A PRESERVATION ARGUMENT FOR DOUBLE SUCCESSORS OF SINGULARS

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In this document, we provide a correct argument for Lemma 4.1 from [3]. We would like to thank Šárka Stejskalová for finding the error in the published version. The issue is with Claim 4.5. We fix this by weakening the claim to see that we can perform a certain splitting argument inside the quotient.

We note that there is a similar preservation lemma in independent work of Friedman, Honzik and Stejskalova [1], but preservation property that they obtain is weaker. In particular, the argument here is enough to give "the approximation property" and so for example the principle ISP at the double successor of a singular as in [2].

We assume that the reader is familiar with the set up from [3]. Recall that G is generic for the Mitchell forcing $\mathbb{M}$ and H is generic for the Prikry $\mathbb{R}$. As usual we say that a stem h is compatible with $r \in \mathbb{R}$, if there is $r' \leq r$ with stem h.

For simplicity we assume that $\mathbb{A}$ is trivial and remark at the end how to modify the argument when it is not.

Lemma 1. *Work in $V[G]$. Suppose that $\bar{r} \in \mathbb{R}, (p, f, \dot{r}) \in k(\mathbb{M} * \mathbb{R})/G$ are such that $s(\bar{r})$ extends $s(\dot{r})$, $\bar{r} \Vdash (p, f, \dot{r}) \in \dot{\mathbb{N}}$ and $(p, f) \Vdash s(\bar{r}) \setminus s(\dot{r})$ is contrained by $\dot{r}$. Then there are $f' \leq_{k(\mathbb{Q})} f$, and a direct extension $\bar{r}' \leq^* \bar{r}$, such that*

1. *$\bar{r}' \Vdash (p, f', \dot{r}) \in \dot{\mathbb{N}}$,*
2. *for every stem h compatible with $\bar{r}'$, there is $p' \leq p$, such that $(p', f') \Vdash h \setminus s(\dot{r})$ is constrained by $\dot{r}$.*

Proof. Using that the term forcing is closed enough, construct a $\leq_{k(\mathbb{Q})}$ decreasing sequence $\langle f^h \mid h \text{ a stem extending } s(\bar{r}) \rangle$ below f, according to some enumeration of the stems as follows.

For every stem h, let f' be a $\leq_{k(\mathbb{Q})}$-lower bound of the conditions constructed so far. If $(p, f') \not\Vdash h$ is incompatible with $\dot{r}$, then pick $f^h \leq_{k(\mathbb{Q})} f'$, so that for some $p' \leq p$, we have that $(p', f^h) \Vdash h \setminus s(\dot{r})$ is constrained by $\dot{r}$. Otherwise, set $f^h = f'$.

Let $f'' \leq_{k(\mathbb{Q})} f^h$ for all h. There is a direct extension of $\bar{r}' \leq^* \bar{r}$ forcing that $(p, f'', \dot{r}) \in \dot{\mathbb{N}}$. Then if h is a stem compatible with $\bar{r}'$, $(p, f'') \not\Vdash h$ is incompatible with $\dot{r}$. So in the construction, we must have picked f^h, so that

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for some $p' \leq p$, $(p', f^h) \Vdash h \backslash s(\dot{r})$ is constrained by $\dot{r}$. Then $(p', f^*) \Vdash h \backslash s(\dot{r})$ is constrained by $\dot{r}$. $\square$

Next we have the following revision of Claim 4.5 of [3].

Lemma 2. *In $V[G * H]$, there is a condition $(p, f, \dot{r}) \in \mathbb{N}$ such that for all $p' \leq p$, x, $\alpha < \mu$, $f' \leq_{k(\mathbb{Q})} f$, $\dot{r}'$ if $(p', f', \dot{r}') \leq_{\mathbb{N}} (p, f, r)$ and forces $\dot{\tau} \restriction \alpha = x$, then there is $f'' \leq_{k(\mathbb{Q})} f'$, such that $(p, f'', \dot{r})$ is in $\mathbb{N}$ and forces $\dot{\tau} \restriction \alpha = x$.*

Proof. We start with a claim that provides one step of an inductive construction.

Claim 3. *Suppose that in $V[G]$, $\bar{r} \in \mathbb{R}$ forces the failure of the claim and $\bar{r}$ forces that $(p, f, \dot{r})$ is in the quotient. Then there are $\alpha, x_i, p_i, f^*, \dot{r}_i$ for $i = 0, 1$ and $\bar{r}^* \leq \bar{r}$ such that*

- *$f^* \leq_{k(\mathbb{Q})} f$*
- *$s(\bar{r}^*)$ is equal to $s(\dot{r}_1)$ and extends $s(\dot{r}_0)$*
- *(p_0, f^*) forces that $s(\bar{r}^*)$ is constrained by $\dot{r}_0$.*
- *$\bar{r}^*$ forces that:*
  - *for $i = 0, 1$, $(p_i, f^*, \dot{r}_i)$ are in the quotient and below $(p, f^*, \dot{r})$,*
  - *$(p_i, f^*, \dot{r}_i)$ forces that $\dot{\tau} \restriction \alpha = x_i$, for $i = 0, 1$,*
  - *x_0 and x_1 are incompatible.*

Proof. Fix $\bar{r}$ and $(p, f, \dot{r})$ as in the statement of the claim. By our assumption there are $(p', f', \dot{r}_0)$, $f' \leq_{k(\mathbb{Q})} f$, x_0, α, and $\bar{r}' \leq \bar{r}$ which forces that:

(1) $(p', f', \dot{r}_0)$ is in $\dot{\mathbb{N}}$ below $(p, f, \dot{r})$,
(2) $(p', f', \dot{r}_0)$ forces that $\dot{\tau} \restriction \alpha = x_0$, and
(3) for all $f^* \leq_{k(\mathbb{Q})} f'$, such that $(p, f^*, \dot{r})$ is in the quotient, $(p, f^*, \dot{r})$ does not force $\dot{\tau} \restriction \alpha = x_0$.

By extending $\bar{r}'$ if necessary, we may assume that $s(\bar{r}')$ extends $s(\dot{r}_0)$.

Let $(p_0, f_0) \leq (p', f')$, $f_0 \leq_{k(\mathbb{Q})} f'$, be such that

$$(p_0, f_0) \text{ forces } s(\bar{r}') \setminus s(\dot{r}_0) \text{ is constrained by } \dot{r}_0$$

Find a direct extension $\bar{r}'' \leq^* \bar{r}'$ forcing that $(p_0, f_0, \dot{r}_0)$ is in $\dot{\mathbb{N}}$ below $(p', f', \dot{r}_0)$. We note that properties (2) and (3) above are preserved.

By Lemma 1 applied to $\bar{r}''$ and $(p, f_0, \dot{r})$, let $f'' \leq_{k(\mathbb{Q})} f_0$ and $\bar{r}''' \leq^* \bar{r}''$ be such that $\bar{r}'''$ forces that $(p_0, f'', \dot{r}_0) \in \dot{\mathbb{N}}$ and for every stem h compatible with $\bar{r}'''$ there is some $p' \leq p$, such that $(p', f'') \Vdash h \setminus s(\dot{r}_0)$ is constrained by $\dot{r}_0$.

Now, pass to a generic extension containing $\bar{r}'''$. Let $f^* \leq_{k(\mathbb{Q})} f''$, x_1, p_1, be such that

- $(p_1, f^*, \dot{r}_1) \in \mathbb{N}$ and below $(p, f'', \dot{r})$,
- $(p_1, f^*, \dot{r}_1)$ forces that $\dot{\tau} \restriction \alpha = x_1$,
- x_0 and x_1 are incompatible.

Let $\bar{r}^* \leq \bar{r}'''$ force the above items, and set $h := s(\bar{r}^*)$. We can assume that $s(\dot{r}_1) = h$. Since h is compatible with $\bar{r}'''$, we have that for some $p_0' \leq p_0$,

$(p_0', f'') \Vdash h$ is constrained by $\dot{r}_0$. In particular, $(p_0', f^*) \Vdash h$ is constrained by $\dot{r}_0$.

Then there is a direct extension $\bar{r}^{**} \leq^* \bar{r}^*$, forcing that $(p_0', f^*, \dot{r}_0) \in \dot{\mathbb{N}}$.

Then, $\bar{r}^{**}$, $(p_0', f^*, \dot{r}_0)$, $(p_1, f^*, \dot{r}_1)$, α, x_0, x_1 are exactly as desired. $\square$

Claim 3 is enough to complete the recursive construction in Claim 4.5 to obtain a contradiction. $\square$

Finally Lemma 2 is enough to complete the coding argument used in the proof of Lemma 4.1 of [3].

Remark 4. We describe how to modify the argument when $\mathbb{A}$ is nontrivial. In Claim 1, we work in $V[G]$ and suppose that $(a, \bar{r}) \in \mathbb{A} * \mathbb{R}$ forces the failure of the Lemma. We use the closure of the term ordering to go over antichains A_h in $\mathbb{A}$ such that each member of A_h forces that there is some p for which (p, f_h) forces h is compatible with r if possible. At the end, the have an extension $(a', \bar{r}')$ of $(a, \bar{r})$ with $\bar{r}' \leq^* \bar{r}$ such that $(a', \bar{r}')$ forces the conclusion.

In the splitting argument used to finish Lemma 2, we pick arbitrary $a \in \mathbb{A}$ which witness the instances of Claim 1. Many of them will be compatible by the chain condition of $\mathbb{A}$.

The remainder of the argument is as in [3].

References

1. Sy-David Friedman, Radek Honzik, and Sarka Stejskalova, *The tree property at $\aleph_{\omega+2}$ with a finite gap*, (2018), submitted.
2. Sherwood Hachtman and Dima Sinapova, *ITP, ISP and SCH*, (2018), submitted.
3. Dima Sinapova and Spencer Unger, *The tree property at $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$*, J. Symb. Log. **83** (2018), no. 2, 669–682. MR 3835083