Talk 4 of 4 on an argument from Mitchell

Spencer Unger

February 24, 2009

1 The final step in Mitchell's proof

1.1 Part 3: ω_2 has the tree property

Before we proceed, I will give the set up one last time. We have κ, which is a measurable cardinal. $j : V \to N$ witnesses κ's measurability. We have G, $\mathbb{M}$-generic over V. By our factorization and because Mitchell conditions are small, we get a lifted elementary embedding, which we will also call $j : V[G] \to N[H]$, where H is $j(\mathbb{M})$-generic over V. By our new factorization of Mitchell, we know that we can think of $N[H]$ as $N[G][H']$, where H' is $\mathbb{M}(V^{\mathbb{P}(\kappa)}, j(\kappa))_{N[G]}$-generic over $N[G]$. Let T be an ω_2-tree in $V[G]$. By the usual argument, T has a branch in $N[G][H']$. By our factor analysis we have to show that $\mathbb{M}(V^{\mathbb{P}(\kappa)}, j(\kappa))_{N[G]}$ doesn't add branches over N[G].

Recall that Last time we defined Mitchell's poset.

Definition 1.1. *Let $\mathbb{P} = Add(\omega, \kappa)$ and for all $\alpha < \kappa$ let $\mathbb{P} \restriction \alpha$ be $Add(\omega, \alpha)$. As I said before we will call mitchell's poset, $\mathbb{M}$. $\mathbb{M}$ is the collection of pairs, (p, f), such that p is a finite partial function from κ to 2 and f is a function with domain a countable subset of κ and for all $\alpha \in dom(f), f(\alpha)$ is a $\mathbb{P} \restriction \alpha$-name for a condition in $Add(\omega_1, 1)_{V^{\mathbb{P}\restriction\alpha}}$. For the ordering, let $(p, f) \leq (p', f')$ iff $p' \subseteq p$, $dom(f') \subseteq dom(f)$ and for all $\alpha \in dom(f')$, $p \restriction \alpha \Vdash^V_{\mathbb{P}\restriction\alpha} f(\alpha) \leq f'(\alpha)$ in $Add(\omega_1, 1)_{V^{\mathbb{P}\restriction\alpha}}$.*

And proved the first two parts of the theorem. We proved that $(2^\omega = \kappa = \omega_2)^{V^{\mathbb{M}}}$ and that ω_1 is preserved in the extension. Today we are going to move on to sketch the proof that ω_2 has the tree property in the extension.

We have arrived at the final section with the majority of the work still to be done. It turns out that the hardest part of the argument is showing that the mitchell poset factors nicely in to an initial segment of Mitchell star a final segment that is Mitchellesque. For our purposes, I am going to blackbox that argument and state it as a theorem. I will however give the definition of the mitchellesque poset and wave my hands at the argument.

Recall from our discussion of the outline of Mitchell's argument that we are interested in how to get from our original poset up to j of the poset.

Do do this factorization argument, we are actually going to work over V and tell the story as a story about the factorization of our original Mitchell poset.

I claim that to understand the factorization of $j(\mathbb{M})$ it is enough to understand a factorization of $\mathbb{M}$ at an arbitrary α which is innacessible below κ. Note that $\phi : \mathbb{M} \to \mathbb{M} \restriction \alpha$ given by $\phi(p, f) = (p \restriction f \restriction \alpha)$ is a projection. Then given a generic for $\mathbb{M} \restriction \alpha$, $G \restriction \alpha$, we can form $\mathbb{M}/(G \restriction \alpha)$ and this is the forcing that we want to understand. This is enough, because we can just apply j to the whole analysis and then it will work on the j-side of the embedding with $\kappa < j(\kappa)$. Important to note is that $\mathbb{M}/(G \restriction \alpha)$ might not be the Mitchell-like thing that I have been mentioning up until now.

Note that we could tell a similar sort of story with $j(\mathbb{M})$ instead of $\mathbb{M}$ $\phi' : j(\mathbb{M}) \to \mathbb{M}$, given by $\phi'(p, f) = (p \restriction \kappa, f \restriction \kappa)$ is a projection. So from our discussion of projections we can form $j(\mathbb{M})/G$, where G is our $\mathbb{M}$-generic over V.

Notationally this approach with just factoring Mitchell is going to be a little easier. In order to show that the mitchell poset factors nicely around α, I want to show that there is an order preserving map with a dense range from $\mathbb{M}$ to an initial segment of mitchell star a modified mitchell poset computed in the extension by that initial segment. This will show that given a generic for either poset, the original mitchell or the factored version, we can compute a generic for the other. This shows that forcing with either poset really just does the same thing.

In order to proceed, I need to generalize the definition of mitchell that I gave last time. In the definition of $\mathbb{M}$, we really only used that κ was innacessible. So my first modification is to define a class of Mitchell posets. Let α be innacessible and let $\mathbb{M}(\alpha)$ be the Mitchell forcing of height α. I think that before I may have called this poset $\mathbb{M} \restriction \alpha$. Notice that $\mathbb{M} = \mathbb{M}(\kappa)$.

Next, I want to make a slightly deeper modification. I concerns where we took output values (which were names) of our second coordinates from. Let $V_0 \subseteq V$ such that ${}^{\omega}V_0 \subseteq V_0$, then define $\mathbb{M}(V_0, \alpha)$ to be Mitchell forcing of height α where the only modification is that we took $f(\alpha) \in V_0$ to be a $\mathbb{P} \restriction \alpha$-name for a condition in $Add(\omega_1, 1)_{V_0^{\mathbb{P} \restriction \alpha}}$.

Let me motivate the definition of $\mathbb{M}(V_0, \alpha)$. What we really want to do is split up a mitchell condition at an innaccessible $\alpha < \kappa$. So given a condition (p, f), the part before α is easy. We just take $(p \restriction \alpha, f \restriction \alpha)$. For the second bit we would like to just take the upper part of the condition, but we need to make sure that we get the right kind of object. The domain of the upper part of f is a subset of $[\alpha, \kappa)$ and I really need it to have it just be a subset of κ. Every ordinal bigger than α can be viewed as $\alpha + \beta$ for some $\beta < \kappa$. So this is how we are going to reorganize our conditions.

Theorem 1.2. *There is an order preserving map with a dense range $h : \mathbb{M} \to \mathbb{M}(\alpha) * \mathbb{M}(V^{\mathbb{P}(\alpha)}, \kappa)_{V^{\mathbb{M}(\alpha)}}$.*

This theorem shows that over the extension $V[G \restriction \alpha]$, the extension $\mathbb{M}/(G \restriction \alpha)$ is really the same as $\mathbb{M}(V^{\mathbb{P}(\alpha)}, \kappa)_{V[G \restriction \alpha]}$.

As modified Mitchell is really quite Mitchell-like, in $V[G \restriction \alpha]$, the extension by $\mathbb{M}(V^{\mathbb{P}(\alpha)}, \kappa)$ lives inside an extension by $\mathbb{P}^* \times \mathbb{R}^*$, where $\mathbb{P}^*$ is just modified cohen forcing and $\mathbb{R}^*$ is a countably closed version of $\mathbb{R}$ from before. We are now going use the elementarity of j on all of this this information and use it to analyse how we get from $\mathbb{M}$ up to $j(\mathbb{M})$ on the j side of our generic embedding.

As I said before, it can be shown that $\mathbb{M}(V^{\mathbb{P}(\kappa)}, j(\kappa))_{N[G]}$ really lives inside an extension by $\mathbb{P}^{**} \times \mathbb{R}^{**}$, which we know exist by elementarity. Recall that $\mathbb{R}^{**}$ is countably closed and $\mathbb{P}^{**}$ is ccc.

Let H_1 be $\mathbb{P}^{**}$-generic over $N[G]$ and H_2 be $\mathbb{R}^{**}$-generic over $N[G]$. Recall that when forcing with a product we can forcing in either order. We want to show that going from $N[G]$ to $N[G][H_2]$ doesn't add any branches to our tree T.

Proposition 1.3. *If $2^\omega = \omega_2$ then countably closed forcing cannot add branches to an ω_2-tree*

So therefore T has no new branches in $N[G][H_2]$. There is a slight problem however. As $\mathbb{M}(V^{\mathbb{P}(\kappa)}, j(\kappa))_{N[G]}$ is Mitchell-like, we would expect it to collapse $\omega_2 = \kappa$. So ω_2 must be collapsed, by $\mathbb{P}^{**} \times \mathbb{R}^{**}$. In fact we can show that it was collapsed by $\mathbb{R}^{**}$. So in $N[G][H_2]$, the height of T is now some ordinal between ω_1 and ω_2, call it η. We can show that $cf(\eta) = \omega_1$.

As $cf(\eta) = \omega_1$, choose a cofinal sequence of ordinals in η, $\langle \alpha_i : i < \omega_1 \rangle$. Define the tree T_s to be a tree of hieght ω_1 such that for all $i < \omega_1$ $Lev_i(T_s) = Lev_{\alpha_i}(T)$, whose ordering is the obvious one inhereted from T. Notice that T_s is NOT an ω_1-tree! It might have some levels of size ω_1.

Remark 1.4. T has a branch if and only if T_s has one.

So we are trying to show that forcing with $\mathbb{P}^{**}$ over $N[G][H_2]$ doesn't add branches to T_s. To show this we need the following definition.

Definition 1.5. *A poset $\mathbb{Q}$ is λ Knaster iff $\forall \langle q_i : i < \lambda \rangle$ such that $\forall i < \lambda$ $q_i \in \mathbb{Q}$, there is $I \subseteq \lambda$ such that $|I| = \lambda$ and $\forall i, j \in I$, $i \neq j \Rightarrow q_i$ and q_j are compatible.*

Remark 1.6. Being λ Knaster is obviously stronger than being λ-cc, because given an antichain of size λ, by Knasterness there are λ many pairwise compatible elements in it, a contradiction.

Remark 1.7. Often when we do the sort of freezing out arguments to create a delta system, we are really showing Knasterness and not just the chain condition.

Proposition 1.8. *$N[G][H_2] \models \mathbb{P}^*$ has the ω_1 Knaster property.*

With this property in hand we show,

Proposition 1.9. *If T^* is a tree of height ω_1, then ω_1 Knaster forcing cannot add branches through T^*.*

And this finishes the proof, because we know that there is a branch through T in $N[G][H']$, but we just showed that forcing with $\mathbb{M}(V^{\mathbb{P}(\kappa)}, j(\kappa))_{N[G]}$ could not have added it. Therefore our branch must be in $N[G]$ and therefore it is in $V[G]$, as required.