A NEW PROOF OF LACZKOVICH'S CIRCLE SQUARING THEOREM I

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Abstract. We give a new, elementary, and self-contained proof of Laczkovich's solution to Tarski's circle squaring problem: the result that a disk and a square of the same area in $\mathbb{R}^2$ are equidecomposable by translations. More generally, we give a new proof of Laczkovich's 1992 theorem that any two bounded Lebesgue measurable subsets of $\mathbb{R}^k$ with the same positive Lebesgue measure are equidecomposable by translations, provided their topological boundaries have upper box dimension less than k. Our proof is based on flows in infinite graphs. We consider a graph whose vertex set is $[0,1)^k$ and whose edges are generated by finitely many translations. In this graph, we show there is a flow between the characteristic functions of the two sets. We then use the Axiom of Choice and the integral flow theorem to construct an equidecomposition from this flow. To show that this flow converges, we give a new proof of Laczkovich's discrepancy estimates for certain translation actions on the k-torus using only the Erdős-Turán inequality.

Our proof gives a new sufficient condition for when two sets are equidecomposable by translations whose coordinates are integer linear combinations of finitely many given real numbers. We use this to answer a 1990 question of Laczkovich by showing that the circle can be squared by translations whose coordinates are algebraic irrational numbers. In our subsequent paper [MU2], we build on this result to give improved upper bounds on the number of pieces needed to square the circle.

1. Introduction

The Banach-Tarski paradox states that, assuming the Axiom of Choice, a single unit ball in $\mathbb{R}^3$ can be partitioned into finitely many sets which can be rearranged by isometries to partition two disjoint unit balls. That is, one unit ball and two unit balls are *equidecomposable by isometries*. However, in two dimensions, the analogue of the Banach-Tarski paradox is false. One unit disk in $\mathbb{R}^2$ is not equidecomposable by isometries with two unit disks. The reason for this is the existence of *Banach measures*, which extend Lebesgue measure to *all* subsets of $\mathbb{R}^2$ and are finitely additive and isometry invariant (see [TW, Corollary 12.9]). Thus, if two sets $A, B \subseteq \mathbb{R}^2$ are Lebesgue measurable and equidecomposable by isometries, they must have the same Lebesgue measure. To see this, let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be partitions of A and B respectively, where there is an isometry mapping each A_i to B_i. If μ is a Banach measure extending Lebesgue measure λ, then $\lambda(A) = \lambda(B)$ since

$$\lambda(A) = \mu(A) = \sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{n} \mu(B_i) = \mu(B) = \lambda(B).$$

Here we are using that μ extends λ for the first equality, μ is finitely additive for the second equality, and μ is isometry invariant for the third equality.

In 1925, Tarski posed the question of whether a disk and a square of the same area are equidecomposable in $\mathbb{R}^2$. The question was motivated by this difference between equidecomposability in $\mathbb{R}^2$ and $\mathbb{R}^3$, and the problem of understanding which subsets of $\mathbb{R}^2$ of the same Lebesgue measure are equidecomposable.

Question 1.1 (Tarski's circle squaring problem [T])**.** *Suppose $A \subseteq \mathbb{R}^2$ is a closed disk and $B \subseteq \mathbb{R}^2$ is a closed square so that A and B have the same area. Are A and B equidecomposable by isometries?*

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Tarski's circle squaring problem remained open for many years until it was answered positively by Laczkovich in 1990 [L90] who showed that in fact there is an equidecomposition by translations. Laczkovich's proof used the Axiom of Choice, but unlike the Banach-Tarski paradox where the sets used in the equidecomposition are necessarily nonmeasurable, in Tarski's circle squaring problem there is no measure-theoretic paradox so it was plausible that the pieces in the equidecomposition could be chosen to be Lebesgue measurable. Such a Lebesgue measurable equidecomposition was recently shown to exist by Grabowski, Máthe, and Pikhurko [GMP], though their proof still used the Axiom of Choice to complete the equidecomposition on a Lebesgue null set. Soon after, the authors of this note showed there is a completely constructive solution to Tarski's circle squaring problem (without the Axiom of Choice) using Borel pieces. Part of that proof relies on a new way of squaring the circle using flows in graphs as an intermediate step.

In this paper, we adapt these tools to give a new and simpler flow-based proof of Laczkovich's theorem that a disk and a square of the same area in $\mathbb{R}^2$ are equidecomposable by translations. We have made the proof as self-contained and elementary as possible. The only prerequisites are some basic knowledge of graph theory, algebra (group actions), topology (Tychonoff's theorem), and measure theory (Lebesgue measure). Throughout, we discuss connections between this proof and other parts of mathematics including combinatorics, ergodic theory, Diophantine approximation, and Fourier analysis. Although we will often give references to sources that can be used to better understand the broader context for the results we are proving, none of them are required to understand our proof.

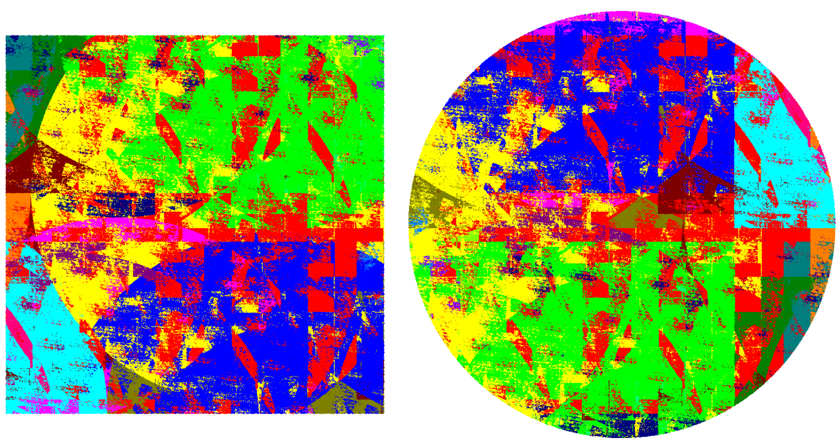

FIGURE 1. An image of an equidecomposition of a disk and a square. Each piece in the partition is represented by a different color. For more details on how this diagram was generated, see [Ma].

The two main differences between our proof and earlier work are as follows. First, the flow we define between the disk and the square (in Lemma 3.1) has a simpler definition than the flow in [MU, Section 4], and its properties are easier to check. Second, our proof of Laczkovich's discrepancy lemma (Lemma B) in Sections 5 and 6 is new. We first use the Erdős-Turán inequality to give a simple 1-dimensional discrepancy estimate for certain finite sets arising in translation actions on $\mathbb{T}$ in Section 5. We then prove a lemma lifting these 1-dimensional discrepancy estimates to k-dimensional discrepancy estimates in product actions on $\mathbb{T}^k$ (Lemma 6.2). This is the crucial new ingredient, and we formulate a new sufficient condition for equidecomposability via a certain set of translations that comes from this new approach (Theorem D). Our proof also contains many other small improvements and simplifications.

This new proof also yields a new corollary. We show that the circle can be squared by translations with algebraic irrational coordinates (as opposed to the random translations originally used by

Laczkovich). This answers an open problem of Laczkovich [L90, Section 10.3] of whether the circle can be squared using effectively given translations. The proof of this result uses a deep result of number theory to verify the sufficient condition for equidecomposability in Theorem D: Schmidt's theorem on simultaneous approximation of algebraic irrationals [S70].

In our second paper, we build on this to prove [MU2] improved upper bounds on the number of pieces needed to square the circle. Previously, Laczkovich had shown that a disk and square can be equidecomposed using 10^{40} pieces [L02, Page 114]. We improve this upper bound to fewer than 100,000 pieces using our new proof, progress in number theory on the problem of proving effective bounds for Roth's theorem on Diophantine approximation of algebraic irrationals, and computer assistance. In this paper we keep track of the number of pieces used in our equidecomposition to aid in that calculation.

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2. Preliminaries and proof outline

If f, g are functions from $\mathbb{R}$ to $\mathbb{R}$ or $\mathbb{N}$ to $\mathbb{R}$, we will use Vinogradov's asymptotic notation $f \ll g$ to denote that there are constants m and $C > 0$ so that $f(n) \leq Cg(n)$ for all $n \geq m$. If f and g depend on some additional parameters, we will use subscripts to denote the parameters on which C and m depend in this notation. So for example, if $k > 0$, then $\log(kn) \ll_k n$, meaning that for any $k > 0$ there exist m_k and $C_k > 0$ so that for all $n \geq m_k$, $\log(kn) \leq C_k n$.

If X is a set, a *finite partition* of X is a finite sequence $A_1, \ldots, A_n$ of disjoint subsets of X whose union is X. We call the sets $A_1, \ldots, A_n$ the *pieces* of the partition. If $a\colon G \curvearrowright X$ is an action of a group G on a space X, then we say that $A, B \subseteq X$ are *a-equidecomposable* if there exist a finite partition $A_1, \ldots, A_n$ of A and group elements $g_1, \ldots, g_n \in G$ so that $g_1 \cdot A_1, \ldots, g_n \cdot A_n$ is a partition of B. For example, the group $\mathbb{R}^k$ acts on itself by translations: each group element $g \in \mathbb{R}^k$ acts on $x \in \mathbb{R}^k$ by $g \cdot x \coloneqq g + x$. We say $A, B \subseteq \mathbb{R}^k$ are *equidecomposable by translations* if A and B are equidecomposable in this translation action. See [TW] for more context and an introduction to the theory of equidecomposability.

Our first goal is to formulate equidecomposability in graph theoretic terms. The graphs we will consider are the Schreier graphs associated to group actions:

Definition 2.1 (Schreier graph of a group action)**.** *If $a\colon G \curvearrowright X$ is an action of a group G on a space X and $S \subseteq G$, then we let $Sch(a, S)$ be the undirected graph with vertex set $V = X$ where $\{x, y\}$ is an edge in $Sch(a, S)$ if $x \neq y$ and $g \cdot x = y$ for some $g \in S \cup S^{-1}$.*

If H is an undirected graph on X, we let $d_H(x, y)$ be the usual graph metric where $d_H(x, y)$ is equal to the shortest length of a path from x to y in the graph H if one exists, and it is equal to ∞ otherwise.

Now we can restate equidecomposability in more graph theoretic terms: two sets A and B are equidecomposable in some group action if and only if there is a bijection between A and B that moves points a bounded distance in some Schreier graph of the action:

Proposition 2.2. *If $a\colon G \curvearrowright X$ is an action of a group G on a space X, then $A, B \subseteq X$ are a-equidecomposable if and only if there exist a finite set $S \subseteq G$, a bijection $f\colon A \to B$, and some $m \in \mathbb{N}$ so that $d_{Sch(a,S)}(x, f(x)) \leq m$ for every $x \in A$.*

Proof. First, suppose A and B are a-equidecomposable, so there is a partition $A_1, \ldots, A_n$ of A and group elements $g_1, \ldots, g_n \in G$ so that $g_1 \cdot A_1, \ldots, g_n \cdot A_n$ is a partition of B. Let $B_i = g_i \cdot A_i$. Now define the function $f\colon A \to B$ by $f(x) = g_i \cdot x$ where i is the unique value such that $x \in A_i$. We claim that f is a bijection from A to B. First note that for each i, $f(A_i) = g_i \cdot A_i = B_i$ by the definition of f. So f is onto B since $\operatorname{ran}(f) = \bigcup_i f(A_i) = \bigcup_i B_i = B$ because the B_i partition B. So it remains to show that f is an injection. To begin, note $f \upharpoonright A_i$ is an injection from A_i to B_i since its inverse is given by acting by g_i^{-1}. Now suppose $x, x' \in A$ are not equal. Then $x \in A_i$ and $x' \in A_j$ for a unique i and j. If $i = j$, then $f(x) \neq f(x')$ since $f \upharpoonright A_i$ is an injection. If $i \neq j$, then $f(x) \in B_i$ and $f(x') \in B_j$, and B_i and B_j are disjoint, so $f(x) \neq f(x')$. So $x \neq x'$ implies $f(x) \neq f(x')$ and so f is an injection. To finish, if we let $S = \{g_1, \ldots, g_n\}$, then $d_{\mathrm{Sch}(a,S)}(x, f(x)) \leq 1$ for all $x \in A$.

Conversely, suppose $S \subseteq G$ is a finite set and $f \colon A \to B$ is a bijection between $A, B \subseteq X$ so that $d_{\mathrm{Sch}(a,S)}(x, f(x)) \leq m$ for all $x \in A$. Since adding inverses to S does not change the graph $\mathrm{Sch}(a, S)$ (whose definition is symmetric), we may assume that S is closed under inverses. Then there is an edge between distinct x and y if there is a $g \in S$ so that $g \cdot x = y$. Let $S^{\leq m} = \bigcup_{k \leq m} S^k$ be all products of at most m elements from S, so $S^{\leq m}$ is finite. By definition, $S^{\leq m}$ includes the identity (which we define to be the empty product). Note that if $d_{\mathrm{Sch}(a,S)}(x, y) \leq m$, then $g \cdot x = y$ for some $g \in S^{\leq m}$. Let $g_1, \ldots, g_n$ enumerate the elements of $S^{\leq m}$. Now let $A_i = \{x \colon f(x) = g_i \cdot x \text{ and } f(x) \neq g_j \cdot x \text{ for any } j < i\}$. Then the sets A_i partition A. Since $f(x) = g_i \cdot x$ for all $x \in A_i$ and f is an injection, the sets $g_i \cdot A_i$ are disjoint. Their union covers B since $B = \mathrm{ran}(f) = \cup_i g_i \cdot A_i$. So the sets $g_i \cdot A_i$ partition B, and so A and B are a-equidecomposable. ☐

Remark 2.3. We make a remark about how many pieces such equidecompositions use. Suppose $f \colon A \to B$ is a bounded distance bijection in $\mathrm{Sch}(a, S)$. Let $T \subseteq G$ be a set of group elements such that for all $x \in A$, there exists some $g \in T$ such that $f(x) = g \cdot x$. Then A and B are equidecomposable using at most $|T|$ pieces: the equidecomposition is by the sets $A_i = \{x \colon f(x) = g_i \cdot x \text{ and } f(x) \neq g_j \cdot x \text{ for any } j < i\}$ as above, where $\{g_1, \ldots, g_n\}$ enumerates the elements of T.

To show the circle and the square are equidecomposable, we will use the above formulation of equidecomposability and find such a bounded distance bijection.

If A is a set in a topological space such as $\mathbb{R}^k$, we use the notation $\partial A = \mathrm{cl}(A) \setminus \mathrm{int}(A)$ to denote the topological boundary of a set A: its closure minus its interior. So if A is a disk in $\mathbb{R}^2$, then ∂A is the circle that is its boundary. Laczkovich's theorem giving a sufficient condition for when A, B are equidecomposable in $\mathbb{R}^k$ requires a bound on the upper box dimension of the topological boundaries of A and B. So next we recall the definition of box dimension and prove some basic properties of it.

The sets $g + [0, 1)^k$ for $g \in \mathbb{Z}^k$ partition $\mathbb{R}^k$ into unit cubes. So scaling by a real number $c > 0$, the sets $c(g + [0, 1)^k)$ for $g \in \mathbb{Z}^k$ partition $\mathbb{R}^k$ into cubes of side length c. We call the elements of this partition *c-lattice cubes*. The *upper box dimension* or *upper Minkowski dimension* of a bounded set $A \subseteq \mathbb{R}^k$ is defined by

$$\overline{\dim}_{\mathrm{box}}(A) = \limsup_{n \to \infty} \frac{\log N(n, A)}{\log n}$$

where $N(n, A)$ is the number of $1/n$-lattice cubes that intersect A. So if $N(n, A) \ll n^c$, then $\overline{\dim}_{\mathrm{box}}(A) \leq c$. See [F, Section 3.1] for an introduction to box dimension. Note that if A is bounded and lies in $[-m, m)^k$ where $m \in \mathbb{Z}$, then $\overline{\dim}_{\mathrm{box}}(A) \leq k$ since $N(n, A) \leq (2m)^k n^k$. Next, we show that the upper box dimension of the boundary of a disk or a square is at most 1. (In fact, it is an easy exercise to show it is exactly equal to 1).

Proposition 2.4. *Let A be a circle or the perimeter of a square in $\mathbb{R}^2$. Then $\overline{\dim}_{\mathrm{box}}(A) \leq 1$.*

Proof. If A is a circle of radius r, any c-lattice cube that intersects A is contained in the annulus with inner radius $r - \sqrt{2}c$ and outer $r + \sqrt{2}c$ (where $\sqrt{2}c$ is the length of the diagonal of a c-lattice cube). The total area of this annulus is $\pi(r + \sqrt{2}c)^2 - \pi(r - \sqrt{2}c)^2 = 4\pi\sqrt{2}cr$, and so since each c-lattice cube has area c^2, the annulus contains at most $4\pi\sqrt{2}cr/c^2 = 4\pi\sqrt{2}r/c$ such c-lattice cubes.

So if $c = 1/n$, then $N(n, A) \leq 4\pi\sqrt{2}rn$ so

$$\overline{\dim}_{\text{box}}(A) \leq \limsup_{n\to\infty} \frac{\log\left(4\pi\sqrt{2}rn\right)}{\log n} = 1$$

Similarly, if A is the perimeter of a square of side length r, any c-lattice cube that intersects A is contained in the "square annulus" region enclosed by the concentric squares with parallel sides where the larger square has side length $r+2c$ and smaller square has side length $r-2c$. This region has total area $(r+2c)^2 - (r-2c)^2 = 8rc$. So since each c-lattice cube has area c^2, it contains at most $8r/c$ such c-lattice cubes. So $\overline{\dim}_{\text{box}}(A) \leq \limsup_{n\to\infty} \frac{\log 8rn}{\log n} = 1$. □

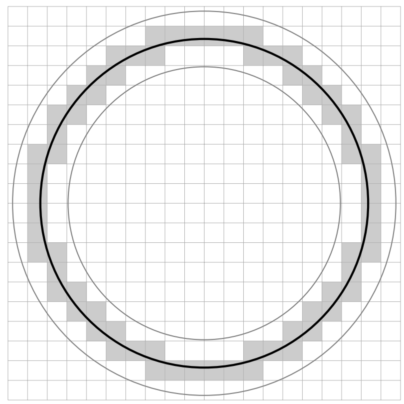

FIGURE 2. A proof that the upper box dimension of a circle (the boundary of the disk) is less than or equal to 1. Every box in the grid that intersects the black circle, A, is shaded in gray. The number of these squares is defined to be $N(n, A)$, where $1/n$ is the side length of the cubes in the grid. These cubes are all contained in the annulus bounded by the circles of radius $r - \sqrt{2}/n$ and $r + \sqrt{2}/n$, so the area of this annulus gives an upper bound on $N(n, A)$.

We let λ denote Lebesgue measure on $\mathbb{R}^k$. Laczkovich's first proof that a disk and square are equidecomposable is the 1990 paper [L90]. The goal of this paper is to give a new proof of the following more general result proved by Laczkovich in 1992 [L92]: if $A, B \subseteq \mathbb{R}^k$ are bounded Lebesgue measurable sets such that $\lambda(A) = \lambda(B) > 0$, and $\overline{\dim}_{\text{box}}(\partial A) < k$ and $\overline{\dim}_{\text{box}}(\partial B) < k$, then A and B are equidecomposable by translations (see Theorem C). This implies a positive solution to Tarski's circle squaring problem since by Proposition 2.4, both the disk and square have topological boundaries of upper box dimension at most 1.

We let $\mathbb{T}^k$ denote the k-dimensional torus $\mathbb{R}^k/\mathbb{Z}^k$. Under the quotient group structure it is an abelian group. We will often identify $\mathbb{T}^k$ with the fundamental domain $[0,1)^k$, which contains exactly one point from each coset of $\mathbb{Z}^k$ in $\mathbb{R}^k$. For example, this identification lets us define Lebesgue measure on $\mathbb{T}^k$, by using the restriction of Lebesgue measure to $[0,1)^k$, so $\lambda(\mathbb{T}^k) = 1$. Now if $x \in \mathbb{T}^k$, then multiplication of x by an integer n makes sense: we define nx to be the group sum of x with itself n times. However, we caution the reader that scalar multiplication by arbitrary real numbers is not defined on $\mathbb{T}^k$, so cx is not defined when $c \in \mathbb{R} \setminus \mathbb{Z}$ and $x \in \mathbb{T}^k$.

A key insight of Laczkovich was to work in $\mathbb{T}^k$ instead of $\mathbb{R}^k$. The following proposition shows this does not change which sets are equidecomposable by translations. However, a benefit to this change in ambient space is that we can naturally use ideas and intuitions coming from ergodic theory. For example, for almost every $v \in \mathbb{T}^k$, the transformation $T(x) = v + x$ is an *ergodic* transformation (see [H] for a definition of an ergodic transformation). By the Birkhoff ergodic theorem, this is equivalent to the statement that for any measurable set $A \subseteq \mathbb{T}^k$, we have $\lim_{N\to\infty} \frac{|\{0 \leq i < N \colon T^i(x) \in A\}|}{N} = \lambda(A)$

for almost every $x \in \mathbb{T}^k$, where λ is Lebesgue measure. That is, the proportion of points in $x, T(x), \dots, T^{N-1}(x)$ that are in A converges to the measure of A as $N \to \infty$. Since this holds for any measurable set A, intuitively the forward orbits of T are "uniformly distributed" or "randomly distributed" throughout the space in the limit as $N \to \infty$. We will not directly use the notion of ergodicity or the ergodic theorem in this paper. However, precise quantitative bounds on the left hand side of the above equation will play a key role in our proof (see Lemma B). The fact that such quantitative bounds should exist is motivated by easier qualitative results such as the ergodic theorem.

Proposition 2.5. *$A, B \subseteq [0,1)^k$ are equidecomposable in the translation action of $\mathbb{R}^k$ on $\mathbb{R}^k$ if and only if A, B are equidecomposable in the translation action of $\mathbb{T}^k$ on $\mathbb{T}^k$ (viewing A and B as subsets of $\mathbb{T}^k$). Moreover, if A and B are equidecomposable in the translation action of $\mathbb{T}^k$ on $\mathbb{T}^k$ using n pieces, then A and B are equidecomposable in the translation action of $\mathbb{R}^k$ on $\mathbb{R}^k$ using $2^k n$ pieces.*

Proof. The forward implication is trivial so we prove the reverse implication. First, if $A, B \subseteq [0,1)^k$ and B is a translation of A inside the torus, then we claim that in $\mathbb{R}^k$, A and B are equidecomposable using at most 2^k pieces. To see this, suppose first that B is a translation of A in the torus so $B = v + A$ (where the sum is computed in $\mathbb{T}^k$). Now we partition A into at most 2^k pieces depending on which coordinates "overflow mod 1" when we add v. Let $S = \{0,1\}^k = \{g \in \mathbb{Z}^k : g(j) \in \{0,1\}$ for all $j\}$. For each $g \in S$, let $C_g = g + [0,1)^k$, so $\bigcup_g C_g = [0,2)^k$ contains every sum of the form $v + w$ for $v, w \in [0,1)^k$ (where the sum is computed in $\mathbb{R}^k$). For each $g \in S$, let $A_g = \{x \in A : v + x \in C_g\}$ (where the sum is computed in $\mathbb{R}^k$). Then $(A_g)_{g \in S}$ is a partition of A into 2^k pieces, and the sets $(v - g) + A_g$ for $g \in S$ partition $B = v + A$. Hence A and B are equidecomposable by 2^k pieces in $\mathbb{R}^k$. Intuitively, this partition of A corresponds to which coordinates of $x \in A$ have overflowed mod 1 after we translate by v.

It follows that if A and B are equidecomposable in $\mathbb{T}^k$ and $A_1, \dots, A_n$ is a partition of A, $B_1, \dots, B_n$ is a partition of B, and B_i is a translation of A_i in $\mathbb{T}^k$ then we can further partition each A_i and B_i into at most 2^k pieces $A_{i,g}$ and $B_{i,g}$ for $g \in S$ so that each $B_{i,g}$ is a translation of $A_{i,g}$ in $\mathbb{R}^k$. So A and B are equidecomposable by translations in $\mathbb{R}^k$ by at most $2^k n$ pieces. □

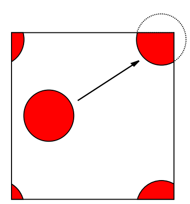

FIGURE 3. Laczkovich's first key insight is to work in the torus $\mathbb{R}^2/\mathbb{Z}^2$ instead of $\mathbb{R}^2$. This change in ambient space does not change what sets are equidecomposable by translations since any translation of a set A in the torus can be replicated in $\mathbb{R}^2$ by partitioning A into at most 2^2 pieces (based on whether or not the x and y coordinates have overflowed mod 1 after translating), and then translating these four pieces separately. In the figure above, we show a translation of a disk in the torus, which can be replicated in $\mathbb{R}^2$ by partitioning the disk into four pieces, and translating them separately.

Next, we introduce notation for the action of the group generated by finitely many translations of $\mathbb{T}^k$ on $\mathbb{T}^k$. Since any two translations commute, we can view d many translations as generating an action of $\mathbb{Z}^d$ on $\mathbb{T}^k$. Suppose $u_1, \dots, u_d \in \mathbb{T}^k$, and let $u = (u_1, \dots, u_d) \in (\mathbb{T}^k)^d$ be the d-tuple containing all the u_i. Let $a_u\colon \mathbb{Z}^d \curvearrowright \mathbb{T}^k$ be the action of $\mathbb{Z}^d$ on $\mathbb{T}^k$ where the ith generator of $\mathbb{Z}^d$ acts by translating by u_i. We use the symbol $\cdot_u$ to denote this action, so

$$(n_1, \ldots, n_d) \cdot_u x = n_1 u_1 + \ldots + n_d u_d + x.$$

Say that this action a_u is *free* if for all $g \in \mathbb{Z}^d$ with $g \neq 0$ and all x, $g \cdot_u x \neq x$ or equivalently, $n_1 u_1 + \ldots + n_d u_d \neq 0$ for all $(n_1, \ldots, n_d) \neq 0$. Note that the action a_u is free for almost every $u \in (\mathbb{T}^k)^d$. This is because for every nonzero $(n_1, \ldots, n_d) \in \mathbb{Z}^d$, the set of $(u_1, \ldots, u_d) \in (\mathbb{R}^k)^d$ such that $n_1 u_1 + \ldots + n_d u_d = 0$ is Lebesgue null since it is a hyperplane of dimension less than kd. Hence its projection to $\mathbb{T}^k$ is also Lebesgue null.

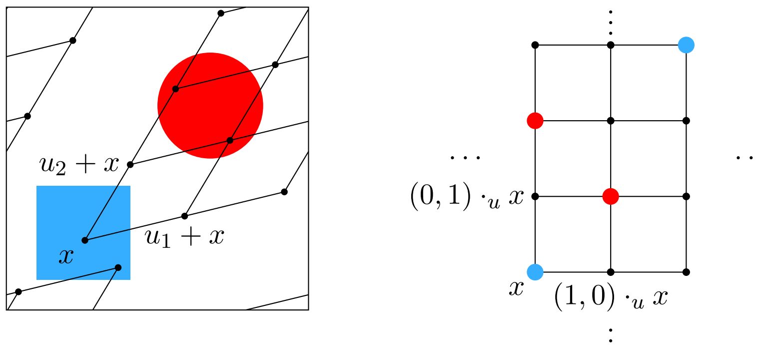

FIGURE 4. Combining Propositions 2.5 and 2.2 we get a new perspective on the problem of circle squaring as follows. Given $u_1, \ldots, u_d \in \mathbb{T}^2$ consider the Schreier graph of the action a_u of $\mathbb{Z}^d$ that they generate on $\mathbb{T}^2$ where $x, y \in \mathbb{T}^2$ are adjacent if there exists some u_i so $y = u_i + x$ or $x = u_i + y$. We can visualize each connected component of this graph as a copy of $\mathbb{Z}^d$. We need to show there is some choice of $u_1, \ldots, u_d$ so that there is a bounded distance bijection between the points of the circle and the square, where distance is measured in the graph metric. In the figure above we have drawn a small part of this graph and how we visualize the connected component of some x in an action of $\mathbb{Z}^2$ by two vectors u_1 and u_2.

Fix some $u \in (\mathbb{T}^k)^d$. Key to the proof will be studying translates of the hypercube $\{0, \ldots, N-1\}^d \subseteq \mathbb{Z}^d$ of side length N under the action a_u, and we define specific notation for these sets. Let

$$F_N(u, x) = \{n_1 u_1 + \ldots + n_d u_d + x \colon 0 \leq n_i < N\} = \{0, \ldots, N-1\}^d \cdot_u x.$$

for any $x \in \mathbb{T}^k$ and $N \in \mathbb{N}$ so $F_N(u, x) \subseteq \mathbb{T}^k$ is a finite set. Note that $\cdot_u$ encodes the generators $u_1, \ldots, u_d$ of the action on the right hand side of this equation.

If F is a finite set, we let $|F|$ denote the cardinality of F. We now have the following proposition, which motivates the main part of the circle squaring proof. It shows that if A and B are equidecomposable by translations, then for all N and all x, the number of points in $F_N(u, x) \cap A$ and $F_N(u, x) \cap B$ must be very close to each other:

Proposition 2.6. *Suppose $u \in (\mathbb{T}^k)^d$, $a_u\colon \mathbb{Z}^d \curvearrowright \mathbb{T}^k$ is a free action, and $A, B \subseteq \mathbb{T}^k$ are equidecomposable in the action a_u. Then there exists a constant C so that for all N and all x,*

$$\left| \frac{|F_N(u, x) \cap A|}{N^d} - \frac{|F_N(u, x) \cap B|}{N^d} \right| \leq C N^{-1}. \tag{1}$$

Proof. By Proposition 2.2, there is a finite $S \subseteq \mathbb{Z}^d$, a bijection $f\colon A \to B$, and some m so that $d_{\mathrm{Sch}(a_u, S)}(x, f(x)) \leq m$ for all $x \in A$. Let $b > 0$ be such that $\|g\|_\infty \leq b$ for all $g \in S$. That is, b is an upper bound on the absolute value on the coordinates of every $g \in S$.

If $x \in A$, then $f(x) = h \cdot x$ for some $h \in S^{\leq m}$. Now h is a sum of at most m elements of S and so each coordinate of h is at most mb in absolute value. Hence, for all N, $f(A \cap \{0, \ldots, N-1\}^d \cdot_u x) \subseteq B \cap \{-mb, \ldots, N-1+mb\}^d \cdot_u x$. So since f is a bijection, $|A \cap \{0, \ldots, N-1\}^d \cdot_u x| \leq |B \cap \{-mb, \ldots, N-1+mb\}^d \cdot_u x|$, and since $\{-mb, \ldots, N-1+mb\}^d \setminus \{0, \ldots, N-1\}^d$ contains $(N+2mb)^d - N^d \ll_{m,b,d} N^{d-1}$ elements, there is a constant C so that

$$|A \cap F_N(u,x)| \leq |B \cap F_N(u,x)| + (N+2mb)^d - N^d \leq |B \cap F_N(u,x)| + CN^{d-1},$$

A similar argument using the bijection $f^{-1}\colon B \to A$ in place of f shows that we must have $|B \cap F_N(u,x)| \leq |A \cap F_N(u,x)| + CN^{d-1}$, so we are finished. Figure 5 gives an illustration of this proof. □

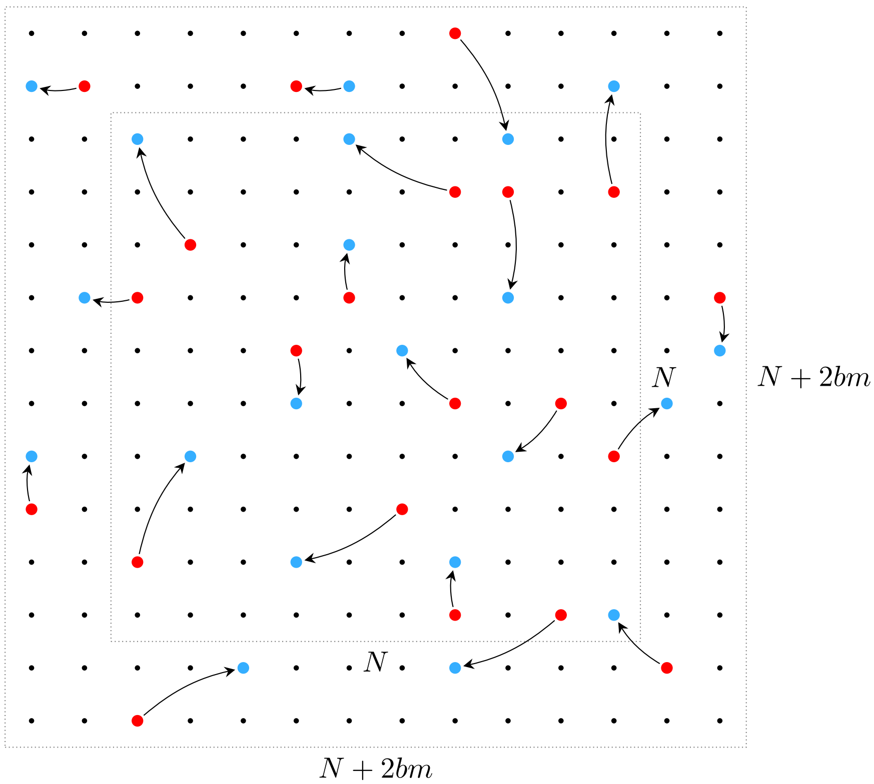

FIGURE 5. An illustration of the proof of Proposition 2.6 in the case $d = 2$. The grid of points represents vertices in a connected component of the graph $\mathrm{Sch}(a_u, S)$. We have colored points in A red, and points in B blue. An equidecomposition corresponds to a bijection of bounded distance in $\mathrm{Sch}(a_u, S)$ by Proposition 2.2. Hence, if we take an $N \times N$ square in the grid, the points that are in A map to points of B that are in a $(N+2bm) \times (N+2bm)$ square centered around it, where bm is the largest distance (in the sup metric) a point can be moved by f. The "square annulus" of points that are in the $(N+2bm) \times (N+2bm)$ square but not in the $N \times N$ square has size at most $4(bm)^2 + 4bmN \leq CN$ for all $N \geq 1$ where $C = 4bm + 4(bm)^2$. Thus, for all N, $|A \cap F_N(u,x)| \leq |B \cap F_N(u,x)| + CN$ and $|B \cap F_N(u,x)| \leq |A \cap F_N(u,x)| + CN$.

Since equation (1) must be true if A and B are equidecomposable by Proposition 2.6, it is natural that a key part of the proof of circle squaring is proving tight bounds on the number of points of

A and B that intersect the set $F_N(u,x)$. Indeed, what Laczkovich shows is that a slightly stronger condition than equation (1) suffices to prove that A and B are equidecomposable.

Suppose that A is a Lebesgue measurable subset of $\mathbb{T}^k$ and F is a finite subset of $\mathbb{T}^k$. We define the *discrepancy* of A with respect to F as

$$D(F,A) = \left| \frac{|A \cap F|}{|F|} - \lambda(A) \right|.$$

where $|A \cap F|$ denotes the cardinality of the set $A \cap F$. If we randomly selected points to be in F, then we would expect that the fraction of points of F that are inside A to be equal to $\lambda(A)$. Discrepancy measures the difference between the actual number of points of F that are inside A compared to this expected value $\lambda(A)$. It is a quantitative measure of the equidistribution of the set F with respect to A. This notion of discrepancy has connections to many areas of mathematics such as number theory, harmonic analysis, numerical integration, and optimal transport. See the books [BC], [DT], and [KN] for introductions to the theory of discrepancy and its applications.

Note that if $\lambda(A) = \lambda(B)$, then a bound on discrepancy of $F_N(u,x)$ with respect to A and B gives a bound on the quantity $\left| \frac{|F_N(u,x) \cap A|}{N^d} - \frac{|F_N(u,x) \cap B|}{N^d} \right|$ from Proposition 2.6 since

$$\begin{aligned} (2) \qquad \left| \frac{|F_N(u,x) \cap A|}{N^d} - \frac{|F_N(u,x) \cap B|}{N^d} \right| &= \left| \frac{|F_N(u,x) \cap A|}{N^d} - \lambda(A) + \lambda(B) - \frac{|F_N(u,x) \cap B|}{N^d} \right| \\ &\leq D(F_N(u,x),A) + D(F_N(u,x),B) \end{aligned}$$

The new proof we give of Laczkovich's theorem on equidecompositions breaks down into the following two lemmas, just as in Laczkovich's proof [L92]. First, a lemma showing that good estimates on the discrepancy of $F_N(u,x)$ with respect to A and B imply that they are equidecomposable. This is "almost" a converse to the necessary condition in Proposition 2.6; it replaces the exponent 1 in that proposition with $1+\delta$ for an arbitrarily small δ.

Lemma A ([L92, Theorem 1])**.** Suppose $u \in (\mathbb{T}^k)^d$, $a_u \colon \mathbb{Z}^d \curvearrowright \mathbb{T}^k$ is a free action, $A, B \subseteq \mathbb{T}^k$ have the same positive Lebesgue measure $\lambda(A) = \lambda(B) > 0$, and there are constants C and $\delta > 0$ so that $D(F_N(u,x),A) \leq CN^{-1-\delta}$ and $D(F_N(u,x),B) \leq CN^{-1-\delta}$ for all N that are powers of 2. Then A and B are equidecomposable in the action a_u.

This lemma was originally proved by Laczkovich [L92, Theorem 1] using an ingenious counting argument and the Hall-Rado matching theorem. The new proof we give of Lemma A uses ideas from the theory of flows in graphs, and compactness arguments to turn finite combinatorics into infinite combinatorics. We prove this lemma in Sections 3 and 4. In Section 3 we introduce flows in graphs and show that under the assumptions of Lemma A, there is a bounded flow ϕ between A and B. This flow ϕ is in a Schreier graph of the form $\mathrm{Sch}(a_u, S)$ as in Definition 2.1. Then in Section 4 we show this bounded flow ϕ can be transformed into a bounded distance bijection, and hence an equidecomposition by Proposition 2.2.

The second lemma shows we can find $u \in (T^k)^d$ making the discrepancy estimates needed in the hypothesis of Lemma A true in the case where $A, B \subseteq \mathbb{T}^k$ have boundaries with upper box dimension less than k:

Lemma B ([L92, Lemma 2])**.** For every k and $\epsilon > 0$, there exist a positive integer d, $u \in (\mathbb{T}^k)^d$, and $\delta > 0$ such that for all sets $A \subseteq \mathbb{T}^k$ with $\overline{\dim}_{\mathrm{box}}(\partial A) < k - \epsilon$, there is a C such that

$$D(F_N(u,x),A) \leq CN^{-1-\delta}$$

for every $x \in \mathbb{T}^k$ and N that is a power of 2.

The proof of this second lemma uses ideas from Diophantine approximation, discrepancy theory, and Fourier analysis. We give the proof in Sections 5, 6 and 7. In Section 5 we use the Erdős-Turán inequality to prove discrepancy bounds for sets $F_N(u,x)$ in the 1-dimensional torus $\mathbb{T}$ for a.e. $u \in (\mathbb{T})^d$. In Section 6 we lift these 1-dimensional discrepancy estimates to the k-dimensional discrepancy estimates needed to prove Lemma B. Finally, in Section 7 we give a proof of the Erdős-Turán inequality for self-containedness.

We note that in [L92], Laczkovich proves a stronger result than Lemma B using the Erdős-Turán-Koksma inequality, which shows that the conclusion of Lemma B holds for almost every $u \in (\mathbb{T}^k)^d$, for all N instead of just N that are powers of 2, and with a better bound on the right hands size. We have proved the version above because it is all that is needed for the proof of the main theorem, proving the above version simplifies many calculations. These simpler estimates also yield our new corollaries.

Together the above two lemmas yield the following theorem of Laczkovich which is the main result we give a new proof of:

Theorem C ([L92, Theorem 3])**.** Suppose $k \geq 1$ and suppose $A, B \subseteq \mathbb{R}^k$ are bounded Lebesgue measurable sets such that $\lambda(A) = \lambda(B) > 0$, and $\overline{\dim}_{\mathrm{box}}(\partial A) < k$ and $\overline{\dim}_{\mathrm{box}}(\partial B) < k$. Then A and B are equidecomposable by translations.

Proof. By scaling and translating A and B, we may assume $A, B \subseteq [0,1)^k$, since scaling both sets by the same factor does not change whether they are equidecomposable by translations. Viewing A and B as subsets of $\mathbb{T}^k$, by Lemma B, there exist a positive integer d, $u \in (\mathbb{T}^k)^d$, $\delta > 0$, and $C > 0$ such that $D(F_N(u,x), A) + D(F_N(u,x), B) \leq CN^{-1-\delta}$ for all x and all N that is a power of 2. Note the constant C here is the sum of the two constants C obtained in Lemma B for the sets A and B.

So by Lemma A, the sets A and B are equidecomposable in the action a_u, and hence A and B are equidecomposable by translations in $\mathbb{T}^k$. Finally by Proposition 2.5 this implies A and B are equidecomposable by translations in $\mathbb{R}^k$. □

A positive solution to Tarski's circle squaring problem is a corollary of this theorem, since by Proposition 2.4 the topological boundaries of a disk and square have upper box dimension 1, which is strictly less than 2.

Finally, our proof also gives a new sufficient condition for when two sets can be equidecomposed using translations whose coordinates are integer linear combinations of finitely many real numbers. In this theorem, $\|x\|$ denotes the distance from x to the nearest integer.

Theorem D. Suppose $\epsilon > 0$ and $A, B \subseteq \mathbb{R}^k$ are bounded Lebesgue measurable sets such that $\lambda(A) = \lambda(B) > 0$, and $\overline{\dim}_{\mathrm{box}}(\partial A) \leq k - \epsilon$ and $\overline{\dim}_{\mathrm{box}}(\partial B) \leq k - \epsilon$. Suppose $c > 0$ and $u_1, \ldots, u_d \in \mathbb{R}$ are irrational numbers linearly independent over $\mathbb{Q}$ so that for all N that are powers of 2,

$$\sum_{n=1}^{N} \frac{1}{n \prod_i \|nu_i\|} \ll_u N^c,$$

and $c < d\epsilon - 1$. Then A and B are equidecomposable in $\mathbb{R}^k$ by finitely many translations whose coordinates are integer linear combinations of $1, u_1, \ldots, u_d$.

We note that summations of the above form: $\sum_{n=1}^{N} \frac{1}{n \prod_i \|nu_i\|} \ll_u N^c$ are studied in number theory and are known as "sums of products of fractional parts" and are related to diophantine approximation and the Littlewood conjecture. See for example [LV], [BHV], and [F19].[1] The new content of Theorem D compared to Laczkovich's work in [L92] is that [L92] relies on upper bounds on "higher-dimensional" sums of this form obtained from the Erdős-Turán-Koksma inequality (see [L92, Lemma 2]). It is an open problem to find reasonable upper bounds on those higher dimensional

[1]Recall that the Littlewood conjecture is the famous open problem in Diophantine approximation that for any two real numbers a and b, $\liminf_{n\to\infty} n\|na\|\|nb\| = 0$.

sums in the case when the numbers are algebraic, which is why squaring the circle with algebraic translations had previously been an open problem. (See the discussion in [MU2, Section 3].)

We can use Theorem D to answer an open 1990 question of Laczkovich [L90, Section 10.3], by showing that the equidecompositions in Theorem C can be achieved using translations with algebraic irrational coordinates. This is an immediate corollary of Schmidt's theorem on simultaneous approximations of algebraic irrationals [S70]. Schmidt's theorem is a generalization of Roth's fundamental theorem on Diophantine approximation that we discuss briefly in Section 5.

Theorem 2.7. *Suppose $k \geq 1$ and suppose $A, B \subseteq \mathbb{R}^k$ are bounded Lebesgue measurable sets such that $\lambda(A) = \lambda(B) > 0$, and $\overline{\dim}_{\text{box}}(\partial A) < k$ and $\overline{\dim}_{\text{box}}(\partial B) < k$. Then A and B are equidecomposable by translations whose coordinates are algebraic irrationals.*

Proof. Schmidt's theorem on Diophantine approximation of algebraic irrational numbers [S70] states that if $u_1, \ldots, u_d \in \mathbb{R}$ are algebraic irrationals that are linearly independent over $\mathbb{Q}$, then for every $\delta > 0$, there is a constant C so that for every integer $n > 0$, $\prod_{i=1}^{d} \|nu_i\| > Cn^{-1-\delta}$. In this case

$$\sum_{n=1}^{N} \frac{1}{n \prod_i \|nu_i\|} \leq \sum_{n=1}^{N} n^{\delta} \ll N^{1+\delta}.$$

So if we choose d sufficiently large and δ sufficiently small, then the result follows by Theorem D. □

In our subsequent paper [MU2] we also prove better upper bounds on these sums of products of fractional parts for algebraic irrational numbers than those that come from this naive application of Schmidt's theorem. This yields improved bounds on the dimension d of the action of $\mathbb{Z}^d$ in which one can square the circle using algebraic irrational coordinates.

3. A flow from the circle to the square

In this section we introduce flows on graphs, and show that under the hypothesis of Lemma A, there is a certain kind of flow between the characteristic functions of A and B.

Suppose $G = (V, E)$ is a simple graph (an undirected graph without loops or multiple edges). We define the associated directed graph $\vec{G} = (V, \vec{E})$ by replacing each undirected edge $\{x, y\}$ in G with a pair of directed edges (x, y) and (y, x) in $\vec{G}$ going from x to y and from y to x. A *flow* of G is a real-valued function $\phi \colon \vec{E} \to \mathbb{R}$ on the directed edges of $\vec{G}$ so that $\phi(x, y) = -\phi(y, x)$ for all edges $(x, y) \in \vec{E}$. A *potential function* on G is a real-valued function $f \colon V \to \mathbb{R}$ on the vertices of the graph. The *flow divergence operator* assigns to each flow $\phi \colon \vec{E} \to \mathbb{R}$ a potential $\operatorname{div} \phi \colon V \to \mathbb{R}$ by the formula:

$$\operatorname{div} \phi(x) = \sum_{\{x,y\} \in E} \phi(y, x).$$

That is, the potential at the vertex x in $\operatorname{div} \phi$ is the sum of the flow on all incoming edges to x. Since $\phi(x, y) = -\phi(y, x)$ for all edges $(x, y) \in \vec{E}$, we can also think of $\operatorname{div} \phi(x)$ as being the sum of all incoming positive flow minus all positive outgoing flow. So $\operatorname{div} \phi(x)$ measures the difference between the inflow and outflow at x. If we think of a potential function f as assigning an amount of some commodity to each vertex, then given a flow ϕ on G, $\operatorname{div} \phi + f$ is the potential function that measures the new amount of commodity at each vertex after we transport the amount $\phi(x, y)$ over each edge from x to y. Note that $\operatorname{div} \phi$ defines a homomorphism from the set of flows to the set of potential functions, regarding both as additive abelian groups under the operation of pointwise addition. That is, $\operatorname{div}(\phi + \psi) = \operatorname{div} \phi + \operatorname{div} \psi$ for all flows ϕ and ψ.

These types of flows are often studied in graph theory, such as in the max-flow min-cut theorem where flows ϕ with the property that $\operatorname{div} \phi = 0$ are called *circulations*. In the setting of homology of graphs, a flow is called a 1-chain, a potential function is called a 0-chain, and the differential

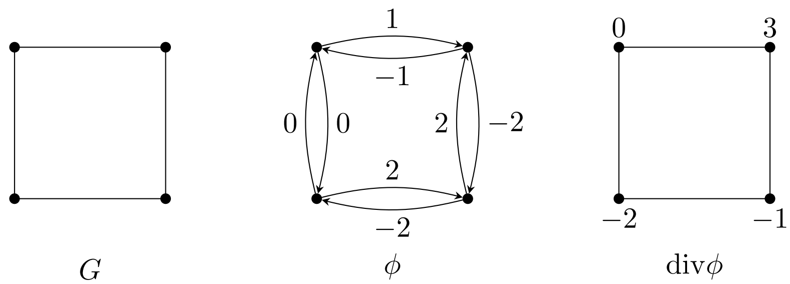

FIGURE 6. An undirected graph G, a flow ϕ on the graph G (mapping each directed edge of $\vec{G}$ to a real number), and the flow divergence operator applied to the flow $\operatorname{div}\phi$, which gives a function measuring the net inflow at each vertex.

operator is called the boundary operator. However, we will not use any ideas or language from homology in what follows.

Let $A, B \subseteq \mathbb{T}^k$ be two sets, $a_u : \mathbb{Z}^d \curvearrowright \mathbb{T}^k$ be an action, and S be a generating set for $\mathbb{Z}^d$. In this section, under certain conditions we will show that there is a flow ϕ of $\operatorname{Sch}(a_u, S,)$ such that $\operatorname{div}\phi + 1_A = 1_B$. Recall here that if X is a space and $A \subseteq X$, we let $1_A \colon X \to \{0, 1\}$ be the *characteristic function* of A, where $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$. To motivate our study of this flow problem, we note that flow can be viewed as giving a kind of real-valued generalization of equidecompositions. Instead of partitioning A into pieces and translating them by group elements to obtain a partition of B as in an equidecomposition, we can more generally write the characteristic function 1_A of A as a sum of finitely many bounded real-valued functions and translate these functions by group elements so their sum becomes 1_B. So an equidecomposition corresponds to the case of such a decomposition when the functions are $\{0, 1\}$-valued.

Flows correspond to bounded real-valued equidecompositions as follows: if $a \colon G \curvearrowright X$ is an action of a group G on a space X, $A, B \subseteq X$, and $S = \{g_1, \ldots, g_n\} \subseteq G$ is finite, then there is a flow ϕ of $\operatorname{Sch}(a, S)$ such that $\operatorname{div}\phi + 1_A = 1_B$ if and only if there are bounded functions $f_0, \ldots, f_n \colon X \to \mathbb{R}$ such that $1_A(x) = \sum_{i=0}^n f_i(x)$ and $1_B(x) = \sum_{i=0}^n f_i(g_i^{-1} \cdot x)$ where g_0 is the identity. We will not use this fact in our proof, so we leave it as an exercise for the reader.[2]

The idea of studying flows as an intermediate step towards finding an equidecomposition is an instance of a general theme: when studying an integer optimization problem, it is often helpful to study the relaxation of the problem obtained by removing the constraints that the solution must be integer and working in this large space to make progress towards solving the problem. For example, this idea is used in graph theory to prove theorems about matchings by studying the matching polytope, and in computer science to make fast algorithms for solving many optimization problems. We will see in Lemma 4.1 one method of converting real-valued solutions of such problems to integer solutions: by taking an extreme point in the convex set of all real-valued solutions.

We now show that given a sufficiently small bound $\rho(N)$ on $D(F_N(u, x), A) + D(F_N(u, x), B)$ (in particular such that $\sum_{n \geq 0} 2^n \rho(2^n) < \infty$), we can find a bounded flow ϕ of a Schreier graph $\operatorname{Sch}(a_u, S)$ so that $\operatorname{div}\phi + 1_A = 1_B$. Below we use $\|\phi\|_\infty$ to denote the supremum norm of ϕ: $\|\phi\|_\infty = \sup_{(x,y) \in \vec{E}} |\phi(x, y)|$.

Lemma 3.1. *Suppose $A, B \subseteq \mathbb{T}^k$ are sets, $u \in (\mathbb{T}^k)^d$, a_u is a free action, and $\rho \colon \mathbb{N} \to \mathbb{R}^+$ is such that $\sum_{n \geq 0} 2^n \rho(2^n) < \infty$ and*

$$D(F_N(u, x), A) + D(F_N(u, x), B) \leq \rho(N)$$

[2]A simple case is where the action is free and there is no $g \in S$ so that $g^2 = 1$ (this is the case in this paper). Then we can let $f_i(x) = \phi(x, g_i \cdot x)$ for $i \geq 1$, and $f_0(x) = 1_A - \sum_{i=1}^n \phi(x, g_i \cdot x)$.

for all $x \in \mathbb{T}^k$ and N that are powers of 2. Let $S = \{0,1\}^d \setminus \{0\}^d$ be the nonidentity group elements whose coordinates are all 0 or 1. Then there is a flow ϕ of Sch(a_u, S) such that $\operatorname{div}\phi + 1_A = 1_B$ *and* $\|\phi\|_\infty \leq 2^{-d}\sum_{n\geq 0} 2^n \rho(2^n)$.

Proof. Our proof will be by giving an explicit formula for ϕ and then proving that it works. The idea behind the definition of ϕ will be based on iteratively averaging the difference $1_A - 1_B$ over nested cubes of side length 2^n and pushing mass along the directions in S to yield this averages at larger and larger scales.

We will use the notation

$$R(n,d) = \{0, \dots, n-1\}^d$$

for the hypercube in $\mathbb{Z}^d$ of side length n at the origin. Let

$$f_n(x) = \sum_{z \in (-R(n,d)) \cdot_u x} \frac{1_A(z) - 1_B(z)}{n^d}. \tag{3}$$

So $f_n(x)$ is the average value of $1_A - 1_B$ over the set $(-R(n,d)) \cdot_u x$ which has size n^d. By $-R(n,d)$ we mean the set $\{-v \colon v \in R(n,d)\}$ which is just the reflection of this hypercube around the origin. Note $f_1 = 1_A - 1_B$, since $R(1,d)$ contains the single element $0 \in \mathbb{Z}^d$. Now for all $x \in \mathbb{T}^k$, we have $|f_n(x)| \leq \rho(n)$. To see this, note first that $(-R(n,d)) \cdot_u x = R(n,d) \cdot_u y$ where $y = (-n+1, -n+1, \dots, -n+1) \cdot_u x$. That is, the reflection of a hypercube around the origin is also a shift of the hypercube. So

$$|f_n(x)| = \left| \sum_{z \in R(n,d) \cdot_u y} \frac{1_A(z) - 1_B(z)}{n^d} \right| = \left| \frac{|F_n(u,y) \cap A|}{|F_n(u,y)|} - \frac{|F_n(u,y) \cap B|}{|F_n(u,y)|} \right| \tag{4}$$
$$\leq D(F_n(u,y), A) + D(F_n(u,y), B) \leq \rho(n).$$

Define the flow ϕ of Sch(a_u, S) by

$$\phi(x, g \cdot_u x) = 2^{-d} \sum_{n \geq 0} \sum_{0 \leq m < 2^n} f_{2^n}((-mg) \cdot_u x)$$

for all $x \in \mathbb{T}^k$ and $g \in S$. We will decompose this infinite sum defining ϕ as follows:

$$\phi = \sum_{n \geq 0} \phi_{2^n}$$

where ϕ_n is the flow defined by

$$\phi_n(x, g \cdot_u x) = 2^{-d} \sum_{0 \leq m < n} f_n((-mg) \cdot_u x). \tag{5}$$

for all $x \in \mathbb{T}^k$ and $g \in S$. Note that the series defining ϕ converges since by the triangle inequality, (4), and (5) we have

$$|\phi_n(x, g \cdot_u x)| \leq 2^{-d} n \rho(n). \tag{6}$$

Hence,

$$\|\phi\|_\infty \leq \sum_{n \geq 0} \|\phi_{2^n}\|_\infty \leq 2^{-d} \sum_{n \geq 0} 2^n \rho(2^n)$$

which converges by assumption and is our desired bound on ϕ.

The key property of ϕ_n we will show below is that

$$\operatorname{div}\phi_n + f_n = f_{2n}. \tag{7}$$

This implies that $\operatorname{div}(\phi_{2^0} + \phi_{2^1} + \ldots + \phi_{2^{n-1}}) + f_1 = f_{2^n}$ by induction. So

$$\operatorname{div}\phi + (1_A - 1_B) = \left(\sum_{n \geq 0} \operatorname{div}\phi_{2^n}\right) + f_1 = \left(\lim_{m\to\infty} (\sum_{0 \leq n < m} \operatorname{div}\phi_{2^n}) + f_1\right) = \lim_{m\to\infty} f_{2^{m+1}} = 0$$

using (10). The last equality is since $\sum_{n\geq 0} 2^n \rho(2^n)$ converges, so $\rho(2^n) \to 0$ as $n \to \infty$ and $f_{2^{m+1}}(x) \leq \rho(2^{m+1})$ for all x by (4). So all that remains is to prove (7).

Figure 7 illustrates how the flows ϕ_n are defined and why the key property (7) is true. The definition of ϕ_n is based on choosing a 2^d-to-1 map from the set $R(2n, d)$ to $R(n, d)$, and ϕ_n flows along paths from each point to its image under this map to convert averages over hypercubes of side length n in f_n to averages over hypercubes of side length $2n$ in f_{2n}. The particular map we have used here from $R(2n, d)$ to $R(n, d)$ is applying the mod n map in every coordinate. Then the difference between any point and its image under this map is ng where $g \in \{0, 1\}^d$, so we can map along a straight line of length n in the direction g. This is why we have chosen the particular generating set S that we are using.

We now prove (7). To begin, note that $\operatorname{div}\phi_n + f_n(x) = f_n(x) + \sum_{g\in S}(\phi_n((-g)\cdot_u x, x) - \phi_n(x, g\cdot_u x))$. Now $\phi_n((-g)\cdot_u x, x) - \phi_n(x, g\cdot_u x)$ is telescoping, using equation (5):

(8)

$$\begin{aligned}
\phi_n((-g)\cdot_u x, x) - \phi_n(x, g\cdot_u x) &= 2^{-d}\left(\sum_{0\leq m<n} f_n((-mg)\cdot_u(-g\cdot_u x)) - \sum_{0\leq m<n} f_n((-mg)\cdot_u x)\right)\\
&= 2^{-d}\left(\sum_{1\leq m\leq n} f_n((-mg)\cdot_u x) - \sum_{0\leq m<n} f_n((-mg)\cdot_u x)\right)\\
&= 2^{-d}\left(f_n((-ng)\cdot_u x) - f_n(x)\right)
\end{aligned}$$

To simplify the last expression on the right hand size, of (8), note first that $R(2n, d) = \bigsqcup_{g\in\{0,1\}^d}(ng + R(n, d))$ since these sets partition the hypercube of side length $2n$ into 2^d hypercubes of side length n. Now since $f_{2n}(x)$ is the average of the function $f_1(x)$ over the set $(-R(2n, d))\cdot_u x$, by breaking up the set $R(2n, d)$ as above, and using that $f_n(x)$ is the average of f_1 over the set $(-R(n, d))\cdot_u x$, we get the formula $2^d f_{2n}(x) = \sum_{g\in\{0,1\}^d} f_n((-ng)\cdot_u x) = f_n(x) + \sum_{g\in S} f_n((-ng)\cdot_u x)$.

$$\sum_{g\in S}(f_n((-ng)\cdot_u x) - f_n(x)) = 2^d f_{2n}(x) - 2^d f_n(x). \tag{9}$$

since $|S| = 2^d - 1$ and so $\sum_{g\in S} f_n(x) = (2^d - 1)f_n(x)$.

Combining (8) and (9), we have

$$\begin{aligned}
\operatorname{div}\phi_n + f_n(x) &= f_n(x) + \sum_{g\in S}(\phi_n((-g)\cdot_u x, x) - \phi_n(x, g\cdot_u x))\\
&= f_n(x) + 2^{-d}\sum_{g\in S}(f_n((-ng)\cdot_u x) - f_n(x))\\
&= f_n(x) + f_{2n}(x) - f_n(x)\\
&= f_{2n}(x).
\end{aligned} \tag{10}$$

which concludes the proof of (7). □

Remark 3.2. We remark that the proof above actually gives a slightly stronger theorem. If we replace the assumption that $D(F_N(u, x), A) + D(F_N(u, x), B) \leq \rho(N)$ in Lemma 3.1 with the weaker

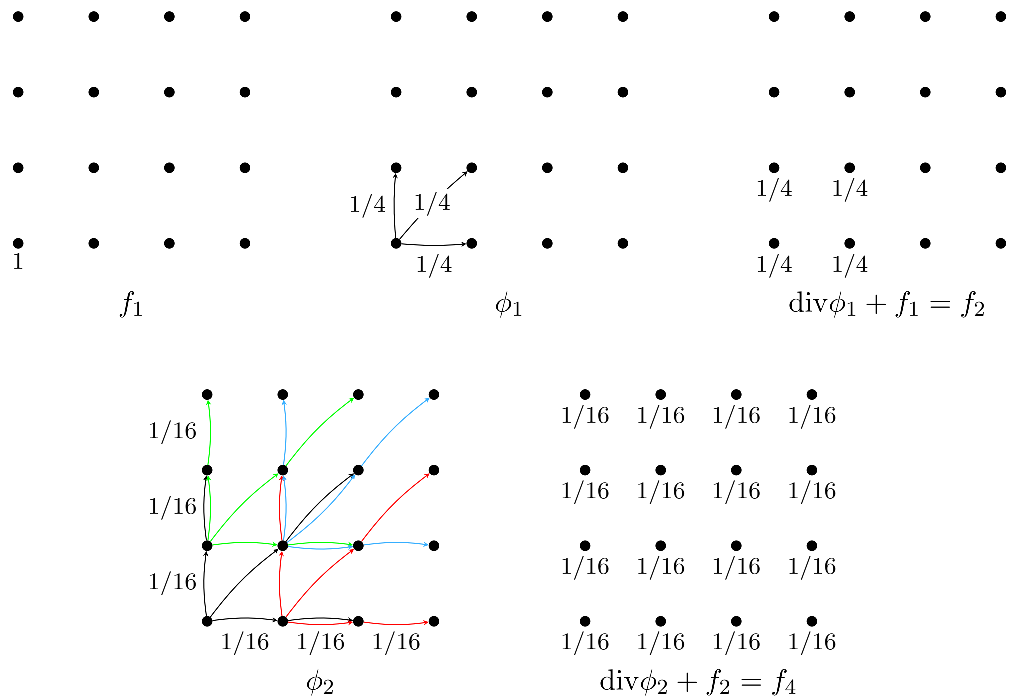

FIGURE 7. A depiction of the flows ϕ_1, ϕ_2 in the case when $d = 2$ and f_1 is supported at a single point x where $f_1(x) = 1$, showing how $\operatorname{div} \phi_1$ averages out the value of f_1 over a 2×2 square, and $\operatorname{div} \phi_1 + \phi_2$ averages out the value of f_2 over a 4×4 square. The flow ϕ_1 splits the value 1 into 4 parts, and flows $1/4$ to each of the points in the 2×2 box whose bottom left corner is x (also keeping $1/4$ stationary at the point). Then ϕ_2 splits the value of $1/4$ at each point in the 2×2 square into 4 parts once more, and flows an amount of $1/16$ along a path of length 2 to 4 of the 16 points in the 4×4 box whose bottom left corner is x. Black, red, green, and blue are used to show these four different starting point in the flow. And each arrow corresponds to a flow amount of $1/16$, so the paths where there are two arrows correspond to a total flow of $1/8$ along this edge. In the case when $f_1(x)$ has a different value, then the values in the flow are scaled by $f_1(x)$. And in general if $f_1(x)$ is supported on more than one point, then the flows ϕ_n consists of many shifted copies of this picture added together: one starting at each x where $f_1(x)$ is nonzero.

assumption

$$\left| \frac{|F_N(u,x) \cap A|}{|F_N(u,x)|} - \frac{|F_N(u,x) \cap B|}{|F_N(u,x)|} \right| \leq \rho(N).$$

Then we still have the same result that there is a flow ϕ of $\operatorname{Sch}(a_u, S)$ such that $\operatorname{div} \phi + 1_A = 1_B$ and $\|\phi\|_\infty \leq 2^{-d} \sum_{n=0}^{\infty} 2^n \rho(2^n)$. This is because our bound $D(F_N(u,x), A) + D(F_N(u,x), B) \leq \rho(N)$ in the lemma above is just used to establish the inequality $\left| \frac{|F_N(u,x) \cap A|}{|F_N(u,x)|} - \frac{|F_N(u,x) \cap B|}{|F_N(u,x)|} \right| \leq \rho(N)$ in equation (4) in the proof above. We will use this slightly stronger version of Lemma 3.1 in [MU2].

For context, recall Cauchy's condensation test for the convergence of a series: if $\rho\colon \mathbb{N} \to \mathbb{R}$ is a decreasing sequence of non-negative real numbers, then $\sum_{n \geq 1} \rho(n)$ converges if and only if $\sum_{n \geq 0} 2^n \rho(2^n)$ converges. So a sufficient condition for there to be a bounded flow from A to B is for $D(F_n(u,x), A) + D(F_n(u,x), B) \leq \rho(n)$ for a decreasing function ρ so that $\sum_{n=1}^{\infty} \rho(n)$ converges. We have kept the "condensed" version $\sum_{n=0}^{\infty} 2^n \rho(2^n)$ of the sum in Lemma 3.1 because later in

Lemma 6.2, it will be easier to bound $D(F_N(u,x),A)$ and $D(F_N(u,x),B)$ for only N that are powers of 2.

To define the flow above, we have used the unusual generating set $S = \{0,1\}^d \setminus \{0\}^d$ for $\mathbb{Z}^d$ instead of the standard generating set $E = \{e_1, \ldots, e_d\}$ where the ith coordinate of e_i is 1 and all other coordinates of e_i are 0. We finish by noting that it is easy to convert the bounded flow ϕ in the graph $\mathrm{Sch}(a,S)$ to a bounded flow ψ in the graph $\mathrm{Sch}(a,E)$ with the standard generating set, but keeping $\operatorname{div}\phi = \operatorname{div}\psi$. Indeed, if $a : G \curvearrowright X$ is any free action of a group G, and S and T are two different finite sets of generators for G, then we can convert any bounded flow ϕ in the graph $\mathrm{Sch}(a,S)$ into a bounded flow ψ in the graph $\mathrm{Sch}(a,T)$ so that $\operatorname{div}\phi = \operatorname{div}\psi$. To prove this, pick for each $(x,y) \in \mathrm{Sch}(a,S)$ a path $p(x,y)$ of minimal length in $\mathrm{Sch}(a,T)$ from x to y. Then add an amount of $\phi(x,y)$ to ψ along each edge in the path $p(x,y)$. Intermediate vertices z along this path will have an amount of $\phi(x,y)$ flowing both into and out of z, so the total contribution to $\operatorname{div}\psi$ is 0. However, the endpoints of this path still have a factor of $\phi(x,y)$ in the sum defining $\operatorname{div}\psi$ and so $\operatorname{div}\psi = \operatorname{div}\phi$.

We will give a specific formula that works in the case that we need, and which we'll use in our subsequent paper [MU2]. If $g \in \{0,1\}^d$, then we use the obvious path from 0 to g which is $e_{i_1}, e_{i_2}, \ldots, e_{i_n}$ where i_j is the jth coordinate where g is 1. Here we introduce a small piece of notation. If $g \in \{0,1\}^d$, let $g_{<k} \in \{0,1\}^d$ be defined by letting its ith coordinate be $g_{<k}(i) = g(i)$ if $i < k$ and $g_{<k}(i) = 0$ if $i \geq k$. So for example if $g = (1,0,1,0)$, then $g_{<3} = (1,0,0,0)$.

Proposition 3.3. *Suppose a is a free action of $\mathbb{Z}^d$, and ϕ is a bounded flow of Sch(a,S) where $S = \{0,1\}^d \setminus \{0\}^d$. Then there is a bounded flow ψ of Sch(a,E) where $\operatorname{div}\psi = \operatorname{div}\phi$ defined by*

$$\psi(x, e_i \cdot x) = \sum_{\{g \in S \colon g(i)=1\}} \phi(-g_{<i} \cdot x, g \cdot (-g_{<i}) \cdot x)$$

Proof. As we explained above, we'll show that when we compute $\operatorname{div}\psi$, the terms from intermediate steps along these paths cancel in a telescoping sum, and the only terms that remain are the ones in $\operatorname{div}\phi$. Formally,

$$\begin{aligned}
\operatorname{div}\psi &= \sum_{i \leq d} \big(\psi(-e_i \cdot x, x) - \psi(x, e_i \cdot x)\big) \\
&= \sum_{i \leq d} \sum_{\{g \in S \colon g(i)=1\}} \Big(\phi\big((-g_{<i} - e_i) \cdot x, g \cdot (-g_{<i} - e_i) \cdot x\big) - \phi\big(-g_{<i} \cdot x, g \cdot (-g_{<i}) \cdot x\big)\Big) \\
&= \sum_{g \in S} \sum_{\{i \colon g(i)=1\}} \Big(\phi\big((-g_{<i} - e_i) \cdot x, g \cdot (-g_{<i} - e_i) \cdot x\big) - \phi\big(-g_{<i} \cdot x, g \cdot (-g_{<i}) \cdot x\big)\Big) \\
&= \sum_{g \in S} (\phi(-g \cdot x, g \cdot (-g) \cdot x) - \phi(x, g \cdot x)) \\
&= \operatorname{div}\phi.
\end{aligned}$$

Where the second-to-last step above follows because if $g \in S$, then $\sum_{\{i \colon g(i)=1\}} \phi((-g_{<i} - e_i) \cdot x, g \cdot (-g_{<i} - e_i) \cdot x) - \phi(-g_{<i} \cdot x, g \cdot (-g_{<i}) \cdot x)$ is telescoping. This is because if $g(i) = 1$, and $j > i$ is least so that $g(j) = 1$, then $g_{<i} + e_i = g_{<j}$. The only two terms which do not cancel in this sum are the ones where i is least such that $g(i) = 1$ where $g_{<i} = 0$ and the one where i is greatest such that $g(i) = 1$ where $g_{<i} + e_i = g$. So the remaining terms are $\phi(-g \cdot x, g \cdot (-g) \cdot x) - \phi(x, g \cdot x)$.

We finish by noting that since the total number of elements of S that contain a 1 in the ith position is 2^{d-1}, this new flow is bounded by $\|\psi\|_\infty \leq 2^{d-1}\|\phi\|_\infty$. □

4. Converting flows into equidecompositions

Now we convert this bounded flow ϕ so that $\operatorname{div}\phi + 1_A = 1_B$ into an equidecomposition of A and B. First, we show how we can turn real-valued flows into integer-valued flows. Recall that a graph is *locally finite* if every vertex has finitely many neighbors.

Lemma 4.1 (The integral flow theorem)**.** *Suppose that $G = (V, E)$ is a locally finite directed graph, $f\colon V \to \mathbb{R}$ is a potential function on G taking values in $\mathbb{Z}$, ϕ is a flow of G such that $\operatorname{div}\phi = f$, and $c\colon \vec{E} \to \mathbb{N}$ is such that $|\phi(e)| \leq c(e)$ for all e. Then there is some integer-valued flow $\phi^*\colon \vec{E} \to \mathbb{Z}$ with the same properties: $\operatorname{div}\phi^* = f$ and $|\phi^*(e)| \leq c(e)$ for all e.*

Proof. For finite graphs, this is a standard fact in graph theory called the integral flow theorem (e.g. [D, Corollary 6.2.3]). Solutions to such flow problems can be found using the Ford-Fulkerson algorithm which gives integer solutions provided the constraints (the function f and the capacity function c) are integer. Note that if we add a single source s to the graph G and a single sink t, and an edge from s to x with capacity $f(x)$ if $f(x) > 0$, and an edge from x to t with capacity $-f(x)$ if $f(x) \leq 0$, then finding a flow ϕ of G so that $\operatorname{div}\phi = f$ is exactly the type of flow problem considered in the max-flow min-cut theorem. Hence if a flow problem has an $\mathbb{R}$-valued solution obeying integer constraints, it has a $\mathbb{Z}$-valued solution.

An alternate proof of this fact for finite graphs G is as follows. For any $f\colon V \to \mathbb{Z}$, the set $C_f = \{\phi\colon \operatorname{div}\phi = f \text{ and } (\forall e \in \vec{E})|\phi(e)| \leq c(e)\}$ of all flows satisfying these conditions is a compact convex subset of the finite dimensional vector space $\mathbb{R}^{\vec{E}}$. Convexity follows since if ϕ and ψ are such that $\operatorname{div}\phi = f$ and $\operatorname{div}\psi = f$, then for every $t \in [0, 1]$,

$$\operatorname{div}(t\phi + (1-t)\psi) = t(\operatorname{div}\phi) + (1-t)(\operatorname{div}\psi) = tf + (1-t)f = f.$$

The set C_f is nonempty since it contains ϕ. By the special case of Krein-Milman theorem that any convex compact set in $\mathbb{R}^n$ is the convex hull of its extreme points, set C_f therefore has an extreme point ϕ'. We claim ϕ' must take only integer values. If it does not, then there must be a cycle C in G so that each edge in the cycle has $\phi'(e) \notin \mathbb{Z}$. This is since any vertex x incident to a non-integer edge must be incident to at least two non-integer edges since $\sum_{\{x,y\}\in E}\phi(y, x) = f(x)$ is an integer. So following such edges must eventually give a cycle in a finite graph. Now we can represent ϕ' as a nontrivial convex combination. Let $\psi\colon \vec{E} \to \mathbb{R}$ be the function that takes values 1 on the edges around the cycle C, -1 on these same edges in the opposite direction, and 0 on all other edges. Take $\epsilon > 0$ smaller than the distance from $\phi'(e)$ to the nearest integer for every $e \in C$. Then both $\phi' + \epsilon\psi$ and $\phi' - \epsilon\psi$ are in C_f, and ϕ' is the nontrivial convex combination $1/2(\phi' + \epsilon\psi) + 1/2(\phi' - \epsilon\psi)$.

Now that we have proved this lemma for finite graphs, the case when G is infinite follows from the finite case by a standard compactness argument that is similar to the proof of the De Bruijn-Erdős theorem that a graph G has a k coloring if and only if all its finite subgraphs have a k-coloring [D, Theorem 8.1.3]. Let $\Phi_{\leq c} = \{\phi\colon \phi \text{ is a flow of } G \text{ and } (\forall(x, y) \in \vec{E})|\phi(x, y)| \leq c(x, y) \wedge \operatorname{ran}(\phi) \subseteq \mathbb{Z}\}$. Endow $\Phi_{\leq c}$ with the product topology, so basic open sets are of the form $\{\phi\colon (\forall i \leq m)\phi(e_i) = n_i\}$ for a finite sequence $e_0, \ldots, e_m \in \vec{E}$ of directed edges and values $n_0, \ldots, n_m \in \mathbb{Z}$. The space $\Phi_{\leq c}$ is compact by Tychonoff's theorem.

For each finite set $F \subseteq V$ of vertices, consider the closed set $A_F = \{\phi \in \Phi_{\leq c}\colon (\forall x \in F)\operatorname{div}\phi(x) = f(x)\}$. We will show that A_F is nonempty by using the integral flow theorem for finite graphs. To see this, consider the finite graph G_F with vertex set $F \cup \{x_0\}$ consisting of F and one new point x_0 defined as follows, where intuitively x_0 will represent all other vertices in G. Given $x \in F$, there is an edge $\{x, x_0\}$ from x to x_0 in G_F if x is G-adjacent to y for some $y \notin F$. An illustration of the construction of G_F from G and F is shown in Figure 8. Now let $\phi_F\colon \vec{G_F} \to \mathbb{R}$ be the flow of G_F defined by $\phi_F(x, y) = \phi(x, y)$ if $x, y \in F$, and $\phi_F(x, x_0) = \sum_{\{y\colon (x,y)\in G\wedge y\notin F\}}\phi(x, y)$, so the flow from x to x_0 in ϕ_F is the total flow from x to all its neighbors not in F. Similarly define

$c_F \colon \vec{E_F} \to \mathbb{Z}$ by $c_F(x,y) = c(x,y)$ if $x, y \in F$ and $c_F(x, x_0) = \sum_{\{y \colon (x,y) \in G \wedge y \notin F\}} c(x,y)$. Now $\operatorname{div} \phi(x) = \operatorname{div} \phi_F(x)$ for all $x \in F$. Since Lemma 4.1 is true for finite graphs, G_F has an integer flow bounded by c, and so A_F is nonempty.

To finish, the collection of sets A_F is a family of closed sets with the finite intersection property. Hence, since $\Phi_{\leq c}$ is compact, $\bigcap_{F \subseteq V \text{ finite}} A_F$ is nonempty. An element of $\bigcap_{F \subseteq V \text{ finite}} A_F$ gives the desired ϕ^*. □

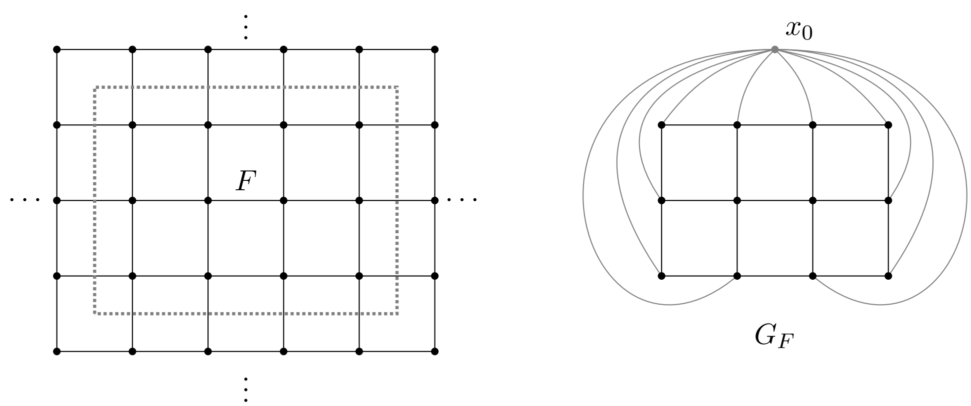

FIGURE 8. A picture of the construction of G_F from G and F in the proof of Lemma 4.1. The new vertex x_0 and edges we add are drawn in brown.

For the specific case of the graph $\operatorname{Sch}(a_u, S)$ in Lemma 3.1, there are other ways of proving Lemma 4.1. Suppose ϕ is a flow in a graph G, $f \colon V \to \mathbb{R}$, and $\operatorname{div} \phi = f$. Then if C is a cycle in G and we add ϵ to a flow ψ along each edge of C to obtain a new flow ϕ', then the divergence is the same: $\operatorname{div} \phi' = f$ since there is both a new inflow of ϵ and outflow of ϵ at each vertex of the cycle, which cancel. This process can be used to make a single edge (x, y) in the flow ϕ integer-valued by letting ϵ equal the fractional part of $\phi(x, y)$. Then by taking a suitable sequence of cycles and iteratively performing this process one can make the flow integer valued in each connected component. For example, one can use an Euler path on the 3-cycles of the graph which renders two 3-cycles adjacent if they intersect in exactly one edge.

We are now ready to convert flows into equidecompositions. We can do this provided there is a number n so that $F_n(u, x) = \{0, \ldots, N-1\}^d \cdot_u x$ contains sufficiently many point of A and B for all x. Note that below $E = \{e_1, \ldots, e_d\}$ is the usual generating set for $\mathbb{Z}^d$ where the ith coordinate of e_i is $e_i(i) = 1$ and $e_i(j) = 0$ if $j \neq i$.

Theorem 4.2. *Suppose $A, B \subseteq \mathbb{T}^k$ are sets, $E = \{e_1, \ldots, e_d\}$ is the standard generating set for $\mathbb{Z}^d$, and there is a flow ψ of $Sch(a_u, E)$ such that $\operatorname{div} \psi + 1_A = 1_B$ and $\|\psi\|_\infty \leq b$ where $b \in \mathbb{N}$. Suppose also that there is an n so that for all $x \in \mathbb{T}^k$, $|F_n(u, x) \cap A| \geq 2dn^{d-1}b$ and $|F_n(u, x) \cap B| \geq 2dn^{d-1}b$. Then A and B are equidecomposable in $\mathbb{T}^k$ by at most $(2d+1)n^d$ pieces.*

Proof. In what follows, write $\cdot$ instead of $\cdot_u$. By the Axiom of Choice, let $C \subseteq \mathbb{T}^k$ be a set that meets each connected component of $\operatorname{Sch}(a_u, E)$ in exactly one point. As in Lemma 3.1, write $R(n, d) = \{0, \ldots, n-1\}^d$ for the hypercube in $\mathbb{Z}^d$ of side length n at the origin. Let $Y = \{(ng + R(n, d)) \cdot x \colon x \in C \wedge g \in \mathbb{Z}^d\}$, so Y is a partition of $\mathbb{T}^k$ into sets of the form $Q = R(n, d) \cdot y$ where $y \in \mathbb{T}^k$. Our plan is to, as much as possible, map the points of A in each set Q in Y to the points of B in Q. However, each set Q may have a different number of points of A and B. To fix these "errors", we will map excess points of A to points of B in adjacent cubes, and excess points of B to points of A in adjacent cubes. Solving this problem is a flow problem, which one can use to motivate our original definitions of flows.

Precisely, define a graph H on Y where $Q_0 \in Y$ is adjacent to $Q_1 \in Y$ if there is an edge $(x, y) \in \mathrm{Sch}(a_u, E)$ so that $x \in Q_0$ and $y \in Q_1$. So H is the graph minor obtained from G by contracting each set of vertices of the form $(ng + R(n, d)) \cdot x$ in $\mathrm{Sch}(a_u, E)$ into the single vertex $Q = (ng + R(n, d)) \cdot x$ in H.

Let ψ' be the flow of H where

$$\psi'(Q_0, Q_1) = \sum_{(x_0,x_1)\in E\colon\, x_0\in Q_0 \wedge x_1\in Q_1} \psi(x_0, x_1).$$

Since there are exactly n^{d-1} edges between Q_0 and Q_1, ψ' is bounded by bn^{d-1}. Applying Lemma 4.1, we may assume ψ' takes integer values so $\mathrm{ran}(\psi') \subseteq \mathbb{Z}$ while it is still bounded by bn^{d-1}. Since each $Q \in H$ is adjacent to $2d$ other vertices (two for each e_i), the total outflow from each vertex is at most $2dn^{d-1}b$.

Let $f_A, f_B \colon Y \to \mathbb{Z}$ be defined by $f_A(Q) = \sum_{z\in Q} 1_A(z)$ and $f_B(Q) = \sum_{z\in Q} 1_B(z)$. So by assumption, $f_A(Q), f_B(Q) \geq 2dn^{d-1}b$ are both at least as large as the total outflow given by ψ'.

So we can construct a partial injection f from A to B so that for every $Q_0, Q_1 \in Y$ if $\psi'(Q_0, Q_1) > 0$, then $|\{(x, y) \in Q_0 \times Q_1 \colon f(x) = y\}| = \psi'(Q_0, Q_1)$, and otherwise $|\{(x, y) \colon f(x) = y\}| = 0$. After doing this, since $\mathrm{div}\,\psi' + (f_A - f_B) = 0$ for every $Q \in Y$, $|A \cap Q \setminus \mathrm{dom}(f)| = |B \cap Q \setminus \mathrm{dom}(f)|$. Hence, we can extend f to a bijection from $A \to B$ by mapping the remaining points of A to points of B in the same element Q of the partition Y. Note that for every $x \in A \cap Q_0$, $f(x)$ is either in Q_0 or in some Q_1 adjacent to Q_0 in H. Since each $Q \in Y$ has n^d elements, the number of elements of Q_0 plus the number of elements of the $2d$ many sets Q_1 adjacent to Q_0 is a total of $(2d + 1)n^d$ elements. So by Remark 2.3, the equidecomposition uses $(2d + 1)n^d$ pieces. □

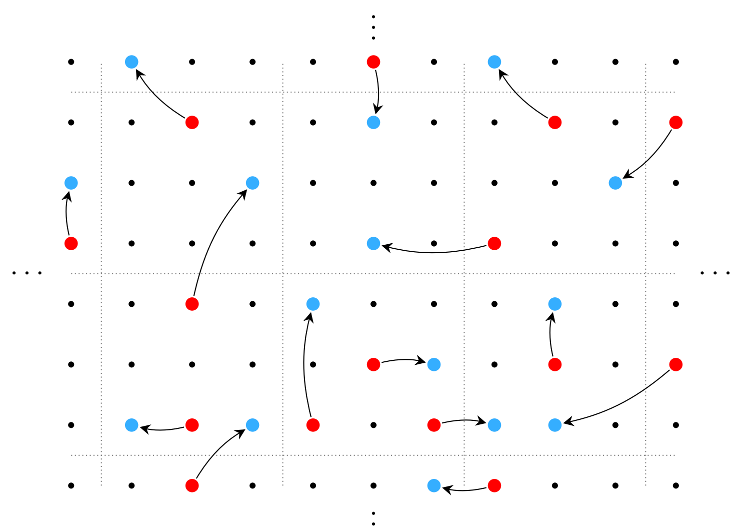

FIGURE 9. An illustration of the proof of Theorem 4.2. We partition the action into hypercubes of side length n. We use red and blue in our diagram to represent the points that are in the sets A and B. There is an integer flow ψ' which tells us how many points of A in each hypercube to map to points of B in each neighboring hypercube, and vice versa. After this process is finished, the number of points in each hypercube of A and B that remain are equal, and we map them to each other. This gives a bounded distance bijection from A to B (where each point of A is mapped inside either the same hypercube or an adjacent one).

So we can now finish proving the first key Lemma A, which we restate for convenience:

Lemma A. Suppose $u \in (\mathbb{T}^k)^d$, $a_u \colon \mathbb{Z}^d \curvearrowright \mathbb{T}^k$ is a free action, $A, B \subseteq \mathbb{T}^k$ have the same positive Lebesgue measure $\lambda(A) = \lambda(B) > 0$, and there are constants C and $\delta > 0$ so that $D(F_N(u,x), A) \leq CN^{-1-\delta}$ and $D(F_N(u,x), B) \leq CN^{-1-\delta}$ for all N that are powers of 2. Then A and B are equidecomposable in the action a_u.

Proof of Lemma A. By Lemma 3.1 there is a bounded flow ϕ of $\mathrm{Sch}(a_u, S)$ such that $\mathrm{div}\,\phi + 1_A = 1_B$ where $S = \{0,1\}^d - \{0\}^d$. By Proposition 3.3 there is also a bounded flow ψ of $\mathrm{Sch}(a_u, E)$ where $E = \{e_1, \ldots, e_d\}$ is the standard generating set such that $\mathrm{div}\,\psi + 1_A = 1_B$. Let b be an integer such that $\|\psi\|_\infty \leq b$. Now $\frac{|F_N(u,x) \cap A|}{|F_N(u,x)|} \geq \lambda(A) - D(F_N(u,x), A)$ by the definition of discrepancy. So since $|F_N(u,x)| = N^d$, our bound on the discrepancy $D(F_N(u,x), A)$ above gives that $|F_N(u,x) \cap A| \geq \lambda(A)N^d - CN^{d-1-\delta}$. An identical argument shows $|F_N(u,x) \cap B| \geq \lambda(B)N^d - CN^{d-1-\delta}$. Let $m = \lambda(A) = \lambda(B)$. To finish, since b, C, d, m are all fixed, if N is large enough then $mN^d - CN^{d-1-\delta} \geq 2dN^{d-1}b$ and $mN^d - CN^{d-1-\delta} \geq 2dN^{d-1}b$. So A and B are equidecomposable by Theorem 4.2. □

So we have finished proving Laczkovich's circle squaring theorem, modulo proving the second key Lemma B, which we prove in the remaining sections of the paper.

To finish this section, we make a few remarks about recent work showing that there is a constructive solution to Tarski's circle squaring problem without using the Axiom of Choice and with explicitly definable Borel pieces in the partition (see [GMP], [MU], and [MNP]). In particular, we discuss how the proof of Theorem C can be modified to yield these constructive solutions.

Note first that the flow between the circle and square constructed in Lemma 3.1 has a simple and explicit definition. Precisely, the flow is a pointwise limit of continuous functions. To prove Theorem C, the only place where we needed the Axiom of Choice is in the proofs of Lemma 4.1, and in Theorem 4.2 above. Now it is not possible to make a partition Y as in Theorem 4.2 of the space $\mathbb{T}^k$ into hypercubes of the form $\{0, \ldots, n-1\}^d \cdot_u x$ in a Borel (or even Lebesgue measurable) way. This can be seen using a straightforward argument of ergodic theory. However, the proof of Theorem 4.2 can be modified where instead we use a tiling by hypercubes whose sides are all of length n or $n+1$. There is a Borel tiling of this form that can be found using tiling machinery of Gao-Jackson [GJ]. These kinds of tilings of group actions are an important topic of study in ergodic theory, dating back to Ornstein and Weiss [OW].

The most difficult step to make constructive in the proof of the circle squaring theorem is Lemma 4.1. It is false in general that if $a \colon G \curvearrowright X$ is a continuous action of a countable group G, $f \colon X \to \mathbb{Z}$ is Borel, and there is a bounded Borel flow ϕ of $\mathrm{Sch}(a, S)$ so that $\mathrm{div}\,\phi + f = 0$, then there is an integer-valued Borel flow ϕ^* so that $\mathrm{div}\,\phi^* + f = 0$. However, there is a "Borel integral flow theorem" in the case when the group is $\mathbb{Z}^d$ for $d \geq 2$, and S is a finite generating set. Proving this requires using a recent theorem of Gao, Jackson, Krohne, and Seward giving particularly nice witnesses to the *hyperfiniteness* of actions of $\mathbb{Z}^d$ (see [MU, Appendix A] and [GJKS]). Hyperfiniteness is a way of writing the orbit equivalence relation of the action as an increasing union of equivalence relations with finite classes, and is an important notion of "tameness" of group actions in descriptive set theory and ergodic theory[K25]. Hyperfiniteness also has deep connections to related notions in topological dynamics and operator algebras. These types of hyperfiniteness witnesses are now called "toasts" in the literature, and have since been used extensively to study the Borel combinatorics of these actions (see e.g. [BBLW], [BKS], [BPZ], and [GJKS]).

5. A discrepancy bound in $\mathbb{T}$

If $J = [a, b) \subseteq \mathbb{R}$ where $0 \leq b - a \leq 1$, then we call the quotient $I = J/\mathbb{Z}$ of J modulo 1 an *interval* in the torus $\mathbb{T}$. If $F \subseteq \mathbb{T}$ is finite, we define $D(F)$ to be the supremum of the discrepancy of F over all intervals:

$$D(F) = \sup_{\text{intervals } I \subseteq \mathbb{T}} D(F, I) = \sup_{\text{intervals } I \subseteq \mathbb{T}} \left| \lambda(I) - \frac{|F \cap I|}{|F|} \right|.$$

Recall here we already defined the discrepancy $D(F, A)$ of F relative to a set A in Section 3 as $D(F, A) = \left| \lambda(A) - \frac{|F \cap A|}{|F|} \right|$. Now taking the complement of A does not change discrepancy: $D(F, A) = D(F, \mathbb{T} \setminus A)$ since $\lambda(A) = 1 - \lambda(\mathbb{T} \setminus A)$ and $\frac{|F \cap (\mathbb{T} \setminus A)|}{|F|} = 1 - \frac{|F \cap A|}{|F|}$. So we also have that $D(F) = \sup_{0 \leq a < b \leq 1} D(F, [a, b))$ since if I is an interval, then either I or its complement is of the form $[a, b)$ where $0 \leq a < b \leq 1$.

Note also that $D(F) \geq \frac{1}{|F|}$ for any nonempty finite set F since we can take an arbitrarily small interval around a single point.

Remark 5.1. Note that discrepancy is "shift-invariant". That is, if $F \subseteq \mathbb{T}$ is any finite set, and $x \in \mathbb{T}$, then $D(F) = D(x + F)$. For this reason, and notational convenience, we define

$$F_N(u) = F_N(u, 0) = \{n_1 u_1 + \ldots + n_d u_d \colon 0 \leq n_i < N\} = \{0, \ldots, N-1\}^d \cdot_u 0.$$

to be one particular representative of the set $F_N(u, x)$ for all $x \in \mathbb{T}$. So $D(F_N(u, x)) = D(F_N(u))$ for all N, u, and x.

In this section we will prove a bound on the discrepancy $D(F_N(u))$ of the sets $F_N(u)$ in $\mathbb{T}$, when the d many translations $u \in (\mathbb{T})^d$ we use are random translations (i.e. we will prove that a measure one set of translations have the property we want). In the next section, we will then use this to establish the second key Lemma B which will conclude our proof the main theorem of the paper.

We will use the Erdős-Turán inequality to bound the discrepancy $D(F_N(u))$. The Erdős-Turán inequality bounds the discrepancy of a finite set F by certain Fourier coefficients. We will prove this inequality in Section 7. As is customary in number theory, we write $e(x) \coloneqq e^{2\pi i x}$ for $x \in \mathbb{R}$. We regard e as a function from $\mathbb{T}$ to $\mathbb{C}$ since it is periodic with period 1.

Theorem 5.2 (Erdős-Turán [ET], see also [KN, Theorem 2.2.5] and [M, Corollary 1.1])**.** *There are constants C_1, C_2 such that for all finite $F \subseteq \mathbb{T}$ and all $m \in \mathbb{N}$,*

$$D(F) \leq C_1 \frac{1}{m+1} + C_2 \sum_{k=1}^{m} \frac{1}{k|F|} \left| \sum_{x \in F} e(kx) \right|$$

For $x \in \mathbb{R}$, let $\|x\|$ be the distance from x to the nearest integer, so $0 \leq \|x\| \leq 1/2$ for all $x \in \mathbb{R}$. We will use only this notation in the next two sections, and there should be no confusion with the supremum norm $\|\cdot\|_\infty$ used in Sections 3 and 4.

In the specific case of the finite sets $F_N(u)$, we have the following bound:

Lemma 5.3. *For any $u = (u_1, \ldots, u_d) \in (\mathbb{T})^d$, for all m,*

$$D(F_N(u)) \leq C_1 \frac{1}{m+1} + C_2 \frac{1}{2^d N^d} \sum_{k=1}^{m} \frac{1}{k \prod_{i=1}^{d} \|k u_i\|}$$

where C_1, C_2 are the constants from the Erdős-Turán inequality.

Proof. To begin, recall the standard bound on a finite geometric series with base $e(z)$:

$$\left| \sum_{n=0}^{N-1} e(nz) \right| = \left| \frac{e(Nz) - 1}{e(z) - 1} \right| = \left| \frac{e(Nz/2)(e(Nz/2) - e(-Nz/2))}{e(z/2)(e(z/2) - e(-z/2))} \right| = \left| \frac{e(Nz/2)\sin(\pi N z)}{e(z/2)\sin(\pi z)} \right|$$

$$= \left| \frac{\sin(\pi N z)}{\sin(\pi z)} \right| \leq \frac{1}{|\sin(\pi z)|} \leq \frac{1}{2\|z\|}$$

Above we are using the formula for the sum of a geometric series, the fact that Euler's identity implies $e(x) - e(-x) = 2i\sin(2\pi x)$, and the fact that $\sin(\pi x) \geq 2\|x\|$ for all x.

Hence, we have

$$\left|\sum_{y \in F_N(u)} e(ky)\right| = \left|\sum_{(n_1,\ldots,n_d)\in\{0,\ldots,N-1\}^d} e(k(n_1u_1 + \ldots + n_du_d))\right|$$
$$= \left|\prod_{i=1}^{d}\sum_{n=0}^{N-1} e(nku_i)\right| \leq \prod_{i=1}^{d}\frac{1}{2\|ku_i\|}$$

by using the bound on finite geometric series with base $e(z)$ above, and the fact that $|e(kx)| = 1$ since kx is a real number.

The lemma follows by substituting the above formula into the Erdős-Turán inequality. □

So our goal now is to bound summations of the form $\sum_{k=1}^{m}\frac{1}{k\prod_{i=1}^{d}\|ku_i\|}$ for a.e. $u_1,\ldots,u_d$. Below we change the index in this summation to n, and we begin with a simple lower bound on the size of $\|nu\|$:

Lemma 5.4. *For every $\epsilon > 0$, for almost all u in $\mathbb{T}$, there is a constant C such that for all positive integers n, $\|nu\| \geq Cn^{-1-\epsilon}$.*

Proof. It is enough to show that for almost every $u \in \mathbb{T}$, there are finitely many $n > 0$ such that $\|nu\| < n^{-1-\epsilon}$. For each $n > 0$, let E_n be the set of $u \in \mathbb{T}$ such that $\|nu\| < n^{-1-\epsilon}$. Clearly, u is in E_n if and only if there is an integer $0 \leq m \leq n$ such that

$$|nu - m| < n^{-1-\epsilon}.$$

For each m, the above condition implies that u lies in the interval of length $2n^{-2-\epsilon}$ around m/n. Summing over all integers $0 \leq m \leq n$, we have that the measure of E_n is at most $2(n+1)/n^{2+\epsilon}$. Note that these measures are summable: $\sum_{n=1}^{\infty}\frac{2(n+1)}{n^{2+\epsilon}} \leq \sum_{n=1}^{\infty}\frac{4}{n^{1+\epsilon}} < \infty$.

One can finish now by quoting the Borel-Cantelli lemma to conclude that the set of u that lie in infinitely many E_n has measure 0. To prove this directly, for each positive integer m, the set of $u \in \mathbb{T}$ that are contained in infinitely many E_n is contained in $\bigcup_{n\geq m} E_n$. By the summability of the measures of the E_n, the measure of $\bigcup_{n\geq m} E_n$ goes to 0 as m goes to infinity. So the set of $u \in \mathbb{T}$ that are contained in infinitely many E_n must have measure 0. So for almost every $u \in \mathbb{T}$, there are only finitely many $n \in \mathbb{N}$ such that $\|nu\| < n^{-1-\epsilon}$ as desired. □

The above lemma is a basic argument from the theory of Diophantine approximation. See [C] for an introduction to this subject. Diophantine approximation studies how well irrational numbers can be approximated by rational numbers. Note that $\|qu\| < c$ if there is an integer p so that $|qu - p| < c$, and so $|u - \frac{p}{q}| < \frac{c}{q}$. So $\|qu\|$ being small means that u has a close approximation by a rational number with denominator q. Dirichlet's theorem on Diophantine approximation (which is proved by an application of the pigeonhole principle) shows that for any irrational x, there are infinitely many p and q such that $|x - \frac{p}{q}| < \frac{1}{q^2}$. Equivalently (after multiplying this equation by q), there are infinitely many q such that $\|qx\| < \frac{1}{q}$ (see e.g. [HW, Theorem 185]). So Lemma 5.4 shows that the exponent in Dirichlet's theorem is as good as possible in general. A deep and fundamental theorem of Roth in Diophantine approximation states that if u is an algebraic irrational number then conclusion of Lemma 5.4 holds for u: there is a constant C so that for all h, $\|hu\| > Ch^{-1-\epsilon}$. We use progress finding effective bounds for special cases of Roth's theorem in our second paper [MU2] to improve the upper bound on the number of pieces needed to square the circle.

Recall that our goal is to study the discrepancy of the set $F_N(u)$. In the case $d = 1$ where $u \in \mathbb{T}$, the discrepancy of $F_N(u)$ (which is equal to $D(\{u, 2u, \ldots, Nu\})$ by Remark 5.1) is highly studied, dating back to work of Behnke, Ostrowski, Hardy and Littlewood, Hecke, and Khintchine in the 1920s. See [DT, Section 1.4.1] for a detailed history and bibliography. If u has a close rational approximation $|u - \frac{p}{q}| < \epsilon$ where $\epsilon \ll 1/q^2$, then this will make $D(\{u, 2u, \ldots, Nu\})$ large (i.e on the order of $1/q$ for $N \ll 1/(q\epsilon)$), since the points of $\{u, 2u, \ldots, Nu\}$ will cluster around the rationals p/q with denominator q. It is a beautiful theorem that the converse is true. If u has *no* good rational approximations, then $D(\{u, 2u, \ldots, Nu\})$ will be small for all N, and in fact there are precise equivalences between how well u is approximated by rationals, and the rate of decay of $D(\{u, 2u, \ldots, Nu\})$ (see e.g. [KN, Chapter 2.3]). We will discuss more about connections between Diophantine approximation and discrepancy in our second paper [MU2].

However, for almost every u, there is an easier way to obtain an upper bound on the summations $\sum_{n=1}^{N} \frac{1}{n \prod_{i=1}^{d} \|nu_i\|}$ and hence get an upper bound on $D(F_N(u))$, which does not go through the theory of Diophantine approximation, and which works for all d. We can simply integrate over all $u_1, \ldots, u_d$ to find a bound that holds for almost every $u_1, \ldots, u_d$. We will use Lemma 5.4 above in this calculation. Similar estimates go back to work of Schmidt [S64] and are an important part of Laczkovich's proofs.

Lemma 5.5. *Let $d \geq 1$. For every $\epsilon > 0$, for almost every $u = (u_1, \ldots, u_d) \in (\mathbb{T})^d$,*

$$\sum_{n=1}^{N} \frac{1}{n \prod_{i=1}^{d} \|nu_i\|} \ll_u \log^{d+1+\epsilon}(N).$$

Proof. Let $\epsilon' = \epsilon/(1+d)$. By Lemma 5.4, for a.e. $u = (u_1, \ldots, u_d)$, there is a C so that for all $i \leq d$, $\|nu_i\| \geq Cn^{-1-\epsilon'}$ (by taking $C = \min C_i$ where $\|nu_i\| \geq C_i n^{-1-\epsilon'}$). Taking the log of both sides and applying the absolute value, we have $|\log(\|nu_i\|)| \geq |\log(C)| + (1+\epsilon')\log(n)$ and so $|\log(\|nu_i\|)| \geq C' \log(n)$ for $n \geq 1$ where $C' = |\log(C)|/\log(2) + (1+\epsilon')$. So for a.e. $u_1, \ldots, u_d$, we have the following asymptotic upper bounds where we regard u as a fixed parameter:

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{n \prod_{i=1}^{d} \|nu_i\|} &\ll_u \log^{d+1+\epsilon}(N) \sum_{n=2}^{N} \frac{1}{n \log^{d+1+\epsilon}(n) \prod_{i=1}^{d} \|nu_i\|} \\ &\ll_u \log^{d+1+\epsilon}(N) \sum_{n=2}^{N} \frac{1}{n \log^{1+\epsilon'}(n)} \prod_{i=1}^{d} \frac{1}{\|nu_i\| |\log(\|nu_i\|)|^{(1+\epsilon')}}.\end{aligned}$$

Note that we have dropped the first term in the summation above since it is just a constant.

To complete the proof it suffices to show that for a.e. $u = (u_1, \ldots u_d)$, the final summation above is bounded by a constant. To do this we show that its integral over all $u_1, \ldots, u_d \in [0,1)$ is finite. (If the summation was infinite for a positive measure set of $u_1, \ldots, u_d$, the integral would be infinite). Now for each positive integer h, using the fact that $u \mapsto \|hu\|$ is periodic with period $1/h$, and $\|hu\| = \|h(1-u)\|$ we have that for any function f, $\int_0^1 f(\|hu\|) \,\mathrm{d}u = 2\int_0^{1/2} f(x) \,\mathrm{d}x$. Hence,

$$\begin{aligned}&\int_0^1 \cdots \int_0^1 \sum_{n=2}^{N} \frac{1}{n \log^{(1+\epsilon')}(n)} \prod_{i=1}^{d} \frac{1}{\|nu_i\| |\log(\|nu_i\|)|^{(1+\epsilon')}} \,\mathrm{d}u_1 \ldots \mathrm{d}u_d \\ &\qquad = \sum_{n=2}^{N} \frac{1}{n \log^{(1+\epsilon')}(n)} \prod_{i=1}^{d} \int_0^{1/2} \frac{1}{x |\log x|^{(1+\epsilon')}} \,\mathrm{d}x \ll \sum_{n=2}^{N} \frac{1}{n \log^{(1+\epsilon')}(n)} \ll 1\end{aligned}$$

In the first equality, we have exchanged the summation and integral, and then using that the integral of the product is the product of the integrals. In the last two steps, $\int_0^{1/2} \frac{1}{x|\log(x)|^{(1+\epsilon')}} \,\mathrm{d}x$ converges, and $\sum_{n=2}^{\infty} \frac{1}{n \log^{(1+\epsilon')}(n)}$ converges by the integral test. □

See [LV] for a more detailed study of these sums, including an upper bound replacing the factor of $\log(N)^\epsilon$ with a $\log\log(N)$ factor, and also lower bounds for a.e. u.

From these results we get a bound on the discrepancy of the sets $F_N(u)$ for a.e. $u \in (\mathbb{T})^d$:

Lemma 5.6. *For all positive integers d, for all real numbers $c < d$, for a.e. $u = (u_1, \ldots, u_d) \in (\mathbb{T})^d$ there is a C so that for all N, $D(F_N(u)) \leq CN^{-c}$.*

Proof. Let $u = (u_1, \ldots, u_d)$ be in the measure one set satisfying Lemma 5.5. Then by applying Lemma 5.3 with $m = N^d$, we get:

$$D(F_N(u)) \ll_u \frac{1}{N^d} + \frac{1}{N^d} \sum_{n=1}^{N^d} \frac{1}{n \prod_{i=1}^d \|nu_i\|} \ll_u \frac{\log^{d+1+\epsilon}(N^d)}{N^d} \ll N^{-c}.$$

□

6. Discrepancy in $\mathbb{T}^k$

The goal of this section is to bound $D(F, A)$ when $F \subseteq \mathbb{T}^k$ is a finite set that is a product of 1-dimensional sets, and where the boundary of A has upper Minkowski dimension less than k: $\overline{\dim}_{\mathrm{box}}(\partial A) < k$. We will use our bound from Lemma 5.6 as part of this proof. We will end by proving the key Lemma B.

We begin with a simple lemma in relating the discrepancy of a set $F = \{x_0, \ldots, x_{n-1}\} \subseteq [0, 1)$, where $x_0 < x_1 < \ldots < x_{n-1}$, to the distance between x_i and i/n.

Lemma 6.1. *Suppose $F = \{x_0, \ldots, x_{n-1}\} \subseteq [0, 1)$ is a finite set where $x_0 < \ldots < x_{n-1}$. Then for all $i < n$, $|x_i - \frac{i}{n}| \leq D(F)$.*

Proof. The interval $[0, x_i)$ has length x_i and contains i points of F. Hence $D(F, [0, x_i)) = \left|x_i - \frac{i}{n}\right|$. Finally, $D(F, [a, b)) \leq D(F)$ for any $0 \leq a \leq b \leq 1$ by definition. □

In fact, there is a very close connection between the quantity $\max_i \max(|x_i - \frac{i}{n}|, |x_i - \frac{i+1}{n}|)$ and the discrepancy $D(\{x_0, \ldots, x_{n-1}\})$. This maximum is in fact equal to the "one-sided discrepancy $D^*(\{x_0, \ldots, x_{n-1}\}) \coloneqq \sup_{x \in [0,1]} D(\{x_0, \ldots, x_{n-1}\}, [0, x))$, and $D(F)$ and $D^*(F)$ are always within a factor of 2 (see [KN, Page 92]). So the lower bound in Lemma 6.1 is quite good.

Suppose $F_1, \ldots, F_k \subseteq \mathbb{T}$ are finite sets in the 1-dimensional torus, and the F_i all have the same cardinality which is a power of 2. We now have the following lemma bounding the discrepancy of their product $\prod_{i=1}^k F_i \subseteq \mathbb{T}^k$ in terms of the 1-dimensional discrepancies of the original sets $D(F_i)$. More strongly, we bound the discrepancy of $F = \prod_{i=1}^k F_i$ with respect to any set A of upper box dimension $\overline{\dim}_{\mathrm{box}}(\partial A) < k$. Recall from Section 2 that upper box dimension is:

$$\overline{\dim}_{\mathrm{box}}(A) = \limsup_{n \to \infty} \frac{\log N(n, A)}{\log n}$$

where $N(n, A)$ is the number of $1/n$-lattice cubes that intersect A. Note that if $\overline{\dim}_{\mathrm{box}}(\partial A) < k - \epsilon$ (our hypothesis in Lemma B), then for sufficiently large m, $N(m, \partial A) \leq m^{k-\epsilon}$. This latter condition is something we assume below.

Lemma 6.2. *For all $\epsilon > 0$ and integers $k > 0$, there is a constant $c_{k,\epsilon}$ so that if $F_1, \ldots, F_k \subseteq \mathbb{T}$ are sets all of the same cardinality n, n is a power of 2, and $N(m, \partial A) \leq m^{k-\epsilon}$ for all $m \geq \frac{1}{2 \sup_i D(F_i)}$, then*

$$D(\prod_i F_i, A) \leq c_{k,\epsilon} (\sup_i D(F_i))^\epsilon$$

Proof. Let $F_i = \{x_{i,0}, \ldots, x_{i,n-1}\}$, where $x_{i,0} < x_{i,1} < \ldots < x_{i,n-1}$. Each point $(x_{1,j_1}, x_{2,j_2} \ldots, x_{k,j_k}) \in \prod_i F_i$ has $\sup_i |x_{i,j_i} - j_i/n| \leq \sup_i D(F_i)$ by Lemma 6.1. So $(x_{1,j_1}, x_{2,j_2} \ldots, x_{k,j_k}) \in F$ is "close" to the lattice point $(j_1/n, \dots, j_k/n)$ in each coordinate. Our proof will be based on counting upper and lower bounds on the number of such lattices points close to A, compared to its measure.

Let m be the greatest power of 2 such that $m \leq 1/\sup_i D(F_i)$. So $\frac{1}{2m} \leq \sup_i D(F_i) \leq \frac{1}{m}$. Since $n = |F_i|$ is also a power of 2 and $D(F_i) \geq \frac{1}{|F_i|}$ by the first paragraph of Section 5, we have $\frac{1}{n} \leq D(F_i) \leq 1$. Hence, $1 \leq m \leq n$ and n/m is an integer by our assumption that n is a power of 2.

Now if $x = (x_{1,j_1}, \dots, x_{k,j_k}) \in F$, then the point $(j_1/n, \dots, j_k/n)$ is either in the same $1/m$-lattice cube as x, or it is in an adjacent $1/m$-lattice cube (where we say distinct $1/m$-lattice cubes $c_g = 1/m(g + [0,1)^k)$ and $c_h = 1/m(h + [0,1)^k)$ are *adjacent* if $|g(i) - h(i)| \leq 1$ for all $i \leq k$). So to upper bound the number of points in $|F \cap A|$ we just need to bound the number of $1/m$-lattice cubes that intersect A or are adjacent to cubes that intersect A.

Let D_1 be the set of $1/m$-lattice cubes that intersect A or are adjacent to cubes that intersect A. Since each $1/m$-lattice cube contains exactly n^k/m^k points of the form $(i_1/n, \dots, i_k/n)$, the number of points in $|F \cap A|$ is at most $|D_1| n^k/m^k$. So $|F \cap A|/|F|$ is at most $|D_1|/m^k$.

Let D_0 be the number of $1/m$-lattice cubes entirely contained in A. Then the measure of A satisfies $\lambda(A) \geq |D_0|/m^k$ since each such cube has volume $1/m^k$. Hence $|F \cap A|/|F| - \lambda(A) \leq |D_1 - D_0|/m^k$. Every cube in $D_1 - D_0$ either intersects ∂A or is adjacent to a cube that intersects ∂A. Since each $1/m$-lattice cube is adjacent to $3^k - 1$ other cubes, and ∂A contains at most $m^{k-\epsilon}$ $1/m$-cubes, $|D_1 - D_0| \leq 3^k m^{k-\epsilon}$. Hence,

$$|F \cap A|/|F| - \lambda(A) \leq \frac{3^k}{m^\epsilon} \leq 2^\epsilon 3^k (\sup_i D(F_i))^\epsilon.$$

This is since $\frac{1}{m} \leq 2\sup_i D(F_i)$. A similar argument giving a lower bound for $|F \cap A|/|F|$ and an upper bound for $\lambda(A)$ shows that $\lambda(A) - |F \cap A|/|F| \leq 2^\epsilon 3^k (\sup_i D(F_i))^\epsilon$. Hence, the theorem is true with the constant $c_{k,\epsilon} = 2^\epsilon 3^k$.

□

Putting the above lemma together with Lemma 5.6, we can now prove Lemma B which concludes our proof of circle squaring, modulo a proof of the Erdős-Turán inequality which we give in the next section. We restate the lemma for convenience:

Lemma B. For every k and $\epsilon > 0$, there exist a positive integer d, $u \in (\mathbb{T}^k)^d$, and $\delta > 0$ such that for all sets $A \subseteq \mathbb{T}^k$ with $\overline{\dim}_{\text{box}}(\partial A) < k - \epsilon$, there is a C such that

$$D(F_N(u,x), A) \leq CN^{-1-\delta}$$

for every $x \in \mathbb{T}^k$ and N that is a power of 2.

Proof. Suppose k and $\epsilon > 0$ are given. Fix an integer d large enough such that $d\epsilon > 1$, and let $c < d$ be so that $c\epsilon > 1$. Then by Lemma 5.6, there is $u = (u_1, \dots, u_d) \in (\mathbb{T})^d$ and C so that $D(F_N(u)) \leq CN^{-c}$.

Let $\{e_1, \dots, e_d\}$ be the usual set of generators of $\mathbb{Z}^d$ and let

$$v = (u_1 e_1, u_2 e_1, \dots, u_d e_1, u_1 e_2, \dots, u_d e_2, \dots, u_1 e_k, \dots, u_d e_k) \in (\mathbb{T}^k)^{dk}$$

so that $a_v \colon \mathbb{Z}^{dk} \curvearrowright \mathbb{T}^k$ is the k-fold product of the action a_u, so $(g_1, \dots, g_k) \cdot_v (x_1, \dots, x_k) = (g_1 \cdot_u x_1, \dots, g_k \cdot_u x_k)$, and so $F_N(v, (x_1, \dots, x_k)) = \prod_{i=1}^k F_N(u, x_i)$.

Now if A is such that $\overline{\dim}_{\text{box}}(\partial A) < k - \epsilon$, there must be some m_0 so that for all $m \geq m_0$, $\frac{\log(N(m,A))}{\log(m)} \leq k - \epsilon$, and hence $N(m, A) \leq m^{k-\epsilon}$. Now since $N^{-c} \to 0$ as $N \to \infty$, and $D(F_N(u)) \leq CN^{-c}$, if N is a sufficiently large power of 2, we have that $\frac{1}{2D(F_N(u))} \geq m_0$. And so by Lemma 6.2, for all sufficiently large N, for all $x \in \mathbb{T}^k$, $D(F_N(v, (x_1, \dots, x_k)), A) \leq c_{k,\epsilon}(CN^{-c})^\epsilon$. Setting $\delta = c\epsilon - 1 > 0$, this is $\ll N^{-1-\delta}$ for sufficiently large powers of 2, and increasing the constant if necessary covers the finitely many smaller powers of 2. □

Note that the bounds we have proved above in Lemma B can be improved in several ways. One can extend this lemma to a bound that is true for all N instead of just N that are powers of 2 using a dyadic decomposition and at the expense of an additional logarithmic factor on the right hand side. It is also possible to use the Erdős-Turán-Koksma inequality (see [DT, Theorem 1.21]) and largely mimic the proof of Lemma 5.6 to obtain a bound in Lemma B for a measure one set of u and for smaller values of d. This is the approach taken by Laczkovich's [L92, Proof of Theorem 3] (see also [GMP, Lemma 6]). We have used the approach above since it is more elementary, and the Erdős-Turán inequality we have used is simpler than the Erdős-Turán-Koksma inequality. This also lets us prove our new sufficient condition for equidecomposability:

Theorem D. Suppose $\epsilon > 0$ and $A, B \subseteq \mathbb{R}^k$ are bounded Lebesgue measurable sets such that $\lambda(A) = \lambda(B) > 0$, and $\overline{\dim}_{\text{box}}(\partial A) \leq k - \epsilon$ and $\overline{\dim}_{\text{box}}(\partial B) \leq k - \epsilon$. Suppose $c > 0$ and $u_1, \ldots, u_d \in \mathbb{R}$ are irrational numbers linearly independent over $\mathbb{Q}$ so that for all N that are powers of 2,

$$\sum_{n=1}^{N} \frac{1}{n \prod_i \|nu_i\|} \ll_u N^c,$$

and $c < d\epsilon - 1$. Then A and B are equidecomposable in $\mathbb{R}^k$ by finitely many translations whose coordinates are integer linear combinations of $1, u_1, \ldots, u_d$.

Proof. By Lemma 5.3, and our assumption on the growth of $\sum_{n=1}^{N} \frac{1}{n \prod_i \|nu_i\|}$, for all integers $m > 0$,

$$D(F_N(u)) \ll \frac{1}{m+1} + \frac{1}{N^d} \sum_{n=1}^{m} \frac{1}{n \prod_{i=1}^{d} \|nu_i\|} \ll \frac{1}{m+1} + \frac{m^c}{N^d}$$

We set $m = \lfloor N^{d/(c+1)} \rfloor$ to optimize this bound so that both terms are equal. This gives

$$D(F_N(u)) \ll \frac{1}{N^{d/(c+1)}} + \frac{N^{cd/(c+1)}}{N^d} \ll \frac{1}{N^{d/(c+1)}}.$$

Now following the proof of Lemma B, let a_v be the action of $\mathbb{Z}^{dk}$ on $\mathbb{T}^k$ which is the k-fold product of the action a_u of $\mathbb{Z}^d$ on $\mathbb{T}$ so $v = (u_1 e_1, u_2 e_1, \ldots, u_d e_1, u_1 e_2, \ldots, u_d e_2, \ldots, u_1 e_k, \ldots, u_d e_k)$. Now since $\overline{\dim}_{\text{box}}(\partial A)$ and $\overline{\dim}_{\text{box}}(\partial B) < k - \epsilon$, for sufficiently large m, $N(m, \partial A) < m^{k-\epsilon}$ and $N(m, \partial B) < m^{k-\epsilon}$. So $D(F_N(v)) \ll \left(N^{-d/(c+1)}\right)^{\epsilon} = N^{-d\epsilon/(c+1)}$.

Finally, by Lemma A, this implies that A and B are equidecomposable in a_v provided $d\epsilon > c+1$. Hence by the proof of Proposition 2.5 they are equidecomposable in $\mathbb{R}^k$ by translations whose coordinates are integer linear combinations of $1, u_1, \ldots, u_d$. □

7. A proof of the Erdős-Turán inequality

The only remaining ingredient we have used in our proof of the circle squaring theorem above is the Erdős-Turán inequality. For the sake of self-containment, we include a proof of this inequality. Our proof largely follows [KN, Theorem 2.2.5] but we give some additional motivation and simplifications.

We'll begin by recalling a couple definitions from Fourier analysis. First, if $f \colon \mathbb{T} \to \mathbb{R}$ is a function, its *kth Fourier coefficient* is

$$\hat{f}(k) = \int_0^1 f(x) e(-kx) \, \mathrm{d}x.$$

Next, recall the definition of the *Fejér kernel* $F_m \colon \mathbb{T} \to \mathbb{R}$ where m is a nonnegative integer:

$$F_m(x) = \sum_{k=-m}^{m} \frac{m+1-|k|}{m+1} e(kx)$$

These kernels are an important tool in Fourier analysis. They are key to showing Fejér's fundamental theorem that the Cesàro means of the Fourier series of a continuous function $f\colon \mathbb{T} \to \mathbb{R}$ converge to f. The interested reader may want to read a proof of this theorem for additional context (see [K, Theorem 2.3]), though we will not use it in what follows. This proof uses the convolutions of f with the Fejér kernels that we calculate in Lemma 7.1.(3) below. Recall that if f and g are functions from $\mathbb{T}$ to $\mathbb{R}$, then their convolution $f * g\colon \mathbb{T} \to \mathbb{R}$ is the function defined by $(f * g)(t) = \int_0^1 f(x)g(t-x)\,\mathrm{d}x$.

We record a few simple properties of Fejér kernels.

Lemma 7.1. *For every integer $m \geq 1$, and every integrable function f,*

(1)
$$F_m(x) = \frac{1}{m+1}\frac{\sin^2((m+1)\pi x)}{\sin^2(\pi x)}.$$

(2) $\int_0^1 F_m(x) = 1$ *and* $\int_0^{1/2} F_m(x)\,dx = \frac{1}{2}$.

(3) $(f * F_m)(t) = \sum_{k=-m}^{m} \frac{m+1-|k|}{m+1} e(kt)\hat{f}(k)$

Proof. We begin with (1). Using the formula for the sum of a finite geometric series and Euler's formula, for $x \neq 0$ we have:

$$\sum_{k=-m}^{m} (m+1-|k|)\, e(kx) = \left(\sum_{j=0}^{m} e((j-m/2)x)\right)^2 = \left(e(-mx/2)\frac{1-e(m+1)x}{1-e(x)}\right)^2$$
$$= \left(\frac{e(-(m+1)x/2) - e((m+1)x/2)}{e(-x/2)-e(x/2)}\right)^2 = \left(\frac{\sin((m+1)\pi x)}{\sin \pi x}\right)^2$$

where the first equality above is by expanding the right hand side.

To prove (2), note that $\int_0^1 e(kx)\,\mathrm{d}x = 0$ for $k \neq 0$, so $\int_0^1 F_m(x) = 1$ by definition of $F_m(x)$. Since $F_m(x)$ is even and $F_m(x) \geq 0$ for all $x \in \mathbb{T}$, we also have that $\int_0^{1/2} F_m(x)\,\mathrm{d}x = \frac{1}{2}\int_{-1/2}^{1/2} F_m(x)\,\mathrm{d}x = \frac{1}{2}$. We can change the limits of integration from $(0,1)$ to $(-1/2,1/2)$ since $F_m(x)$ is periodic.

To prove (3), note that

$$(f * F_m)(t) = \int_0^1 f(x)F_m(t-x)\,\mathrm{d}x = \sum_{k=-m}^{m}\frac{m+1-|k|}{m+1}e(kt)\int_0^1 f(x)e(-kx)\,\mathrm{d}x$$
$$= \sum_{k=-m}^{m}\frac{m+1-|k|}{m+1}e(kt)\hat{f}(k)$$

□

Let $S_n(x) = \sum_{k=-n}^{n} \hat{f}(k)e(kx)$ be the partial Fourier series of f. It is easy to check that the mean $\frac{1}{m+1}\sum_{n=0}^{m} S_n(x)$ of the first $m+1$ partial Fourier series is exactly the convolution $f * F_m$ we have calculated in part (3) of the Lemma above. Fejér's theorem states that if f is continuous, then for all x, the Cesàro limit of the partial Fourier series of f is equal to f, so $\frac{1}{m+1}\sum_{n=0}^{m} S_n(x) \to f(x)$ as $m \to \infty$ [K, Theorem 2.3]. This theorem motivates a key step in the proof below.

We are now ready to prove the Erdős-Turán inequality, which we restate for clarity.

Theorem 7.2 (The Erdős-Turán inequality [ET])**.** *There are constants C_1, C_2 so that for all finite sets $F \subseteq \mathbb{T}$ and all integers m,*

$$D(F) \leq C_1\frac{1}{m+1} + C_2\sum_{k=1}^{m}\frac{1}{k|F|}\left|\sum_{x\in F} e(kx)\right|$$

Proof. We will use the following variant of discrepancy where we only consider intervals whose left endpoint is zero that we defined earlier in Section 6. Recall if $F \subseteq [0,1)$ is a finite set, then

$$D^*(F) = \sup_{x \in [0,1]} D(F,[0,x)) = \sup_{x \in [0,1]} \left| \frac{|[0,x) \cap F|}{|F|} - \lambda([0,x)) \right|.$$

Note that $D(F) \leq 2D^*(F)$, since

$$\begin{aligned} \left| \lambda([x,y)) - \frac{|[x,y) \cap F|}{|F|} \right| &= \left| (\lambda([0,y)) - \lambda([0,x))) - \left(\frac{|[0,y) \cap F|}{|F|} - \frac{|[0,x) \cap F|}{|F|} \right) \right| \\ &\leq \left| \left(\lambda([0,y)) - \frac{|[0,y) \cap F|}{|F|} \right) \right| + \left| \left(\lambda([0,x)) - \frac{|[0,x) \cap F|}{|F|} \right) \right|. \end{aligned}$$

Hence, we may show the inequality in the theorem for $D^*(F)$ instead of $D(F)$ since $D(F) \leq 2D^*(F)$. $D^*(F)$ is easier for us to calculate with since we take a supremum over a single endpoint instead of two endpoints.

Let $f(x) = \frac{|[0,x) \cap F|}{|F|} - \lambda([0,x))$ for $x \in [0,1)$, so $D^*(F) = \sup_x f(x)$. The first step of our proof will be calculating the Fourier coefficients of f. To begin, we consider the case when $\hat{f}(0) = \int_0^1 f(x)\,\mathrm{d}x = 0$. We will show this case suffices at the end of the proof.

We compute the kth Fourier coefficient of f for $k \neq 0$.

$$\begin{aligned} \hat{f}(k) &= \int_0^1 f(x) e(-kx)\,\mathrm{d}x = \int_0^1 \frac{|[0,x) \cap F|}{|F|} e(-kx)\,\mathrm{d}x - \int_0^1 x e(-kx)\,\mathrm{d}x \\ &= \sum_{y \in F} \frac{1}{|F|} \int_y^1 e(-kx)\,\mathrm{d}x + \frac{1}{2\pi i k} = \sum_{y \in F} \frac{1}{|F|} \frac{1}{-2\pi i k}(1 - e(-ky)) + \frac{1}{2\pi i k} = \sum_{y \in F} \frac{1}{|F|} \frac{e(-ky)}{2\pi i k} \end{aligned}$$

Now combining the above equation for $\hat{f}(k)$ with Lemma 7.1, we get that for all t,

$$|(f * F_m)(t)| \leq \left| \sum_{k=-m}^{m} \frac{m+1-|k|}{m+1} e(kt) \hat{f}(k) \right| \leq \sum_{k=-m}^{m} \left| \hat{f}(k) \right| \leq \frac{1}{\pi} \sum_{k=1}^{m} \frac{1}{k|F|} \left| \sum_{y \in F} e(ky) \right| \tag{11}$$

where we've used the triangle inequality, our assumption that $\hat{f}(0) = 0$, that $|\hat{f}(-k)| = |\hat{f}(k)|$ for all k (since $\hat{f}(-k)$ and $\hat{f}(k)$ are complex conjugates), and that $e(-ky)$ and $e(ky)$ are complex conjugates.

Our idea now is to take a value of s so that $|f(s)|$ is close to $D^*(F) = \sup_x f(x)$. Then since most of the mass of $F_m(x)$ is concentrated around $x = 0$, we can use this to get a good lower bound for $|(f * F_m)(s)|$ involving $D^*(F)$. (Indeed, as we have recalled above, Fejér's theorem is that $f * F_m$ converges to f and so we should expect that $|(f * F_m)(s)|$ should also be close to $|f(s)|$ which is close to $D^*(F)$).

For ease of notation, let $D = D^*(F)$. Since $D^*(F) = \sup_x f(x)$, we can choose $s_0 \in \mathbb{T}$ so that $|f(s_0)| \geq 3/4D$. Assume that $f(s_0) \geq \frac{3}{4}D$ (the case where $f(s_0) \leq -\frac{3}{4}D$ is analogous). Now for $0 \leq y < 1$, we have $f(s_0 + y) = f(s_0) + \frac{|F \cap [s_0, s_0+y)|}{|F|} - \lambda([s_0, s_0 + y)) \geq f(s_0) - y \geq 3/4D - y$, since $\mu([s_0, s_0 + y)) \geq 0$. So for $|x| \leq D/4$, we have $f(s_0 + D/4 - x) \geq \frac{1}{4}D$. Let $s = s_0 + D/4$.

Since $f(y) \geq -D$ for all y, and by changing variables to let $y = s - x$, and using periodicity to change the limits of integration, we have

$$(f * F_m)(s) = \int_0^1 f(x) F_m(s - x)\,\mathrm{d}x = \int_{s-1}^{s} f(s-y) F_m(y)\,\mathrm{d}y = \int_{-1/2}^{1/2} f(s-y)\, F_m(y)\,\mathrm{d}y.$$

By our choice of s we have $f(s-y) \geq D/4$ for $|y| \leq D/4$, and in general $f(s-y) \geq -D$. Using this and then evenness of F_m, we obtain

$$\begin{aligned}
(f * F_m)(s) &\geq \int_{|y|\leq D/4} \frac{D}{4} F_m(y)\,\mathrm{d}y + \int_{D/4<|y|\leq 1/2} (-D)\, F_m(y)\,\mathrm{d}y \\
&= \frac{D}{4}\int_{-1/2}^{1/2} F_m(y)\,\mathrm{d}y - \left(\frac{D}{4}+D\right)\int_{D/4<|y|\leq 1/2} F_m(y)\,\mathrm{d}y = \frac{D}{4} - \frac{5D}{4}\cdot 2\int_{D/4}^{1/2} F_m(y)\,\mathrm{d}y \\
&= \frac{D}{4} - \frac{5D}{2}\int_{D/4}^{1/2} F_m(y)\,\mathrm{d}y \geq \frac{D}{4} - \frac{5D}{2}\int_{D/4}^{1/2} \frac{1}{4(m+1)y^2}\,\mathrm{d}y \\
&= \frac{D}{4} - \frac{5}{2}\frac{1}{(m+1)}\left(1-\frac{D}{2}\right) \geq \frac{D}{4} - \frac{5}{2(m+1)}.
\end{aligned}$$

where we have used the standard upper bound $F_m(x) \leq \frac{1}{4(m+1)\,x^2}$ for $x \in [0,1/2]$ since on that domain $\sin(\pi x) \geq 2x$, and also the fact that since $D>0$, we have $1-\frac{D}{2} \leq 1$. By the same argument in the case $f(s) \leq -\frac{3}{4}D$ (replacing f by $-f$), we also obtain

$$|(f * F_m)(s)| \geq \frac{D}{4} - \frac{5}{2(m+1)}.$$

Now combining the above equation with (11), we have that

$$\frac{D}{4} - \frac{5}{2(m+1)} \leq |(f*F_m)(s)| \leq \frac{1}{\pi}\sum_{k=1}^{m} \frac{1}{k|F|}\left|\sum_{y\in F} e(ky)\right|$$

and so the theorem follows in the case $\hat{f}(0)=0$ with $C_1 = 5$ and $C_2 = \frac{8}{\pi}$ (after multiplying by 2 to convert from a bound on $D = D^*(F)$ to a bound on $D(F)$).

It remains to show that the case where $\hat{f}(0)=0$ suffices to prove the theorem. If $a \in \mathbb{T}$, let $F_a = a+F = \{a+x \colon x \in F\}$ be the finite set F "shifted" by a. Let $f_a(x) = \lambda([0,x)) - \frac{|[0,x)\cap F_a|}{|F_a|}$. These shifted sets are useful since the discrepancy $D(F)$ is shift-invariant by definition, so $D(F) = D(F_a)$ for all a. We also have that for all $k \neq 0$ and $a \in T$, that $|\hat{f}_a(k)| = |\hat{f}(k)|$ since

$$\begin{aligned}
\int_0^1 \frac{|[0,x)\cap F_a|}{|F_a|} e(-kx)\,\mathrm{d}x &= \int_{-a}^{1-a} \frac{|[0,x)\cap F_a|}{|F_a|} e(-kx)\,\mathrm{d}x \\
&= \int_0^1 \frac{|[0,x)\cap F|}{|F|} e(-k(x-a))\,\mathrm{d}x = e(ka)\int_0^1 \frac{|[0,x)\cap F|}{|F|} e(-kx)\,\mathrm{d}x.
\end{aligned}$$

So it will suffice to show that there is some a so that the 0th Fourier coefficient of f_a is zero: $\hat{f}_a(0) = 0$.

Since $f_a(x) = \frac{|F\cap[0,x+a)|}{|F|} - \frac{|F\cap[0,a)|}{|F|} - (\lambda([0,x+a)) - \lambda([0,a))) = f(a+x) - f(a)$, we have that $\hat{f}_a(0) = \int_0^1 f(a+x) - f(a)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x - f(a)$. So it suffices to find some a so that $f(a) = \int_0^1 f(x)\,\mathrm{d}x$.

There must be some $b \in [0,1)$ so that $f(b) \geq \int_0^1 f(x)\,\mathrm{d}x$, and $c \in [0,1)$ so that $f(c) \leq \int_0^1 f(x)\,\mathrm{d}x$. Since f is periodic with period 1, by replacing c with $c+1$ if necessary, we may assume $b<c$. Now f is left-continuous and its only discontinuities are jump discontinuities at the finitely many x such that $x \in F$, where f increases by $\frac{1}{|F|}$. So setting $a = \sup\{x \in [b,c] \colon f(x) \geq \int_0^1 f(x)\,\mathrm{d}x\}$, we have that $f(a) = \int_0^1 f(x)\,\mathrm{d}x$. □

This concludes our proof of Laczkovich's circle squaring theorem.

If μ is a measure on $\mathbb{T}$, the *discrepancy of* μ is defined to be $D(\mu) = \sup_{\text{intervals } I} |\mu(I)) - \lambda(I)|$. If $F \subseteq \mathbb{T}$ is a finite set and μ_F is the uniform probability measure supported on F: $\mu_F(A) = \frac{|F \cap A|}{|F|}$, then $D(F) = D(\mu_F)$ by definition. Recall the *kth Fourier coefficient of* a measure μ on $\mathbb{T}$ is defined to be $\hat{\mu}(k) = \int_0^1 e(-kx)\, \mathrm{d}\mu(x)$. So in the case where μ_F is the uniform measure supported on a finite set F, $|\hat{\mu_F}(k)| = \frac{1}{|F|} \left|\sum_{x \in F} e(kx)\right|$. Thus, the Erdős-Turán inequality is the special case of the following theorem for measures of the form $\mu = \mu_F$. This theorem can be proved almost identically to the above

Theorem 7.3. *There are universal constants C_1, C_2 so that for all measures μ on $\mathbb{T}$ and all m,*

$$D(\mu) \leq \frac{C_1}{m+1} + C_2 \sum_{k=1}^{m} \frac{|\hat{\mu}(k)|}{k}$$

A more modern (but technical) proof of the Erdős-Turán inequality shows that for any interval $[a, b)$ and for any m, there are trigonometric polynomials $S^-(x)_m, S^+_m(x)$ of degree at most m such that for all $x \in [0, 1)$

$$S^-_m(x) \leq 1_{[a,b)} \leq S^+_m(x)$$

and so that $\int_x S^-_m(x)dx \geq b - a - \frac{1}{m+1}$ and $\int_x S^-_m(x)dx \leq b - a + \frac{1}{m+1}$. These $S^\pm_m$ are known as the Selberg polynomials and rely crucially on Vaaler's polynomials approximating the saw-tooth function, a calculation of the error in this approximation, and Vaaler's lemma [M, Page 6]. By using $S^\pm_m$ as upper and lower bounds for $1_{[a,b)}$, and writing everything in terms of Fourier series, one can get an upper bound on $|\mu([a, b)) - \lambda([a, b))|$ of $\sum_{0<|k|<m} \hat{S}^+_m(k)\hat{\mu}(k)$. The Erdős-Turán inequality follows easily from this. See [M, Chapter 1] for a detailed proof based on this idea and [DT, Theorem 1.21] for a similar proof of the Erdős-Turán-Koksma inequality.

We mention also that the Erdős-Turán inequality is a quantitative refinement of Weyl's famous theorem on equidistribution which preceded it:

Theorem 7.4 (Weyl [KN, Theorem 1.2.1])**.** *For an infinite sequence of points $(x_n)_{n \in \mathbb{N}}$ in $[0, 1)$, the following are equivalent:*

(1) *(x_n) is equidistributed, that is for all $[a, b) \subseteq [0, 1)$,*

$$\lim_{N \to \infty} \frac{|\{x_i : i < N\} \cap [a, b)|}{N} = b - a.$$

(2) *For all $k \neq 0$,*

$$\lim_{N \to \infty} \sum_{n=0}^{N-1} e(kx_n) = 0.$$

This theme of finding quantitative "hard analysis" refinements of qualitative "soft analysis" theorems (like Erdős-Turán's refinement of Weyl's theorem) is an important recurring theme in many areas such as ergodic theory and harmonic analysis. While the Erdős-Turán theorem was proved after Weyl's theorem, it has also been the case that difficult qualitative results have been proved by first finding quantitative hard analysis estimates which imply them.

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