Talk 2 of 4 on an argument from Mitchell

Spencer Unger

February 20, 2009

As a kind of hold over from last time I wanted to atleast mention a couple of lower bounds for the forcing results that I mentioned last time.

Definition 0.1. *A cardinal κ is strong up to δ iff for every $\alpha < \delta$ there is $j : V \to M$ with $crit(j) = \kappa$ and $j(\kappa) > \alpha$ and $V_\alpha \subseteq M$*

Definition 0.2. *For a subset $A \subseteq \delta$, κ is strong in A up to δ iff for every $\alpha < \delta$ there is j as above such that $j(A)$ and A agree up to α*

Definition 0.3. *A cardinal δ is Woodin iff for every A as above there is $\kappa < \delta$ which is strong in A*

From work of Foreman, Magidor and Schindler we have:

Theorem 0.4. *If $\aleph_n$ has TP for all $2 \leq n < \omega$, then for every n there is an inner model with n Woodin cardinals.*

And also, combining work of Mitchell, Schimmerling, Zeman, Steel and Woodin, we have

Theorem 0.5. *If $\aleph_\omega$ is strong limit and $\aleph_{\omega+1}$ has TP, then there is an inner model with infinitely many woodin cardinals.*

Today we are going to talk about the general forcing facts required for an understanding of Mitchell's argument. Mostly without proof, I will develop most of the forcing technology that will be used in Mitchell's proof. I am going to assume some familiarity with the basics of forcing and develop from there.

A few preliminaries about convention. Stronger conditions for us are going to go 'down' the poset. In general our posets will have top elements and some of our theory will reflect that. I will denote the top element of a poset $\mathbb{P}$ as $\mathbb{1}_{\mathbb{P}}$. I will systematically drop subscripts for readability and ease of writing.

1 Projections

The definition of Mitchell's poset, when I give it, will be quite complicated and one of the main ways that we are going to seek to understand it is through the use of projections. A projection is going to give us a way to move between two posets. In fact given a projection and a generic we will be able to generate a generic for the poset we are projecting on to.

Definition 1.1. *$\pi : \mathbb{P} \to \mathbb{Q}$ is a projection iff The following conditions hold*

1. $\pi(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{Q}}$
2. *π is order preserving*
3. *$\forall p \in \mathbb{P}\ \forall q < \pi(p)\ \exists \bar{p} \leq p$ such that $\pi(\bar{p}) \leq q$*

Proposition 1.2. *If G is V-generic over $\mathbb{P}$, then $G_0 = \{q : \exists p \in G\, \pi(p) \leq q\}$ is V-generic over $\mathbb{Q}$.*

Proof. We need to check that G_0 is a V-generic over $\mathbb{Q}$. Clearly, as defined it is upwards closed. Next we check the compatibility condition. Suppose $q_1, q_2 \in \mathbb{Q}$, then from the definition there are $p_1, p_2 \in G$ with $q_1 \geq \pi(p_1)$ and $q_2 \geq \pi(p_2)$. As $p_1, p_2 \in G$ there is a $p' \in G$ such that $p' \leq p_1, p_2$. Then as π is order preserving, $\pi(p') \leq \pi(p_1), \pi(p_2)$. Also, $\pi(p') \in G_0$. This does it as clearly $\pi(p') \leq q_1, q_2$. Draw the picture.

Next we check genericity. Draw the picture. Let $D \subseteq \mathbb{Q}$ be dense open. We want to show that the inverse image of D is dense in $\mathbb{P}$. Let $p \in \mathbb{P}$. Then as D is dense in $\mathbb{Q}$. There is a $q \in \mathbb{Q}$ with $q \leq \pi(p)$. If $q = \pi(p)$ then we are done. As p was in the inverse image of π. If $q < \pi(p)$, then we apply condition (3) to get a $\bar{p} \leq p$ such that $\pi(\bar{p}) \leq q$. As D is open we know that $\pi(\bar{p}) \in D$ which is exactly what we wanted. So the inverse image of D is dense and therefore $G \cap \pi^{-1}(D) \neq \emptyset$. And thus $G_0 \cap D \neq \emptyset$, as required. □

Remark 1.3. This shows that if you hand me a generic for $\mathbb{P}$ then I can give you back a generic for $\mathbb{Q}$. We often write that $V^{\mathbb{Q}} \subseteq V^{\mathbb{P}}$.

Let G_0 be V-generic for $\mathbb{Q}$. In $V[G_0]$ define $\mathbb{P}/G_0 = \{p \in \mathbb{P} : \pi(p) \in G_0\}$. A poset that inherits it's ordering from $\mathbb{P}$.

Proposition 1.4. *If G is $V[G_0]$-generic over $\mathbb{P}/G_0$, then G is V-generic over $\mathbb{P}$*

In fact the converse of the above is true.

Proposition 1.5. *If G is V-generic for $\mathbb{P}$ and G_0, $\mathbb{P}/G_0$ are as above, then $G \subseteq \mathbb{P}/G_0$ and G is $V[G_0]$-generic for $\mathbb{P}/G_0$.*

Example 1.6. Throughout I will give examples of projections. We are prepared to give a first example here. Let $\mathbb{P}$ and $\mathbb{Q}$ be posets. Then let $\pi : \mathbb{P} \times \mathbb{Q} \to \mathbb{P}$ given by $\pi((p, q)) = p$ is a projection. Given what we know from the product lemma this makes sense and is actually a kind of trivial example.

2 Two Step Iteration

In this section, I will introduce a way of coding two posets together so that forcing with the coded poset is like first forcing with one poset and then forcing with the next poset.

Here is the general set up. We have a poset $\mathbb{P}$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is a poset. So in any extension by the poset $\mathbb{P}$, $\mathbb{Q}$ is a poset. We code the two posets together as follows.

Definition 2.1. *Let $\mathbb{P} * \dot{\mathbb{Q}}$ have the underlying set $\{(p, \dot{q}) : P \in \mathbb{P}$ and $\Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}\}$. And let $(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2)$ iff $p_1 \leq p_2$ and $p_1 \Vdash \dot{q}_1 \leq \dot{q}_2$.*

There are some serious technical considerations here. First, as defined the underlying set is in fact a proper class, because there are a proper class of names for each member of the poset $\mathbb{Q}$. So we need to pick a representative set of $\mathbb{P}$-names for members of $\mathbb{Q}$. So when you think of this poset the above definition is what you should think about, but that's not what is really happening.

Second, as we have defined it $\mathbb{P} * \dot{\mathbb{Q}}$ is only a preorder. We can fairly easily arrange two members of our poset, $(p, \dot{q})$, $(p, \dot{q}_0)$, such that $p \Vdash \dot{q} = \dot{q}_0$, but as names in the ground model $\dot{q} \neq \dot{q}_0$. What this shows is that $(p, \dot{q}) \leq (p, \dot{q}_0)$ and $(p, \dot{q}) \geq (p, \dot{q}_0)$, but $(p, \dot{q}) \neq (p, \dot{q}_0)$. We could say that we don't care and continue forcing anyway, or we could just identify two elements if they are below each other.

Third, we still need to check that this poset does what we want it to do. To this end I will state the appropriate theorems.

Theorem 2.2. *If G is $\mathbb{P}$-generic over V and H is $i_G(\dot{\mathbb{Q}})$-generic over $V[G]$, then $J = \{(p, \dot{q}) : p \in G$ and $i_G(\dot{q}) \in H\}$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V.*

Theorem 2.3. *If J is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V, then $G = \{p \in \mathbb{P} : \exists \dot{q}\ (p, \dot{q}) \in J\}$ is $\mathbb{P}$-generic over V and $H = \{i_G(\dot{q}) : \exists p \in \mathbb{P}\ (p, \dot{q}) \in J\}$ is $i_G(\dot{\mathbb{Q}})$-generic over $V[G]$.*

So forcing with $\mathbb{P} * \dot{\mathbb{Q}}$ is really the same as forcing with $\mathbb{P}$ and then forcing with $\mathbb{Q}$. As expected we have the following example of a projection.

Example 2.4. Let $\pi : \mathbb{P} * \dot{\mathbb{Q}} \to \mathbb{P}$ be given by $\pi(p, \dot{q}) = p$. As expected π is a projection. These may seem like trivial examples but they are actually going to get more difficult and thus more useful.

The two step iteration is going to be important for us, because we can view mitchell's poset as a series of two step iterations. In fact we are going to project Mitchell on to a bunch of two step iterations, but that is for later.

3 Term Forcing

In this section, I will develop an idea that is due to Laver and developed later by Foreman and is used in a modified sort of way in Mitchell's argument. Starting with a two step iteration we make a poset out of the second coordinates. This allows us to decompose the iteration in some sense.

Definition 3.1. *Let the two step iteration, $\mathbb{P} * \dot{\mathbb{Q}}$, be given. Then let $A(\mathbb{P}, \dot{\mathbb{Q}})$ have the underlying set of all canonical $\mathbb{P}$-names for members of $\dot{\mathbb{Q}}$. And let $\dot{q}_1 \leq_{A(\mathbb{P},\dot{\mathbb{Q}})} \dot{q}_2$ iff $\Vdash_{\mathbb{P}} \dot{q}_1 \leq_{\dot{\mathbb{Q}}} \dot{q}_2$.*

Theorem 3.2. *Suppose H is $A(\mathbb{P}, \dot{\mathbb{Q}})$-generic over V and G is $\mathbb{P}$-generic over V, then $H^G = \{i_G(\dot{\tau}) : \dot{\tau} \in H\}$ is $i_G(\dot{\mathbb{Q}})$-generic over $V[G]$.*

If G and H are mutually generic, then we get the following inclusion $V[G][H^G] \subseteq V[H][G]$. Which gives us a nice way of understanding the iteration, because it lives inside an extension where we can understand each of the coordinates separately.

Example 3.3. Here we have our first nontrivial example of a projection. Let $\pi : \mathbb{P} \times A(\mathbb{P}, \dot{\mathbb{Q}}) \to \mathbb{P} * \dot{\mathbb{Q}}$ be given by $\pi(p, \dot{q}) = (p, \dot{q})$! This is a projection! Note that this is nontrivial as the orderings are different. This will be important later as we seek to understand Mitchell's extension by seeing that it lives inside an extension that is a cohen bit cross a variation of term forcing.

Here I will also mention the maximum principle, because it is important for a further result.

Theorem 3.4. *(Maximum Principle) Let $\mathbb{P}$ be a poset and let $\tau_1, ..., \tau_n$ be $\mathbb{P}$-names. If $p \Vdash \exists x \varphi(x, \tau_1, ..., \tau_n)$, then there is a $\mathbb{P}$-name, σ, such that $p \Vdash \varphi(\sigma, \tau_1, ..., \tau_n)$.*

Theorem 3.5. *If $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is countably closed then $A(\mathbb{P}, \dot{\mathbb{Q}})$ is countably closed.*

We will need a similar argument next week.

4 Easton's Lemma

This is a very important lemma that crops up many places in Mitchell's argument. I think that Ernest is going to prove it before next week, so I will state it here as preliminary to the third and final talk of the series. Here I am going to assume familiarity with the concepts of *κ-chain condition* and *κ-closure.*

Theorem 4.1. *(Easton's Lemma) Assume that $\mathbb{P}$ is λ-cc and that $\mathbb{Q}$ is λ-closed, then*

1. *$\Vdash_{\mathbb{Q}} \mathbb{P}$ is λ-cc.*
2. *$\Vdash_{\mathbb{P}} \mathbb{Q}$ adds no $< \lambda$-sequences.*
3. *If $G_{\mathbb{P}}$ is V-generic for $\mathbb{P}$ and $G_{\mathbb{Q}}$ is V-generic for $\mathbb{Q}$, then $G_{\mathbb{P}} \times G_{\mathbb{Q}}$ is V-generic for $\mathbb{P} \times \mathbb{Q}$.*
4. *Forcing with $\mathbb{P} \times \mathbb{Q}$ preserves λ.*

This is going to be important in light of my above remarks. The Mitchell extension is going to live inside an extension that is a ccc bit cross a countably closed bit. ie the cohen part cross the term forcing part from above.

5 Generic Elementary Embeddings

So we have just one more bit of technology that we need before starting in on Mitchell. Generic Elementary embeddings appear in a more general context then the way that I'm stating them, but for our purposes I have toned down that generality. We call certain elementary embeddings generic when they are not definable in the ground model. And we want to know when it is possible to lift an elementary embedding to a generic extension. The following theorem precisely answers that question.

Theorem 5.1. *(Silver) Let $j : V \to N$ be an elementary embedding. Let $\mathbb{P}$ be a poset and G be $\mathbb{P}$-generic over V. Let H be $j(\mathbb{P})$-generic over N. If $j"G \subseteq H$, then there is an elementary embedding $j^+ : V[G] \to N[H]$ such that $j^+(G) = H$ and $j^+ \restriction V = j$.*

Proof. We do the only thing that seems reasonable. For a $\mathbb{P}$-name τ, we let $j^+(i_G(\tau)) = i_H(j(\tau))$. And then thing that saves us is that the pointwise image of G is contained in H. $\square$

6 Why not use the Levy Collapse?

We are now ready to start motivating mitchell's proof. Let's let κ be measurable and show that we can get a generic elementary embedding with critical point $(\kappa = \omega_2)^{V[G]}$, but that this alone is not enough! The first thing that comes to mind in trying to arrange the tree property at ω_2 is the Levy collapse. Why not just collapse all of the ordinals less than κ on to ω_1, then extend the elementary embedding to the generic extension and use the same argument as last time for κ measurable has TP? Well this proof fails for a lot of reasons, but I want to show that we can get an appropriate elementary embedding and even a branch, but this can't help us.

Proposition 6.1. *Let G be V-generic for $Coll(\omega_1, < \kappa)$. Then $V[G] \models CH$.*

I think that this is a modification of an exercise in our forcing class. So already from last time we know that living in $V[G]$ there is a special ω_2-A-tree. So something must go wrong in the elementary embedding proof, but it is instructive to see exactly what goes wrong.

Let $j : V \to N$ witness κ measurable. Then by elementarity $j(Coll(\omega_1, < \kappa)) = Coll(\omega_1, < j(\kappa))$. So in fact, given G $Coll(\omega_1, < \kappa)$-generic over V, we can find $H \subseteq G$ which is $Coll(\omega_1, < j(\kappa))$-generic over N.

Now as the members of $Coll(\omega_1, < \kappa)$ are small and $crit j = \kappa$, for $p \in Coll(\omega_1, < \kappa)$, $j(p) = p$.. So we have $j"G \subseteq H$ and we can apply Silver's theorem from above.

So there is $j : V[G] \to N[H]$ with critical point ω_2 and all of the same things hold as be for. So $N[H] \models$ "there is a branch through T." Essential in the proof of measurable κ has the tree property was that $N \subseteq V$, but note that this is no longer the case. $V[G] \nsubseteq N[H]$! In particular, in $N[H]$ we have added a surjection from ω_1 onto κ. So κ is just some random ordinal between ω_1 and ω_2 in $N[H]$ and it is clear that this is what added the branch. The extended collapse added a branch that was not in $V[G]$.

Furthermore, we can't save the situation by adding reals after the fact with cohen forcing, as ccc forcing couldn't destroy the specialness of our ω_2-A-tree.

So the moral of the story is that when we finally get to the end of mitchell's argument, we want to show that the extra bit forcing in j of the mitchell poset did NOT add a branch through our tree. Ok so next time I will be prepared to give the definition of the Mitchell poset and sketch the argument of why it works.