What is forcing anyway?

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1 Introduction

The purpose of this talk is to answer the question, "What is forcing?" As I said in the abstract, the short answer is that forcing is a method to show that certain statements are indepedent of the axioms of set theory. So let's dig in to the terminolgy of the short answer and then we will slowly work back to the actual construction of forcing.

Let's start off very basically. Since you are all mathematicians, I think that you already understand many of the concepts that I'm going to describe today, but what I am going to do is give names to things that you haven't thought about naming and rename some things that you already know.

Let's start with a mathematical statement. A mathematical statement is exactly what you would think it is, it is a statement in mathematical language. So for example your favorite theorem is a statement of mathematics.

We understand in mathematics that certain statements follow from others, right? So for example, if I am in the middle of a proof in analysis and I say that a certain limit is zero and give a reason. We say that the fact that the limit is zero follows from my reason. This is all dependent on some background context right. If I change the context then the limit being zero may no longer follow from my reason. So for example if you are taking Giovanni's topology class you are learning that certain statements in metric spaces no longer hold in topological spaces. This is because of this kind of 'change of context'

Ok so all of that was appropriately vague. One of the many things that a logicians have done for us is they have formalized the idea of when a certain statement follows from another statement. Let me sketch some of these ideas.

To do this I am going to take an approach that a proof theorist would disapprove of, because I am going to use a theorem as a definition. But this should be sufficient to give the idea.

1.1 Models

Informal Definition 1. *A model is an mathematical object where we can do math, ie prove theorems and just generally do mathy things*

Some examples. The real numbers with all of the usual operations are a model, since the real numbers are a place where we can prove theorems of calculus, take sups and infs etc.

The symmetic group on n elements is a model. It is fairly limited because the only things that I can do with it is compose permutations of my n objects, but it is still a model.

So we have to examples of models some can do a lot and some can't do that much. Now I want to say what it means to say that some mathematical statement follows from some other statement.

Fact. *In a metric space every compact set is closed and bounded.*

What I want to do is turn this in to a way a logician sees things.

Translation: In every model that is a metric space, every compact set is closed and bounded.

This is a very subtle change, but it highlights an important feature of our discussion. Lets replace 'metric space' with 'set of axioms' and our theorem with a statement φ. (This isn't too surprising right? We have a set of axioms for a metric space.)

Informal Definition 2. *φ follows from a set of axioms if and only if in every model of that set of axioms φ is true.*

Ok so now we are ready to talk about the notion of independence.

Informal Definition 3. *A statement φ is independent of a set of axioms if neither φ or $\neg\varphi$ follows from that set of axioms*

The idea of this definition is to capture the idea that we cannot prove φ from our axioms. So let us characterize independence interms of what we have said about models.

Fact. *A statement φ is independent of a set of axioms if and only if there is a model of the set of axioms where φ is true and a model of the axioms in which φ is false.*

So really we run in to this notion of independence all the time. For example think about the statement "Every closed and bounded set is compact." This statement is indepedent of the axioms of metric spaces, because we have a metric space where it is true, the reals, and a metric space where it is false, like L_2.

Now to move on to talk about how we get things to be independent of the axioms of set theory. To do this I need to talk about what a model of set theory is.

1.2 Models of Set theory

Caveat: For the purpose of this talk, I am going to assume that the axioms of Set Theory are consistent. In particular, we are assuming that there are models of set theory out there.

Let me start by saying that the axioms of set theory are really powerful. They were designed to provide a foundation for all of mathematics. What this means is that in a model of set theory, I can prove all of the theorems of classical mathematics.

Now we have to take another step towards metamathematics and ask the question, "What would happen if there were different models of set theory?" If I can have different models of set theory, then can I have different views on classical mathematics? How much can classical mathematics change if I go from one model of set theory to another?

Just like if we change the metric space we get different things to be true. Changing models of set theory makes different things true. Except there is a glitch. When talking about models of set theory, we are talking about metamathematics.

By changing models of set theory, I can change what is true about classical mathematics to a certain extent.

So now we rephrase the short answer to the question "What is forcing?"

Recall the short answer "Forcing is a method so show that certain statements are indepedent of the axioms of set theory."

I can now make this slightly more precise using the notion of indepedence that I developed above. Forcing is a technique for constructing models of set theory in which different things are true. More specifically, say that I want to prove that a statement φ is indepedent of the axioms of set theory. If through the method of forcing I can construct a model of set theory in which φ holds and another one in which φ is false, then I will have shown that φ is indepedent of the axioms of set theory.

2 A survey of big ideas in forcing

Before I go on to the specific construction of forcing. I want to talk briefly about what forcing can do and what forcing can't do.

2.1 What forcing can do

Here is a small list of classical mathematical statements that are indepedent of the set theoretic axioms.

Theorem 1. *Over the axioms of set theory without the axiom of choice, the existence of a non-lebesgue measurable set of reals is independent.*

Recall to show that some thing is indepedent we need to show that there is a model of set theory where we have a non measurable set and one where we don't. The model where we have a non measurable set is obvious it is just the standard model of ZFC. In the standard model of ZFC, we have the usual version of the real numbers and using the axiom of choice we can construst a non measurable set. (I said that we could do all of math in set theory right?)

For a model in which all sets of reals are lebesegue measurable we need a forcing argument together with some other set theoretic machinery. The model that we construct has a very nice version of the real numbers. All sets of reals are lebesgue measurable, have the baire property and the perfect set property.

Theorem 2. *The Whitehead problem is indepedent of the axioms of set theory*

This was a seminal result in the theory of forcing. It is a question in homolgical algebra that algebraists were interested in and it turns out that it is niether provable nor disprovable from the axioms of ZFC. I don't have time to develop this, but it is cool and you should look it up.

Here is list of notable independent statements

1. The Axiom of Choice is independent of ZF
2. Suslin's problem on linearly ordered sets
3. Kaplansky's Conjecture for Banach Algebra's
4. The Borel Conjecture
5. Stronger versions of Fubini's theorem

2.2 What forcing can't do

There are limitations to what we can prove independent with forcing. I briefly want to say something about the notion of absoluteness in set theory. In logic, we have a measure of the logical complexity of statements. One can show that if a statement φ is of a relatively low complexity, then it's truth value cannot be changed by passing to a different model of set theory.

In particular, we cannot use forcing to construct models of set theory in which the truth value of these statements is different.

Here is a list of statements that are absolute (their truth value cannot change) for models of ZFC:

1. The Riemann Hypothesis
2. P equals nP
3. Many of the Millenium problems

3 The Forcing Construction

In the following discussion, I am going to omit alot of details, but I am going to continue to try to continue to give the idea. If you are worried about issues of logical justification then you are going to have to study forcing more indepth.

Let M be a transitive model of set theory. I am going to continue to be vague about what this is. If you know some model theory then you can think of it as a standard model theoretic object.

We are going to define a way of adding an object to our model. The method of adding is called forcing. You could think of this as analgous to field extensions. Starting with a field we can add a root of a certain polynomial and get a new field. Notice that the properties of our field have changed. The base field doesn't have a root of our polynomial and the extension does.

We are going to add an object, G to our model of set theory to get a model $M[G]$, in such a way that $M[G]$ is a model of set theory with different properties.

Remark. There is a large step up in complexity from field extensions to models of set theory.

From last week we have the definition of a poset,

Definition. *A partially ordered set $\mathbb{P}$ is a set P together with a binary relation on P, $<_P$, which is irreflexive and transitive.*

I will routinely right $\mathbb{P}$ for both the poset and the underlying set of the poset. We also have the convention that stronger conditions go down in the poset.

Definition. *A filter F on a poset $\mathbb{P}$ is a subset of $\mathbb{P}$ such that*

1. $F \neq \emptyset$
2. $\forall p, q \in \mathbb{P}$, $\exists r \in \mathbb{P}$ *such that* $r \leq p, q$
3. $\forall p, q \in \mathbb{P}$, *if* $p \leq q$ *and* $p \in F$ *then* $q \in F$

Definition. $D \subseteq \mathbb{P}$ *is dense if and only if* $\forall p \in \mathbb{P}$, $\exists q \in D$ *such that* $q \leq p$

This exemplifies the idea that our posets go down.

Definition. *(Genericity) Let M be a set with $\mathbb{P} \in M$. A filter G on $\mathbb{P}$ is M-generic if $\forall D \in M$ if D is dense in $\mathbb{P}$, then $G \cap D \neq \emptyset$*

In the above definition M is going to be our model of set theory. For a given $\mathbb{P}$ and M as above, generic objects need not exist. Part of making forcing precise is working in a setting where generic objects exist. In fact they exist if M is countable.

Theorem 3 (The Generic Model Theorem)**.** *Let M be a transitive model of ZFC. Let $\mathbb{P} \in M$ be a poset. If $G \subseteq \mathbb{P}$ is a generic filter then there is a transitive model $M[G]$, called a generic extension, such that*

1. $M[G]$ *is a model of ZFC*
2. $M \subseteq M[G]$ *and* $G \in M[G]$
3. M *and* $M[G]$ *have the same ordinals*
4. *if N is a transitive model of ZF such that $M \subseteq N$ and $G \in N$, then $M[G] \subseteq N$.*

This has a very complex proof which I will omit here. If you are interested, I recommend "Set Theory" by Kenneth Kunen as a reference.

Here is a super sketch of how things go. Each member of $M[G]$ has a name that lives in M. Each name requires a generic object to be realized/interpreted. The idea is that through names the ground model M has some limited control over what happens in the extension. The model M has access to $M[G]$ through the forcing relation $\Vdash$.

Definition. *We say that a condition, $p \Vdash^M \varphi$ if and only if for every generic G, an M-generic filter, $M[G] \models \varphi$.*

This definition is external to M. In fact so far we have defined forcing using M all over the place. To make the theory of forcing precise one shows that the external version is equivalent to a version of forcing defined working inside M! This internal notion is the one that gives M some working knowledge of the extension $M[G]$.

4 Set Theoretic Examples

So far we have taken a very short and nontechnical route to a development of forcing. However, we do have all of the tools to actually do some forcing arguments. The first non trivial forcing is called cohen forcing. The idea is to add a subset of ω that is not in the ground model M.

4.1 Cohen Forcing

A subset of ω can be represented as a function from $f : \omega \to 2$. So we will define a poset where the generic object is a function from ω to 2 which is not in the ground model.

Let $\mathbb{P} = \{f : f : n \to 2\}$ under the ordering of reverse containment. So $f, g \in \mathbb{P}$, $f \leq g$ if and only if $f \supseteq g$. Recall that n is just a set of n elements namely zero through $n - 1$.

Let G be an M-generic filter on $\mathbb{P}$.

Claim. *Letting $g = \bigcup G$. $g : \omega \to 2$ and $g \notin M$.*

To show the claim we need to work with the definition of generic filter. First we show that g is a function. Let $f, f' \in G$. Then for all $n \in \mathrm{dom}(f) \cap \mathrm{dom}(f')$ we must have $f(n) = f'(n)$, since $f \cup f' \in \mathbb{P}$.

To see that $\mathrm{dom}(g) = \omega$, we need to define some dense sets. Define $D_n = \{f \in P : n \in \mathrm{dom}(f)\}$. Clearly, D_n is dense for all $n \in \omega$. So for all n, $G \cap D_n \neq \emptyset$.

So for all n, $n \in dom(g)$.

Lastly we show that $g \notin M$. Fix a function $h : \omega \to 2$ with $f \in M$. Define $D_h = \{f \in \mathbb{P} : \exists n \in \omega f(n) \neq h(n)\}$. D_h is dense since given a function in $\mathbb{P}$, I can find an natural number not in it's domain and then extend my function to differ with h on that natural number.

So for all $h \in M$, $D_h \cap G \neq \emptyset$. This implies that for all $h \in M$ $g \neq h$. So we're done.

4.2 Collapsing Cardinals

In general when we pass to a generic extension, we might collapse cardinals. What this means is that in M, κ might have been a cardinal, ie it what the least ordinal such that there is no surjection from an ordinal less than κ on to κ.

However, a generic object can add a surjection from an ordinal less than κ on to κ. Let $\mathbb{P} = \{f : f : \omega \to \omega_1, |\mathrm{dom}(f)| < \omega\}$ ordered under reverse inclusion.

As usual if G is a generic filter then $g = \bigcup G$ is a function with domain ω. However we want to see that g is a surjection from ω onto ω_1. For $\alpha < \omega_1$, define $D_\alpha = \{f : \alpha \in \mathrm{rng}(f)\}$. Then D_α is dense, since for every condition in our poset either α is already in the range or we can add it by mapping the least n not in the domain of our condition to α.

So we see that in $M[G]$ something awful has happened ω_1^M is no longer a cardinal. It is relatively easy to show that the old ω_1 is now a countable ordinal in the generic extension.

4.3 The Continuum Hypothesis

It is not too far from the previous forcing example to forcing the failure of the continuum hypothesis. First, let me say a few things about what the continuum hypothesis is.

Last week, Will talked about ordinals as characterizing the well ordered order types. He also mentioned briefly the idea that there are special ordinals called cardinals. These are the least ordinals where there is a jump in size as measured by whether or not there is a one to one function from a given ordinal to another.

Fact. *For each ordinal α there is a cardinal that we call $\aleph_\alpha$.*

So just like there is no set of all sets, there no set of all ordinals and also no set of all cardinals.

We are all, I hope, acquainted with the fact that $|\mathcal{P}(A)| > |A|$. So in particular $|\mathcal{P}(\omega)| > \aleph_0 = |\omega|$. The continuum hypothesis proposes an answer to the question, "Where does $|\mathcal{P}(\omega)|$ fall on our list of cardinals above?" The continuum hyopthesis says that $|\mathcal{P}(\omega)| = \aleph_1 = |\omega_1|$. That the size of the power set of ω is the least infinite cardinal greater than ω.

Notice that $\mathcal{P}(\omega)$ is the set of subsets of omega. So we want to do a modified version of Cohen forcing from the last example and we want to do it ω_2 times!

Let $\mathbb{P} = \{f : f : \omega \times \omega_2 \to 2, |\mathrm{dom}(f)| < \omega\}$ ordered as before under reverse inclusion.

As before we get that if G is an M-generic filter, then $g = \bigcup G$ is a function from $\omega \times \omega_2 \to 2$. And more over we can show that if we define $g_\alpha(n) = g(\alpha, n)$, then $g_\alpha \notin M$ for all $\alpha < \omega_2$! So in $M[G]$, we found that there are ω_2 many subsets of ω all of which are not in M. So we must have $|\mathcal{P}(\omega)| \geq \aleph_2$!

This only gives the failure of the continuum hypothesis if we can show that $\mathbb{P}$ does not collapse cardinals. This fact takes some more forcing theory, we show that our poset has the so called countable chain condition and that such posets do not collapse cardinals.