A SHORT PROOF OF HALPERN-LAUCHLI

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In this document, we give a short proof of the Halpern-Lauchli theorem. We hope that this is clearer than the proofs found in [2] and [1], although the contents appear to be roughly the same. Thanks to Jing Zhang for explaining the outline of the proof.

The proof is by induction on the dimension d of the product. For each d, we define three versions of the Halpern-Lauchli theorem: SD_d, DS_d and SS_d. These stand for somewhere dense, dense set and strong subtree, respectively.

Every tree T below is countable and finitely branching. For $x \in T$, $\mathrm{ht}(x)$ is the height of x. The words "above" and "below" refer to the natural tree order on whatever product of trees is relevant.

In what follows we use $\prod_{i<d} T_i$ and $\prod_{i<d}^{lev} T_i$ to distinguish between non-level and level products of trees. As the name suggests, a level product is a subset of the full product where every tuple consists of points of the same height/level.

For $\bar{x} \in \prod_{i<d} T_i$ and $k \in \mathbb{N}$, sets $X_i \subseteq T_i$ for $i < d$ form a k-$\bar{x}$-dense matrix if for all $\bar{t} \in \prod_{i<d} T_i(k)$ above $\bar{x}$ there is $\bar{y} \in \prod_{i<d} X_i$ above $\bar{t}$.

Definition 1 (SD_d)**.** *For every coloring $c : \prod_{i<d} T_i \to r$, there are $\bar{x}$ and $k \in \mathbb{N}$ such that there is a monochromatic k-$\bar{x}$-dense matrix.*

Definition 2 (DS_d)**.** *For every coloring $c : \prod_{i<d} T_i \to r$, there is $\bar{x}$ such that for all $k \in \mathbb{N}$ there is a monochromatic k-$\bar{x}$-dense matrix.*

For a tree T, a subset $S \subseteq T$ is a strong subtree if there is $A \subseteq \mathbb{N}$ infinite such that:

(1) For all $s \in S$, $s \in T(n)$ for some $n \in A$ and for all $n \in A$, $S \cap T(n) \neq \emptyset$.
(2) If $m < n$ are consecutive elements of A and $s \in S \cap T(m)$, then every immediate successor of s in T has a unique extension in $S \cap T(n)$.

We call A *the level set* of S.

Definition 3 (SS_d)**.** *For every coloring $c : \prod_{i<d}^{lev} T_i \to r$, there are strong subtrees $S_i \subseteq T_i$ with the same level set such that c is constant on $\prod_{i<d}^{lev} S_i$.*

The workflow is as follows:

(1) Prove SD_1.
(2) For all d, SD_d implies DS_d.
(3) For all d, DS_d implies SS_d.
(4) For all d, DS_d implies SD_{d+1}.

As listed it looks like there is some redundancy but SS_d will be used in the proof of item (4).

Claim 1. SD_1

Proof. Let $c : T \to r$ be a coloring and suppose SD_1 fails for this coloring. Starting at the root of T build an increasing sequence t_i for $i \leq r$ in T such that for all i, $c \restriction (T \restriction t_{i+1})$ avoids color i. Then t_k cannot take any color, a contradiction. □

Claim 2. *SD_d implies SD_d where the sets in the k-$\bar{x}$-dense matrix share a common level.*

We call this a *level* matrix.

Proof. Fix a product of trees $\prod_{i<d} T_i$ and a number of colors r. A straightforward compactness argument shows that there is l such that letting T_i' be the restriction of T_i to the first l levels we have: For all colorings $c : \prod_{i<d} T_i' \to r$ there are $\bar{x} \in \prod_{i<d} T_i'$ and $k < l$ for which there is a monochromatic k-$\bar{x}$-dense matrix.

Let $c^* : \prod_{i<d} T_i \to r$ be a coloring. For each $x \in T_i$ below level l, choose $h_i(x) \in T_i(l)$. Let $c : \prod_{i<d} T_i' \to r$ (with T_i' as above) be given by $c(\langle y_i \mid i < d \rangle) = c^*(\langle h_i(y_i) \mid i < d \rangle)$.

By the choice of l, there are $\bar{x} \in \prod_{i<d} T_i$ and k such that $\prod_{i<d} T_i'$ contains a monochromatic matrix X_i for $i < d$. It is straightforward to check that $h_i[X_i]$ for $i < d$ is a monochromatic matrix for c^* and all of whose points are on level l. □

Remark 1. *The level product is an example of a dense "subproduct" of the full product. The same proof can work to find dense matrices in other dense "subproducts".*

Remark 2. *In versions of Halpern-Lauchli where we end up finding a monochromatic level matrix, it is enough consider colorings of the level product to begin with.*

Claim 3. *For all d, SD_d implies DS_d.*

Proof. We argue by contradiction. Let $c : \prod_{i<d} T_i \to r$ be a coloring for which DS_d fails. Then for all $\bar{x} \in \prod_{i<d} T_i$ there is $k_{\bar{x}}$ such that for all $k \geq k_{\bar{x}}$ no k-$\bar{x}$-dense monochromatic matrix exists.

Construct an increasing sequence of levels l_n by induction on n. Let $l_0 = 0$ and let $l_{n+1} = \max_{\bar{x}} k_{\bar{x}}$ where the maximum is over all nodes in $\bar{x}$ are on level at most l_n.

Let $L = \{l_n \mid n \in \mathbb{N}\}$. Let T_i' be the obvious restriction of T_i to levels from L. We apply SD_d to the restriction of c to $\prod_{i<d} T_i'$ to get $\bar{x} \in \prod_{i<d} T_i'$ and $n \in \mathbb{N}$ such that there is a monochromatic n-$\bar{x}$-dense matrix X_i for $i < d$ in $\prod_{i<d} T_i'$.

It follows that X_i for $i < d$ is a monochromatic l_n-$\bar{x}$-dense matrix in $\prod_{i<d} T_i$, contradicting our choice of l_n. □

We note that the same argument implies a version of DS_d where every witnessing matrix is a level matrix as in Claim 2

Claim 4. *For all d, DS_d implies SS_d.*

This is a straightforward inductive construction.

Proof. Fix a coloring $c : \prod_{i<d} T_i \to r$. Let $\bar{x}$ witness DS_d with level matrices. By induction choose an infinite sequence k_n and monochromatic level matrices X_i^n for $i < d$ which are k_n-$\bar{x}$-dense and such that k_{n+1} is above the level of X_i^n for $i < d$. By the pigeonhole principle, we can assume that every matrix is monochromatic with the same color.

From this sequence it is straightforward to construct a sequence of strong subtrees S_i for $i < d$ with a common level set such that c is monochromatic on $\prod_{i<d}^{lev} S_i$. □

Claim 5. *For all d, DS_d implies SD_{d+1}.*

Proof. Let $c : \prod_{i \leq d} T_i \to r$ be a coloring. For each $t \in T_d$ we can define a coloring $c_t : \prod_{i<d} T_i \to r$ by $c_t(\bar{x}) = c(\bar{x} \frown t)$.

By a straightforward fusion argument, we can find strong subtrees $S_i \subseteq T_i$ for $i < d$ with a common level set and the following property: For all $t \in T_d$ and all $\bar{x} \in \prod_{i<d}^{lev} S_i$ with $\operatorname{ht}(\bar{x}) \geq \operatorname{ht} t$, c_t is constant on $\left(\prod_{i<d} T_i\right) \restriction \bar{x}$. We note here that the $\operatorname{ht}(\bar{x})$ is as computed in $\prod_{i<d}^{lev} S_i$.

Next, we derive a different family of colorings. For each $b \in [T_d]$, we define a coloring $f_b : \prod_{i<d}^{lev} S_i \to r$ by $f_b(\bar{x}) = c(\bar{x} \frown y)$ for the unique $y \in b$ with height $\operatorname{ht}(\bar{x})$. We apply DS_d to this coloring to obtain $\bar{x}_b$ and a color $i_b < r$ such that for all k, there is a k-$\bar{x}_b$-dense level matrix which is monochromatic with color i_b.

We pause for a claim.

Claim 6. *Let T be an infinite, finitely branching tree and $g : [T] \to \mathbb{N}$ be a coloring. There are $t \in T$, a dense subset $D \subseteq T \restriction t$ and $j \in \mathbb{N}$ such that for all $s \in D$, there is $b \in [T]$ such that $s \in b$ and $g(b) = j$.*

Proof. Otherwise, for all t and all colors j there is t' above t such that $c \restriction [T \restriction t']$ avoids color j. From this it is straightforward to construct a branch $b \in [T]$ which is uncolored. □

Apply this claim to the coloring of $[T_d]$ which takes b to $(\bar{x}_b, i_b)$. Let $t \in T_d$, $D \subseteq T_d \restriction t$ dense, $\bar{x} \in \prod_{i<d} S_i$ and $i < r$ witness the claim.

We construct a k-$(\bar{x} \frown t)$-dense matrix as follows. Enumerate the immediate successors of t as t_k for $k < n$. From the conclusion of the previous claim, we can build sequences

(1) $b_k \in [T_d]$ with $t_k \in b_k$,
(2) $\operatorname{ht}(\bar{x}) + 1$-$\bar{x}$-dense level matrices $X_i^k \subseteq S_i$ for $i < d$ such that each $u \in X_i^k$ has a unique extension in X_i^{k+1} and $\prod_{i<d} X_i^k$ gets color i under f_{b_k}.
(3) $s_k \in b_k$ is the unique element whose height in $\prod_{i<d}^{lev} S_i$ is the same as the common height of the X_i^k.

By the choice of the trees S_i, we have that for $k < k'$, c takes color i on the set $\left(\prod_{i<d} X_i^{k'}\right) \times \{s_k\}$ since c_{s_k} takes color i on $\prod_{i<d} X_i^k$ and every element of $\prod_{i<d} X_i^{k'}$ is above an element of $\prod_{i<d} X_i^k$.

It follows that $\left(\prod_{i<d} X_i^n\right) \times \{s_k \mid k < n\}$ is monochromatic under c with color i. □

References

[1] Pandelis Dodos and Vassilis Kanellopoulos, *Ramsey theory for product spaces*, Mathematical Surveys and Monographs, vol. 212, American Mathematical Society, Providence, RI, 2016. MR 3469779

[2] Stevo Todorcevic, *Introduction to Ramsey spaces*, Annals of Mathematics Studies, vol. 174, Princeton University Press, Princeton, NJ, 2010. MR 2603812