mathbb{X}$ be a poset such that*

(1) *for all $\alpha \leq \tau$, $\mathbb{X}$ is α-cc in the extension by $\mathbb{M} \restriction \alpha$ and*
(2) *$\mathbb{X} \simeq \bar{\mathbb{X}} \times \hat{\mathbb{X}}$ where $\bar{\mathbb{X}}$ is λ-cc and $\hat{\mathbb{X}}$ is $< \lambda$-distributive and preserves the λ-cc of $\mathrm{Add}(\rho, \tau) \times \bar{\mathbb{X}}$.*

Then in the extension by $\mathbb{M} \restriction \alpha \times \mathbb{X}$, $\mathbb{M}/\mathbb{M} \restriction \alpha$ has the λ-approximation property.

The lemma is immediate from the proof of Lemma 4.1 of [25]. Unfortunately the proof in [25] has a number of mistakes which are correctable but require additional work. We produce a direct proof of the lemma above, which avoids the complications in [25] with quotients of Prikry type forcing.

Proof. By standard facts about the Mitchell forcing (see [1]), in $V[\mathbb{M} \restriction \alpha]$, $\mathbb{M}/\mathbb{M} \restriction \alpha$ can be written as the projection of a product $\mathbb{P} \times \mathbb{Q}$ where $\mathbb{P}$ is isomorphic to $\mathrm{Add}^V(\rho, \tau)$ and $\mathbb{Q}$ is σ-closed. Moreover, the identity map on $\mathbb{P} \times \mathbb{Q}$ is a projection to $\mathbb{M}/\mathbb{M} \restriction \alpha$. We note that by the assumptions of the lemma, these properties are preserved in $V[\mathbb{M} \restriction \alpha \times \hat{\mathbb{X}}]$.

For ease of notation, we let W be $V[\mathbb{M} \restriction \alpha \times \hat{\mathbb{X}}]$ and work over this model. Let $\dot{A}$ be an $\bar{\mathbb{X}} \times \mathbb{M}/\mathbb{M} \restriction \alpha$-name for a set of ordinals all of whose intersections with sets of size less than λ from $W[\bar{\mathbb{X}}]$ are elements of $W[\bar{\mathbb{X}}]$.

Claim 5.5. *In $W[\bar{\mathbb{X}}]$, there is a condition $(p, f) \in \mathbb{M}/\mathbb{M} \restriction \alpha$ such that for any $(p_0, f_0) \leq (p, f)$ if (p_0, f_0) decides $\dot{A} \cap z$ for some z, then (p, f_0) decides $\dot{A} \cap z$.*

Proof. Suppose otherwise. Then for any (p, f) there are $(p_0, f_0) \leq (p, f)$ and z such that (p_0, f_0) decides $\dot{A} \cap z$, but (p, f_0) does not. We can extend (p, f_0) to (p_1, f_1) which decides $\dot{A} \cap z$ to be a different value from the one given by (p_0, f_0). By a standard construction on names we can find f^* such that for $i \in 2$, p_i forces that f^* is below f_i and $f^* \leq f$ in $\mathbb{Q}$.

We fall back to W and construct antichains A_α in $\bar{\mathbb{X}}$, conditions $f_\alpha \in \mathbb{Q}$, functions $H_\alpha : A_\alpha \to \mathbb{P}^2$ and sets z_α as follows. Suppose that we have constructed each of the above for all $\alpha < \beta$. Let f^* be a lowerbound for $\langle f_\alpha \mid \alpha < \beta \rangle$ in $\mathbb{Q}$. This is possible by the closure of $\mathbb{Q}$ in W. Let $z^* = \bigcup_{\alpha < \beta} z_\alpha$. Using the closure of $\mathbb{Q}$ and the chain condition of $\bar{\mathbb{X}}$, construct a condition $f^{**} \leq f^*$ and a maximal antichain B_α such that for all $x \in B_\alpha$, there is $p_x \in \mathbb{P}$ such that x forces that (p_x, f^{**}) decides the value of $\dot{A} \cap z^*$.

Again using the closure of $\mathbb{Q}$ and the chain condition of $\bar{\mathbb{X}}$, we find $f^{***} \leq f^{**}$ and a refinement of B_α as follows. For each $y \in B_\alpha$, we find a maximal antichain A_y below y such that for any $x \in A_y$ there are (p_0^x, p_1^x) each below p_y and z_x such that x forces that (p_0^x, f^{***}) and (p_1^x, f^{***}) decide different values for $\dot{A} \cap z_x$. This is possible by our assumption for a contradiction. We let $f_\beta = f^{***}$, $A_\beta = \bigcup_{y \in B_\alpha} A_y$, $z_\beta = \bigcup_{x \in A_\beta} z_x$ and H_β be the map $x \mapsto (p_0^x, p_1^x)$.

The closure of $\mathbb{Q}$ means that we can continue the construction for $\lambda \leq \sigma$ many steps. Note that there may not be a lower bound for the sequence $\langle f_\alpha \mid \alpha < \lambda \rangle$.

Now we force with $\bar{\mathbb{X}}$. We consider the set B of all $H_\beta(x)$ where x is the unique element A_β in the generic. We claim that B is an antichain in $\mathbb{P}^2$ of size λ in $W[\bar{\mathbb{X}}]$, a contradiction. For definiteness, let $\alpha < \alpha' < \lambda$, $H_\alpha(x) = (p_0, p_1)$ and $H_{\alpha'}(x') = (p'_0, p'_1)$ be two elements of B and $f^* = f_{\alpha'}$.

By construction (p'_0, f^*) and (p'_1, f^*) decide the same value for $\dot{A} \cap z_\alpha$, but (p_0, f^*) and (p_1, f^*) decide different values. Hence we cannot have that p_0 is compatible with p'_0 *and* p_1 is compatible with p'_1. $\square$

Let (p^*, f^*) witness the previous claim.

We assume that over $W[\bar{\mathbb{X}}]$, it is forced that $\dot{A}$ is not in $W[\bar{\mathbb{X}}]$. Working below (p^*, f^*), we have that for any f_0, f_1 there are extensions f'_0, f'_1 (respectively) and z such that (p^*, f'_0) and (p^*, f'_1) force different values for $\dot{A} \cap z$.

Working for the moment in W, we can use the closure of $\mathbb{Q}$ to find for every f a maximal antichain A_f in $\bar{\mathbb{X}}$, a set z and conditions f_0 and f_1 such that for all $x \in A_f$, x forces that (p^*, f_0) and (p^*, f_1) force different values for $\dot{A} \cap \dot{z}$. The antichain A_f is constructed inductively together with a decreasing sequence of pairs (f_0^α, f_1^α). We initialize the construction by setting $f = f_0^0 = f_1^0$. Suppose that we completed the construction for all $\alpha < \beta$. If the antichain that we have constructed so far is maximal, then we stop the construction and for $i \in 2$ let f_i be a lower bound for $\langle f_i^\alpha \mid \alpha < \beta \rangle$. Otherwise we let x be an element of $\bar{\mathbb{X}}$ which is incompatible with all elements of A_f constructed so far and f_i^* be a lowerbound for $\langle f_i^\alpha \mid \alpha < \beta \rangle$. By density we can find $x' \leq x$ and for $i \in 2$, $f_i^\beta \leq f_i^*$ such that x' forces that (p^*, f_0^β) and (p^*, f_1^β) force different values for $\dot{A} \cap z$ for some set z. By the chain condition of $\bar{\mathbb{X}}$, the construction must terminate after fewer than λ steps with the construction of a maximal antichain.

Working in W, we construct a binary tree of conditions $\langle f_s \mid s \in 2^{<\rho} \rangle$ as follows. To construct $f_{s \frown 0}$ and $f_{s \frown 1}$ from f_s we apply the claim from the previous paragraph and if s has limit length η, then we let f_s be a lower bound in $\mathbb{Q}$ for $\langle f_{s \restriction \alpha} \mid \alpha < \eta \rangle$.

Letting $\dot{b}$ be the characteristic function of the first subset of ρ added by $\mathbb{P}$, we have that $\langle f_{\dot{b} \restriction \eta} \mid \eta < \rho \rangle$ is forced to be a decreasing sequence of elements of $\mathbb{Q}$. A standard argument shows that it has a lower bound f^{**}. We pass to the generic extension by $\bar{\mathbb{X}}$. Let z^* be the union of sets $z_{s,x}$ associated to the unique elements of A_{f_s} in the generic and let $a_{s \frown i, x}$ be the value decided by $(p^*, f_{s \frown i})$ for $z_{s,x} \cap \dot{A}$ as forced by x. Note that $2^{<\rho} = \rho$ implies that z^* has size less than λ. Let $f^{***} \leq f^{**}$ be such that (p^*, f^{***}) decides the value of $\dot{A} \cap z^*$ to be a.

We claim that $\dot{b} \restriction \eta + 1 = s \frown i$ if and only if $a \cap z_{s,x} = a_{s \frown i, x}$ where x is the unique element of A_{f_s} in the generic. This is a straightforward proof by induction on $\eta < \rho$ using the fact that the intersection of a with $z_{s,x}$ must agree with exactly one of $a_{s \frown 0, x}$ and $a_{s \frown 1, x}$ where x is the unique element of A_{f_s} in the $\bar{\mathbb{X}}$ generic object and s is the unique initial segment of $\dot{b}$ given by induction. This is a contradiction, since (p^*, f^{***}) completely determines $\dot{b}$. $\square$

6. The failure of approachability in the final model

Let R be $\mathbb{R}$-generic. We give notation for the generic objects added by R. Let $\langle x_n \mid n < \omega \rangle$ be the Prikry generic sequence. For all $n < \omega$ we let $\lambda_n = x_n \cap \kappa$ and for all $i \leq \omega + 3$, $\lambda_{n,i} = (\alpha \mapsto \alpha_i)(\lambda_n)$ where $\alpha \mapsto \alpha_i$ is the function defined at the beginning of Section 3.

For $n \geq 1$ we have the following generic objects induced by R:

(1) $Q_n = Q_n^0 \times Q_n^1$ which is generic for $\mathbb{Q}^0(x_{n-1}, x_n) \times \mathbb{Q}^1(x_{n-1}, x_n)$.
(2) Q_n^0 can be written as $P_n^0 * S_n^0$ where P_n^0 is generic for $\mathrm{Add}^V(\lambda_{n-1,\omega+2}, \lambda_n)$ and S_n^0 is generic for $\mathbb{C}^+(\mathrm{Add}^V(\lambda_{n-1,\omega+2}, \lambda_n), \lambda_{n-1,\omega+3}, \lambda_n)$ over the extension by P_n^0.
(3) Q_n^1 can be written as $P_n^1 * S_n^1$ where P_n^1 is generic for $\mathrm{Add}^V(\lambda_{n-1,\omega+3}, \lambda_{n,1})$ and S_n^1 is generic for $\mathbb{C}^+(\mathrm{Add}^V(\lambda_{n-1,\omega+3}, \lambda_{n,1}), \lambda_n, \lambda_{n,1})$.
(4) In a cardinal preserving extension, there are generics C_n^0, C_n^1 which are generic for $\mathbb{C}(\mathrm{Add}^V(\lambda_{n-1,\omega+2}, \lambda_n), \lambda_{n-1,\omega+3}, \lambda_n)$ and $\mathbb{C}(\mathrm{Add}^V(\lambda_{n-1,\omega+3}, \lambda_{n,1}), \lambda_n, \lambda_{n,1})$ respectively.

Finally, we let Q_0 be the induced generic for $\mathrm{Coll}(\omega, \lambda_{0,\omega})$.

Lemma 6.1. *In $V[G * H][R]$, $\aleph_{\omega^2+1} \notin I[\aleph_{\omega^2+1}]$.*

In joint work with Sinapova [24], we provided a sufficient condition for the failure of weak square in diagonal Prikry extensions. It is straightforward to check that $\mathbb{R}$ is a *diagonal Prikry forcing* as in Definition 19 and also satisfies the hypotheses of Theorem 26 from that paper. Hence we have the failure of weak square in the extension. To prove the lemma above, we give a direct argument for the failure of approachability and note that the technique generalizes to give a metatheorem for the failure approachability in extensions by diagonal Prikry type forcing.

Proof. Recall that $\vec{K}$ is $\mathbb{Q}^\omega$-generic over $V[G * H]$. Note that in $V[G * H][\vec{K}]$, κ is $\kappa_{\omega+1}$-supercompact as witnessed by U^* the measure on $\mathcal{P}_\kappa(\kappa_{\omega+1})$ derived from j and $\bar{\mathbb{R}}$ is $\kappa_{\omega+1}$-cc. By Remark 5.1, it is enough to show that approachability fails when we force with $\bar{\mathbb{R}}$ over $V[G*H][\vec{K}]$. Assume for a contradiction some condition r^* forces that $\langle \dot{a}_\alpha \mid \alpha < \kappa_{\omega+1} \rangle$ witnesses approachability. Let $k : V[G*H][\vec{K}] \to M$ be the ultrapower by U^*. By the construction of $\mathbb{R}$, we can choose a condition $r \in k(\bar{\mathbb{R}})$ below $k(r^*)$ by selecting a point y with $k(\kappa) \cap y = \kappa$. It follows that r forces that $\kappa_{\omega+1}$ is preserved. We let $\gamma = \sup k``\kappa_{\omega+1}$.

It follows that r forces that γ is approachable with respect to $k(\langle \dot{a}_\alpha \mid \alpha < \kappa_{\omega+1} \rangle)$. So there is a $k(\bar{\mathbb{R}})$-name $\dot{A}$ for a subset of γ all of whose initial segments are enumerated on the sequence $k(\langle \dot{a}_\alpha \mid \alpha < \kappa_{\omega+1} \rangle)$ before stage γ. By standard arguments we can assume that $\dot{A}$ is forced to be closed. Since $\mathrm{cf}(\gamma) = \kappa_{\omega+1}$ and it is forced by r that every club subset of $\kappa_{\omega+1}$ contains a club from the ground model, there is a club subset B of γ which is forced to be a subset of $\dot{A}$. We let $C = \{\alpha \mid k(\alpha) \in B\}$. It is straightforward to see that C is $< \kappa$-club in $\kappa_{\omega+1}$. Let η be the κ_ω-th element in an increasing enumeration of C.

We can assume that there is an index $\bar{\gamma} < \kappa_{\omega+1}$ such that r forces $\dot{A} \cap k(\eta)$ is enumerated before stage $k(\bar{\gamma})$ in $k(\langle \dot{a}_\alpha \mid \alpha < \kappa_{\omega+1} \rangle)$. Now for every $x \subseteq C \cap \eta$ of ordertype ω, there is a condition $r_x \in \bar{\mathbb{R}}$ which forces that $x \subseteq \dot{a}_\alpha$ for some $\alpha < \bar{\gamma}$. Note that for a given x, r witnesses k applied to this statement.

By the chain condition of $\bar{\mathbb{R}}$, we can find a condition which forces that for $\kappa_{\omega+1}$ many x, r_x is in the generic. This is impossible, since we can assume that each $\dot{a}_\alpha$ is forced to have ordertype less than κ and hence $|\bigcup_{\alpha<\bar{\gamma}} \mathcal{P}(\dot{a}_\alpha)| \leq \kappa$. $\square$

Remark 6.2. *Note that none of the specific properties of* $\mathbb{R}$ *are used in the proof above and hence the assumptions of Theorem 26 of* [24] *are enough to show the failure of approachability.*

Next we take care of the successors of singulars below $\aleph_{\omega^2}$.

Lemma 6.3. *There is a condition of length* 0 *in* $\mathbb{R}$ *which forces that for all* $n \geq 1$, *there is a bad scale of length* $\aleph_{\omega \cdot n+1}$ *in some product of regular cardinals.*

It follows that for all $n < \omega$, $\aleph_{\omega\cdot n+1} \notin I[\aleph_{\omega\cdot n+1}]$.

Proof. Working in $V[G*H]$, fix a scale $\vec{g}$ of length $\kappa^{+\omega+1}$ in some product of regular cardinals. By standard arguments there is a U_0-measure one set A such that for all $\delta \in A$, there are stationarily many bad points for $\vec{g}$ of cofinality $\delta_{\omega+1}$. This is absolute to $M[G^* * H^*]$. It follows that κ is in the set given by j applied to $B = \{\gamma \mid$ there is a scale of length $\gamma_{\omega+1}$ such that for all $\delta \in A \cap \gamma$ there are stationarily many bad points of cofinality $\delta_{\omega+1}\}$. It follows that $B \in U_0$.

Now for each $i \geq 1$, there is a U_i-measure one set A_i of x such that $\kappa_x \in A \cap B$. For any $x \prec y$ such that for some $i < i'$ $x \in A_i$ and $y \in A_{i'}$, we have arranged the following property. Since $\kappa_x \in A \cap B \cap \kappa_y$ and by the choice of A and B, there are stationary many bad points of cofinality $\kappa_{x,\omega+1}$ for some scale of length $\kappa_{y,\omega+1}$.

The condition required for the lemma is any condition length 0 whose measure one sets are contained in the A_i. Work below such a condition and fix $n \geq 1$. Let p be a condition of length $n+1$ and $\vec{f}$ be a scale of length $\kappa_{x^p_n,\omega+1}$ such that there is a stationary set S of bad points of cofinality $\kappa_{x^p_0,\omega+1}$. By Corollary 4.19, it is enough to show that $\vec{f}$ remains a scale with stationary set of bad points S in the model $V[G*H][\prod_{i\leq n} Q_i]$. The forcing to add $\prod_{i\leq n} Q_i$ is small relative to $\kappa_{x^p_n,\omega+1}$ and hence it is easy to see that $\vec{f}$ remains a scale and S remains stationary. So it is enough to show that every point in S is still bad for $\vec{f}$ in the extension.

In this extension $\kappa_{x^p_0,\omega+1}$ becomes $\aleph_1$ via $\mathrm{Coll}(\omega, \kappa_{x^p_0,\omega})$ and every $\aleph_1$-sequence of ordinals in $V[G*H][\prod_{i\leq n} Q_i]$ is in the extension by this collapse. Now a standard argument shows that for every $\delta \in S$ and every unbounded subset A of δ in the extension there is an unbounded subset of A in $V[G*H]$. So if A witnesses that $\delta \in S$ is good in the extension, then there is an unbounded subset of A witnessing that δ is good in $V[G*H]$. This is impossible, so we must have the every point in S is bad in $V[G*H][\prod_{i\leq n} Q_i]$. This completes the proof. $\square$

For the cardinals which are not successors of singulars, we apply the scheme from the previous section. Throughout the proofs of the following lemmas, we omit the straightforward but tedious verification that our scheme from Section 5 applies and that the hypotheses of our preservation lemmas hold.

We make some general remarks about the application of our scheme. Suppose that τ is a double successor cardinal in the final model and we wish to prove $\tau \notin I[\tau]$. In each case we have a forced with a Mitchell forcing $\mathbb{M}(\rho,\sigma,\tau)$ for some cardinals ρ and σ. This Mitchell forcing is either defined in the ground model V or defined at some step of the iteration that we did as preparation. By breaking up the preparation forcing, we work to apply the lemma in some τ-closed extension of the model where $\mathbb{M}(\rho,\sigma,\tau)$ is defined. Note that passing to a τ-closed extension preserves the Mahloness of τ and does not change the definition of $\mathbb{M}(\rho,\sigma,\tau)$.

To obtain the appropriate forcings $\bar{\mathbb{X}}$ and $\hat{\mathbb{X}}$, we use Remark 5.1 and pass to an outer model to split any remaining forcing which adds subsets to τ in to a λ-cc part and a λ-closed part for some $\lambda \leq \sigma$. We note that the λ-closed part need not be λ-closed, but only $< \lambda$-distributive and preserve λ-cc. For this it is enough for them to be λ-closed in an inner model obtained by λ-cc forcing.

In the case of the next two lemmas we will also use Theorem 5.2 to remove the influence of some centered forcing.

Lemma 6.4. *In $V[G * H][R]$, $\aleph_{\omega^2+2} \notin I[\aleph_{\omega^2+2}]$*

Proof. Note that by Remark 5.1 it is enough to show the conclusion in an outer model with the same cardinals up to $\aleph_{\omega^2+2}$. So we show that it holds in $V[G * H][\vec{K}][\bar{R}]$ where $\vec{K}$ is generic for $\mathbb{Q}^\omega$ and $\bar{R}$ is generic for $\bar{\mathbb{R}}$ as defined in the extension by $\vec{K}$. Since $\bar{\mathbb{R}}$ is κ_ω-centered in $V[G * H][\vec{K}]$, by Theorem 5.2 it is enough to show that $\kappa_{\omega+2} \notin I[\kappa_{\omega+2}]$ in $V[G * H][\vec{K}]$.

We write H as $P_0 \times P_1 * \prod_{i \leq \omega+1} C_i^+$ where P_1 is generic for $\mathbb{P}_1 \times \mathrm{Add}(\kappa_1, \theta^+ \setminus \theta) \simeq \mathrm{Add}(\kappa_1, \theta^+)$. Again by Remark 5.1, it is enough to show $\kappa_{\omega+2} \notin I[\kappa_{\omega+2}]$ in the extension

$$V[\vec{K}][G][C_{\omega+1}][\prod_{i<\omega} C_i][P_0][P_1 \restriction [\kappa_{\omega+2}, \theta^+)][P_1 \restriction \kappa_{\omega+2} * C_\omega^+].$$

Note that in $V[\vec{K}][G][C_{\omega+1}]$ we have that $\kappa_{\omega+2}$ is still Mahlo. So in this model we apply the scheme from Section 5 and Lemma 5.4 where

(1) $\lambda = \kappa_2$,
(2) $\mathbb{M}$ is the forcing to add $P_1 \restriction \kappa_{\omega+2} * C_\omega^+$,
(3) $\bar{\mathbb{X}}$ is the forcing to add $P_0 \times C_0 \times P_1 \restriction [\kappa_{\omega+2}, \theta^+]$,
(4) $\hat{\mathbb{X}}$ is the forcing to add $\prod_{0<i<\omega} C_i$

□

Lemma 6.5. *In $V[G * H][R]$, $\aleph_{\omega^2+3} \notin I[\aleph_{\omega^2+3}]$.*

The proof is similar to the proof of the previous lemma with some changes of the details.

Proof. Again by Theorem 5.2 and Claim 4.10 it is enough to show that $\kappa_{\omega+3} \notin I[\kappa_{\omega+3}]$ in $V[G * H][\vec{K}]$. By the proof of Claim 4.10, there is a cardinal preserving outer model of $V[G * H][\vec{K}]$ where we have decomposed $\vec{K}$ as $K^0 \times K^1$ which is generic for the product of $\kappa_{\omega+3}$-closed forcing and $\kappa_{\omega+3}$-cc forcing both taken from V. The $\kappa_{\omega+3}$-cc forcing is just $\mathrm{Add}(\kappa_{\omega+2}, \eta)$ for some η.

As in the previous lemma we pass to an outer model where we have decomposed H. In particular it is enough to prove that $\kappa_{\omega+3} \notin I[\kappa_{\omega+3}]$ in the model

$$V[K^0][G][P_0 \times \prod_{i \leq \omega} C_i][K^1][P_1 \restriction [\kappa_{\omega+3}, \kappa_{\omega+3}^+)][P_1 \restriction \kappa_{\omega+3} * C_{\omega+1}^+].$$

Note that in $V[K^0][G]$, $\kappa_{\omega+3} = \theta$ is still Mahlo. So in this model we apply the scheme from Section 5 and Lemma 5.4 where

(1) $\lambda = \kappa_{\omega+2}$,
(2) $\mathbb{M}$ is the forcing to add $P_1 \restriction \kappa_{\omega+3} * C_{\omega+1}^+$,
(3) $\bar{\mathbb{X}}$ as the forcing to add $P_0 \times P_1 \restriction [\kappa_{\omega+3}, \kappa_{\omega+3}^+) \times \prod_{i \leq \omega} C_i$, and
(4) $\hat{\mathbb{X}}$ as the forcing to add K^1.

□

Lemma 6.6. *In $V[G*H][R]$ for each successor τ of a regular cardinal where $\tau \in [\aleph_2, \aleph_{\omega^2})$, $\tau \notin I[\tau]$.*

Proof. There are a few cases based on how close τ is to the collapses between the Prikry points. Some cardinals we must treat individually and others we can treat uniformly.

First we assume that $\tau = \aleph_2$. Note that in $V[G*H][R]$, $\lambda_{0,\omega+1} = \aleph_1$ and $\lambda_{0,\omega+2} = \aleph_2$. Notice that any sequence witnessing that $\aleph_2 \in I[\aleph_2]$ is in the extension $V[G \restriction \lambda_0 + 1][Q_0 \times Q_1]$. Recall that Q_0 is generic for $\mathrm{Coll}(\omega, \lambda_{0,\omega})$ and Q_1 is generic for $\lambda_{0,\omega+2}$-closed forcing from V.

By Theorem 5.2 it is enough to show that $\tau \notin I[\tau]$ in the model $V[G \restriction \lambda_0+1][Q_1]$. The proof of this is simpler than the proof of the next case, so we continue.

Next we assume that $\tau = \lambda_{n,\omega+2}$ where $0 < n < \omega$. Any sequence witnessing that $\tau \in I[\tau]$ in $V[G*H][R]$ is in the extension $V[G \restriction \lambda_n + 1][\prod_{k\leq n} Q_k][Q_{n+1}]$. We let $P_0 \times P_1$ be generic for $\mathbb{P}_0(\lambda_n) \times (\mathbb{P}_1(\lambda_n) \times \mathrm{Add}(\lambda_{n,1}, \theta^+_{\lambda_n} \setminus \theta_{\lambda_n}))$ and $\prod_{k\leq\omega+1} C^+_k$ be generic for $\mathbb{C}^+(\lambda_n)$. These are the generics obtained from step λ_n of the preparation.

By Remark 5.1, it is enough to show that $\tau \notin I[\tau]$ in the model

$$V[G \restriction \lambda_n][C_{\omega+1}][Q_{n+1}][\prod_{k\leq n} Q_k][C_0][P_0][P_1 \restriction [\lambda_{n,\omega+2}, \theta^+_{\lambda_n})][\prod_{0<i<\omega} C_i][P_1 \restriction \lambda_{n,\omega+2} * C^+_\omega]$$

Note that in $V[G \restriction \lambda_n][C_{\omega+1}][Q_{n+1}]$, $\lambda_{n,\omega+2}$ is still Mahlo. So in this model we apply the scheme from Section 5 and Lemma 5.4 where

(1) λ is $\lambda_{n,2}$,
(2) $\mathbb{M}$ is the forcing to add $P_1 \restriction \lambda_{0,\omega+2} * C^+_\omega$,
(3) $\bar{\mathbb{X}}$ is the forcing to add $P_0 \times P_1 \restriction [\lambda_{n,\omega+2}, \theta^+_{\lambda_n}) \times C_0 \times \prod_{k\leq n} Q_k$ and
(4) $\hat{\mathbb{X}}$ is the forcing to add $\prod_{0<i<\omega} C_i$.

This completes the argument that $\lambda_{n,\omega+2} \notin I[\lambda_{n,\omega+2}]$ for $0 < n < \omega$.

Suppose that $\tau = \lambda_{n,\omega+3}$ for $n < \omega$. Any sequence witnessing $\tau \in I[\tau]$ in $V[G*H][R]$ is in the extension by $V[G \restriction \lambda_n + 1][\prod_{k\leq n+1} Q_k]$. Recall that by passing to an outer model with the same cardinals, we can decompose Q_{n+1} as $P^0_{n+1} \times C^0_{n+1} \times P^1_{n+1} \times C^1_{n+1}$. As before we let $P_0 \times P_1$ be generic for $\mathbb{P}_0(\lambda_n) \times (\mathbb{P}_1(\lambda_n) \times \mathrm{Add}(\lambda_{n,1}, \theta^+_{\lambda_n} \setminus \theta_{\lambda_n}))$ and $\prod_{k\leq\omega+1} C^+_k$ be generic for $\mathbb{C}^+(\lambda_n)$.

By Remark 5.1 it is enough to show that $\tau \notin I[\tau]$ in the model

$$V[G \restriction \lambda_n][\prod_{k\leq n} Q_k][P^1_{n+1} \times C^1_{n+1} \times C^0_{n+1}][P^0_{n+1}][P_0][P_1 \restriction [\theta_{\lambda_n}, \theta^+_{\lambda_n})][\prod_{k\leq\omega} C_k][P_1 \restriction \lambda_{n,\omega+3} * C^+_{\omega+1}]$$

We have that $\lambda_{n,\omega+3}$ is Mahlo in the model $V[G \restriction \lambda_n][P^1_{n+1} \times C^1_{n+1} \times C^0_{n+1}]$. So we apply the scheme from Section 5 and Lemma 5.4 in this model where

(1) $\lambda = \lambda_{n,\omega+2}$,
(2) $\mathbb{M}$ is the forcing to add $P_1 \restriction \lambda_{n,\omega+3} * C^+_{\omega+1}$,
(3) $\bar{\mathbb{X}}$ is the forcing to add $P_0 \times P_1 \restriction [\theta_{\lambda_n}, \theta^+_{\lambda_n}) \times \prod_{k\leq\omega} C_k \times \prod_{k\leq n} Q_k$ and
(4) $\hat{\mathbb{X}}$ is the forcing to add P^0_{n+1}.

This completes the argument that $\lambda_{n,\omega+3} \notin I[\lambda_{n,\omega+3}]$ for all $n < \omega$.

Next we assume that $\tau = \lambda_n$ for $n \geq 1$. Any sequence witnessing that $\tau \in I[\tau]$ in $V[G*H][R]$ is in $V[G \restriction \lambda_n + 1][\prod_{i\leq n} Q_i]$. As before by passing to an outer model, we can decompose Q_n as $P^0_n * S^0_n \times P^1_n \times C^1_n$. By Remark 5.1, it is enough to show that there are no such sequences in the outer model

$$V[C_n^1][Y][Z][G \upharpoonright \lambda_{n-1}+1][\prod_{i\leq n-1} Q_i][P_n^1][P_n^0 * S_n^0]$$

where Y is generic for $\mathcal{A}(\mathbb{A} \upharpoonright \lambda_n, \mathbb{A}(\lambda_n))$ and Z is generic for $\mathcal{A}(\mathbb{A}) \upharpoonright [\lambda_{n-1}+1, \lambda_n)$.

By Fact 2.6, λ_n is still Mahlo in $V[C_n^1][Y][Z]$. Moreover, the computation of $\mathbb{Q}_n^0 = \mathbb{M}(\lambda_{n-1,\omega+2}, \lambda_{n-1,\omega+3}, \lambda_n)$ is the same in V and $V[C_n^1][Y][Z]$, since the forcing to add Z is closed beyond the first inaccessible above $\lambda_{n-1,\omega+3}$.

So in this model we apply the scheme from Section 5 and Lemma 5.4 where

(1) $\lambda = \lambda_{n-1,\omega+3}$.
(2) $\mathbb{M}$ is the forcing to add $P_n^0 * S_n^0$,
(3) $\bar{\mathbb{X}}$ is the forcing to add $G \upharpoonright (\lambda_{n-1}+1) \times \prod_{i\leq n-1} Q_i$ and
(4) $\hat{\mathbb{X}}$ is the forcing to add P_n^1.

This finishes the proof that for all $n \geq 1$, $\lambda_n \notin I[\lambda_n]$ in $V[G * H][R]$.

Next we assume that $\tau = \lambda_{n,1}$ for some $n \geq 1$. Any sequence witnessing that $\tau \in I[\tau]$ is in the model $V[G \upharpoonright \lambda_n + 1][\prod_{i\leq n} Q_i]$. By Remark 5.1 it is enough to show that $\tau \notin I[\tau]$ in the model

$$V[Y][G \upharpoonright \lambda_n][P_n^0][C_n^0][Y_{P_0}][P_n^1 * S_n^1]$$

where Y is generic for the poset of $\mathbb{A} \upharpoonright \lambda_n$-names for elements of $\mathbb{P}_1(\lambda_n) \times \mathrm{Add}(\lambda_{n,1}, \theta_{\lambda_n}^+ \setminus \theta_{\lambda_n}) \times \mathbb{C}(\lambda_n)$ and Y_{P_0} is generic for the poset of $\mathbb{A} \upharpoonright \lambda_n$-names for elements of $\mathbb{P}_0(\lambda_n)$. By Fact 2.5, we can take the forcing to add Y_{P_0} to be $\mathrm{Add}(\lambda_{n,0}, \lambda_{n,2})$ as computed in V.

Note that $\lambda_{n,1}$ is still Mahlo in $V[Y]$. So we apply the scheme from Section 5 and Lemma 5.4 in this model where

(1) $\lambda = \lambda_n$,
(2) $\mathbb{M}$ is the forcing to add $P_n^1 * S_n^1$,
(3) $\bar{\mathbb{X}}$ is the forcing to add $G \upharpoonright \lambda_n \times P_n^0 \times C_n^0$ and
(4) $\hat{\mathbb{X}}$ is the forcing to add Y_{P_0}.

This finishes the proof that $\lambda_{n,1} \notin I[\lambda_{n,1}]$ for $n \geq 1$.

Next we assume that $\tau = \lambda_{n,2}$ for $n \geq 1$. Any sequence witnessing that $\tau \in I[\tau]$ in $V[G * H][R]$ is in $V[G \upharpoonright \lambda_n + 1][\prod_{k\leq n} Q_k]$. By Remark 5.1, it is enough to show that $\tau \notin I[\tau]$ in the model

$$V[G \upharpoonright \lambda_n][\prod_{1\leq i\leq \omega+1} C_i][\prod_{k\leq n} Q_k][P_1][P_0 * C_0^+]$$

where as before we have written out the generics for step λ_n of the preparation.

Note that $\lambda_{n,2}$ is still Mahlo in $V[G \upharpoonright \lambda_n][\prod_{1\leq i\leq\omega+1} C_i]$. Hence we apply the scheme from Section 5 and Lemma 5.4 in this model where

(1) $\lambda = \lambda_{n,1}$,
(2) $\mathbb{M}$ is the forcing to add $P_0 * C_0^+$,
(3) $\bar{\mathbb{X}}$ is the forcing to add $\prod_{k\leq n} Q_k$ and
(4) $\hat{\mathbb{X}}$ is the forcing to add P_1.

Next we assume that $\tau = \lambda_{n,i+1}$ for $0 < n < \omega$ and $2 \leq i < \omega$. Any sequence witnessing that $\tau \in I[\tau]$ in $V[G * H][R]$ is in $V[G \upharpoonright \lambda_n + 1][\prod_{k\leq n} Q_k]$. By Remark

5.1, it is enough to show that $\tau \notin I[\tau]$ in the model

$$V[G \restriction \lambda_n][\prod_{k \geq i} C_k][\prod_{k \leq n} Q_k][\prod_{k < i-1} C_k][P_0][P_1 \restriction [\lambda_{n,i+1}, \theta^+_{\lambda_n})][P_1 \restriction \lambda_{n,i+1} * C^+_{i-1}]$$

where as before we have written out the generics for step λ_n of the preparation.

Note that in $V[G \restriction \lambda_n][\prod_{k \geq i} C_k]$, $\lambda_{n,i+1}$ is still Mahlo. Hence we apply the scheme from Section 5 and Lemma 5.4 in this model where

(1) $\lambda = \lambda_{n,i}$,
(2) $\mathbb{M}$ is the forcing to add $P \restriction \lambda_{n,i+1} * C^+_{i-1}$,
(3) $\bar{\mathbb{X}}$ is the forcing to add $\prod_{k \leq n} Q_k \times \prod_{k < i-1} C_k \times P_0 \times P_1 \restriction [\lambda_{n,i+1}, \theta^+_{\lambda_n})$ and
(4) $\hat{\mathbb{X}}$ is the trivial forcing.

This finishes the proof that for $0 < n < \omega$ and $2 \leq i < \omega$, $\lambda_{n,i+1} \notin I[\lambda_{n,i+1}]$ and with it the proof of Lemma 6.6 $\square$

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