THE TREE PROPERTY BELOW $\aleph_{\omega\cdot 2}$

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In this paper we extend the length of the longest interval of regular cardinals which can consistently have the tree property. Neeman [5] has shown that starting from ω supercompact cardinals one can force to obtain the tree property at every regular cardinal in the interval $[\aleph_2, \aleph_{\omega+1}]$. In this paper we prove the following theorem.

Theorem 0.1. *Assuming there is an $\omega + \omega$-sequence of supercompact cardinals, then there is a generic extension in which every regular cardinal in the interval $[\aleph_2, \aleph_{\omega\cdot 2})$ has the tree property.*

An important remark is that in the model for the main theorem $\aleph_\omega$ is not strong limit, in particular $2^{\omega_1} = \aleph_{\omega+2}$. In doing so we avoid a difficult question of Woodin's which asks whether it is consistent to have the failure of both SCH and weak square at $\aleph_\omega$. The main theorem of this paper should be compared with a theorem in [9] where a similar result is obtained for successive failures of weak square. The result makes use of the poset and a some of the main lemmas from Neeman's paper [5]. The reader is advised to have a copy of it on hand. Throughout the paper we have attempted to keep the notation very close to Neeman's. One notable convention is the use of τ-closed to mean every decreasing sequence of length τ has a lower bound.

The paper is organized into sections based on these topics: preliminaries, definition of the main forcing, cardinal structure, the tree property below $\aleph_\omega$, the tree property at $\aleph_{\omega+1}$, the tree property at $\aleph_{\omega+2}$, and the tree property above $\aleph_{\omega+2}$.

1. Preliminaries

We list some essential lemmas which will be used in the proof below. The first two are preservation lemmas due to the author.

Lemma 1.1 ([8])**.** *If $\mathbb{P}\times\mathbb{P}$ is κ-cc, then forcing with $\mathbb{P}$ cannot add a branch through a tree of height κ.*

Lemma 1.2 ([7])**.** *If $\mathbb{P}$ is τ^+-cc, $\mathbb{Q}$ is τ-closed and $2^\tau \geq \eta$, then forcing with $\mathbb{Q}$ over $V[\mathbb{P}]$ cannot add a branch through an η-tree.*

We will also make use of a lemma of Abraham [1], which allows us to preserve the chain condition of certain posets. We take our statement of the lemma from Cummings and Foreman [2].

Lemma 1.3. *Let $\tau < \kappa$ and assume $V \models$ "τ is regular and κ is inaccessible." Let $\mathbb{P} = \mathrm{Add}(\tau, \eta)_V$. If $W \supseteq V$ is a model of set theory where*

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(1) *τ and κ are cardinals in W and*
(2) *every set of ordinals in W of size less than κ is covered by a set of ordinals in V of size less than κ,*

then $\mathbb{P}$ has the κ-Knaster property in W in particular its square is κ-cc.

Remark 1.4. *The above lemma is often stronger than what we need. We will often need to see that the square of a Cohen poset retains some chain condition property in an outer model. However a Cohen poset is isomorphic to its square. So for an application of Lemma 1.1, it is enough to show that the chain condition of the poset is preserved in an outer model (rather than Knasterness).*

We also need Easton's Lemma [3].

Lemma 1.5. *If $\mathbb{P}$ is τ-cc and $\mathbb{R}$ is $< \tau$-closed, then it is forced by $\mathbb{P}$ that $\mathbb{R}$ is $< \tau$-distributive.*

2. The main forcing

In this section we give the definition of the main forcing. The basic idea is add to Neeman's construction a poset which will force the tree property at each $\aleph_{\omega+n}$ for $n > 1$. Neeman's argument requires a specific choice of ω_1 to obtain the tree property and we will need to repeat this argument in the presence of this additional poset.

Working in V fix $\langle \kappa_n \mid n \geq 2 \rangle$ an increasing sequence of supercompact cardinals and $\langle \lambda_n \mid n \geq 2 \rangle$ another increasing sequence of supercompact cardinals greater than $\nu =_{def} \sup_{n \geq 2} \kappa_n$. Let $\sup_{n \geq 2} \lambda_n = \iota$.

- $\mathbb{I}$ is Neeman's forcing without $\mathrm{Coll}(\omega, \mu)$.
- $\mathbb{A}_1^+ = \mathrm{Add}(\kappa_1, \lambda_2 \setminus \kappa_3)^V$.
- For $n \geq 1$ we define $\mathbb{A}_{\omega+n} = \mathrm{Add}(\lambda_n, \lambda_{n+2})_V$ where for ease of notation we set $\lambda_1 = \nu^+$.
- $\mathbb{U}$ is a reverse Easton style preparation over initial segments of $\mathbb{A}_1 \times \mathbb{A}_1^+ \times \prod_{n \geq 1} \mathbb{A}_{\omega+n}$, that is $f \in \mathbb{U}_\omega$ are partial functions with reverse Easton support, which at nontrivial coordinates γ return $\mathbb{A}_1 \times \mathbb{A}_1^+ \times \prod_{0<k<n+1} \mathbb{A}_{\omega+k} \upharpoonright \gamma * \mathbb{U} \upharpoonright \gamma$-names for elements of a γ-directed closed poset. There is no harm in calling this poset $\mathbb{U}$ since we can view it as extending Neeman's poset $\mathbb{U}$. Note however that coordinates in $\mathbb{U}$ above ν are only generic over $\mathbb{A}_1$ from Neeman's forcing. We also have a poset with the same underlying set as $\mathbb{U}$ with a different ordering called $\mathbb{B}$. We also think of $\mathbb{B}$ as extending Neeman's poset denoted by the same letter. In the cases where we need to refer to the coordinates of $\mathbb{B}$ above ν we will write $\mathbb{B}^+$. The full definition of this new poset is easily adapted from Definition 4.1 of [5].
- For $n < \omega$ we define $\mathbb{C}_{\omega+n}$ to be a Mitchell style collapse to bring down λ_{n+2}. Conditions in $\mathbb{C}_{\omega+n}$ are partial functions of size less than λ_{n+1} and at nontrivial coordinates γ they return $\mathbb{A}_1 \times \mathbb{A}_1^+ \times \prod_{0<k<n+1} \mathbb{A}_{\omega+k} \upharpoonright \gamma * \mathbb{U} \upharpoonright \lambda_{n+1}$-names for conditions in $\mathrm{Add}(\lambda_{n+1}, 1)$ of the generic extension. We define $\mathbb{C}$ to be a product of Neeman's $\mathbb{C}$ with the full support product of our posets $\mathbb{C}_{\omega+n}$. In cases where we want to refer to only the coordinates of $\mathbb{C}$ above ν we will write $\mathbb{C}^+$. The full definition of $\mathbb{C}_{\omega+n}$ is easily adapted from Definition of 4.13 of [5].

Each of our posets of terms has *enrichments* as in Definitions 4.3 and 4.14 of Neeman's paper. We will also make use of Neeman's terminology for posets and their generic objects.

- We let $\mathbb{A}_0 = \mathrm{Add}(\omega, \kappa_2)$ and $\mathbb{A}_n = \mathrm{Add}(\kappa_n, \kappa_{n+2})$ for $n \geq 2$. Recall that κ_1 is μ^+ where $\mu \in \mathrm{Index}$ is selected to be ω_1 by the poset. For $n < \omega$ we have A_n which is $\mathbb{A}_n$-generic over V. Further we have objects $A_{[n,m)}$ for $n < m \leq \omega$ with the obvious interpretation and similar notation for closed intervals. We use similar notation for $\mathbb{A}_{\omega+n}$.
- We let $\mathbb{U}_0 = \mathbb{U} \restriction \kappa_2$ and $\mathbb{U}_n = \mathbb{U} \restriction [\kappa_{n+1}, \kappa_{n+2})$ for $n \geq 1$. We also have interval notation for $\mathbb{U}$, $\mathbb{U}_{[0,n)} = \mathbb{U} \restriction \kappa_{n+2}$. Similar to our conventions for $\mathbb{A}$, we have U_n is generic for $\mathbb{U}_n$ and $U_{[0,n)}$ is generic for $\mathbb{U}_{[0,n)}$. Note that $\mathbb{U}$ is not a product of its coordinates. Also note that the $\mathbb{U}$ posets only make sense in generic extensions of V. In particular $\mathbb{U}_0$ is a poset defined in $V[A_0]$ and $\mathbb{U}_n$ for $n \geq 1$ is a poset defined in $V[A_{[0,n]} * U_{[0,n)}]$. We use similar notation for $\mathbb{B}$, but note that $\mathbb{B}$ is a poset in V and hence so are its restrictions. We use similar notation for coordinates of $\mathbb{U}$ above ν.
- Recall that $\mathbb{C}$ is defined to be a full support product of posets $\mathbb{C}_n$ and $\mathbb{C}_{\omega+n}$. We also make use of interval notation for $\mathbb{C}$, so for example $\mathbb{C}_{[n,\omega)}$ is $\mathbb{C} \restriction [\kappa_{n+1}, \nu)$. Again we note the presence of κ_1, but this time in the definition of $\mathbb{C}_0$.

For generics for the new forcing posets we have tried to retain conventions above. We use the same letters but often add a superscript $+$ to distinguish.

The final model is obtained by the following procedure:

(1) Force with $\mathbb{I} \times \mathbb{A}_1^+ \times \prod_{n \geq 1} \mathbb{A}_{\omega+n}$ obtaining an extension $V[A][U][S][A_1^+][A^+]$.
(2) Over this model force with the enriched poset $\mathbb{U} \restriction [\nu^+, \iota)^{+A_1 \times A_1^+ \times A^+}$ to get a generic U^+.
(3) Over $V[A][U][S][A_1^+][A^+][U^+]$ force with the enriched poset
$$(\prod_{n<\omega} \mathbb{C}_{\omega+n})^{+(A_1 \times A_1^+ \times A^+) * U^+}$$
to obtain a generic S^+.
(4) Finally force with $\mathrm{Coll}(\omega, \mu)$ to obtain a generic e.

For ease of notation we denote the final model by W throughout the paper.

3. Cardinal Structure

In this section we prove the following lemma.

Lemma 3.1. *In W we have*

(1) $\mu = \aleph_1$,
(2) *for all* $n \geq 2$, $\kappa_n = \aleph_n$ *and* $\lambda_n = \aleph_{\omega+n}$,
(3) $2^\omega = \omega_2$,
(4) $2^{\omega_1} = \aleph_{\omega+2}$ *and*
(5) *for all* $n \geq 1$, $2^{\aleph_{\omega+n}} = \aleph_{\omega+n+2}$.

We start by proving that the forcing can be split, roughly speaking, into a part above ν and a part below ν. To do so we need a slight strengthening some lemmas from Neeman's section 4. In addition to Claim 4.19, we have the following.

Claim 3.2. *$\mathbb{B}_n$ is κ_{n+2}-cc in V, in particular it is κ_{n+2}-cc in $V_{n+2} = V[A_{[n+2,\omega)} \times B \restriction [\kappa_{n+2}, \nu) \times C \restriction [\kappa_{n+2}, \nu^+)$*

Proof. This is immediate from an easy Δ-system argument. □

Working through Neeman's Claims 4.18 through 4.21 with this extra claim, we have the following strengthening of Claim 4.26.

Claim 3.3. *For every $n < \omega$, the forcing $\mathbb{I} \times \mathrm{Coll}(\omega, \mu)$ is the projection of a product κ_{n+1}-cc with $< \kappa_{n+1}$-closed forcing and moreover the generic for the κ_{n+1}-cc forcing is determined by $\mathbb{I} \times \mathrm{Coll}(\omega, \mu)$.*

Proof. This is immediate. We give the proof for $n \geq 1$. First we see that the model $V_{n+1}[A_{[0,n]}][U \restriction \kappa_{n+1}][S \restriction \kappa_{n+1}][e]$ generates a generic for $\mathbb{I} \times \mathrm{Coll}(\omega, \mu)$. Second V_{n+1} is a $< \kappa_{n+1}$-closed extension of V and the forcing to add $A_{[0,n]}, U \restriction \kappa_{n+1}, S \restriction \kappa_{n+1}$ and e is κ_{n+1}-cc. Finally the forcing to add the κ_{n+1}-cc parts is determined by a generic for $\mathbb{I} \times \mathrm{Coll}(\omega, \mu)$. □

Remark 3.4. *None of the above claims rely on the fact that $\mathbb{A}_1$ adds only κ_3 subsets of κ_1. In particular we can obtain the same conclusion in models which include A_1^+. For $n \geq 1$, A_1^+ takes part in the κ_{n+1}-cc forcing and for $n = 0$ it takes part in the κ_1-closed forcing.*

Remark 3.5. *A similar argument to the above claim establishes a version of Claim 3.3 for the forcing $\mathbb{A} \times \prod_{n<\omega} \mathbb{B}_n \times \mathbb{C}$.*

Corollary 3.6. *For all $n < \omega$, every set of size less than κ_{n+1} in $V[A][U][S][e][A_1^+]$ is covered by a set of size less than κ_{n+1} in V.*

Claim 3.7. *Every $< \nu^+$ sequence from W is in the model $V[A][U][S][A_1^+][e]$.*

Proof. It is enough to show that every sequence of length less than ν is in the desired model. Note that the posets $\prod_{n<\omega} \mathbb{B}_{\omega+n}$ and $\mathbb{C}^+$ are in V and are $< \nu^+$-directed closed by standard arguments. So by forcing over W to refine U^+ and S^+ to generics B^* and C^+ for these two posets, we see that every $< \kappa_{n+1}$-sequence from W is in a κ_{n+1}-cc extension of V which is a submodel of $V[A][U][S][e][A_1^+]$. The claim follows. □

It follows that each κ_n is preserved in W. Moreover since $V[A][U][S][e]$ is an inner model with the same cardinals below ν, we have $\kappa_n = \aleph_n$ for all $n \geq 1$ in W. Further by standard arguments we have that ν^+ is preserved.

Corollary 3.8. *Every set of size ν in W is covered by a set of size ν in V.*

Proof. Suppose X is a set of size ν in the final model. Write X as an increasing union of sets X_n of size κ_n. By the previous claim X belongs to $V[A][U][S][A_1^+][e]$. Using Claim 3.3 for each n there is a set Y_n in V of size κ_n such that $Y_n \supseteq X_n$. The sequence $\langle Y_n \mid n < \omega \rangle$ need not be in V, but it is in $V[A_0][e]$ by another application of Claim 3.3. Using the κ_1-cc of $\mathbb{A}_0 \times \mathrm{Coll}(\omega, \mu)$, we can find (in V) a sequence of sets Z_n where for each n, $|Z_n| = \kappa_n$ and $Z_n \supseteq Y_n$. It follows that $\bigcup_{n<\omega} Z_n \supseteq X$ as required. □

Next we show that each λ_n for $n \geq 2$ is preserved.

Claim 3.9. *$\mathbb{I} \times \mathrm{Coll}(\omega, \mu)$ has size ν^+ in V.*

This is immediate from GCH in V. To prove that each λ_n is preserved we look at W without $\mathbb{I}$, $\mathrm{Coll}(\omega,\mu)$ and $\mathbb{A}_1^+$.

Claim 3.10. *For all $n<\omega$ λ_{n+2} is preserved in the model*

$$V[A][U][S][e][A_1^+][A^+][B^*][C^+]$$

where B^* is generic for $\prod_{n<\omega}\mathbb{B}_{\omega+n}$ and C^+ is generic for $\mathbb{C}^+$.

Proof. The proof is similar to the proof of Claim 3.3, but easier. The forcing in question can be written as a product of λ_n-cc and $<\lambda_n$-closed with no extra forcing required. By Claim 3.9 the generic for $\mathbb{I}\times\mathrm{Coll}(\omega,\mu)$ can be incorporated into the forcing with chain condition with no difficulty. The only part which is not immediate from similarities with Neeman's argument is a relative of Claim 3.2 but for the forcing $\mathbb{B}\restriction[\lambda_{n+1},\lambda_{n+2})$, namely that this forcing is λ_{n+2}-cc. $\square$

Corollary 3.11. *For all $n<\omega$, λ_{n+2} is preserved in W.*

This is immediate, since the model from the previous claim contains the final model. Using the corollary and Claim 3.7, it follows that $2^{\omega_1}=\lambda_2$ in W. It remains to show that $\lambda_{n+2}=\aleph_{\omega+n+2}$ for all $n<\omega$.

Claim 3.12. *For all $n<\omega$, $\lambda_{n+2}=\aleph_{\omega+n+2}$ in W.*

This is the main point of the design of the poset $\mathbb{C}_{\omega+n}$ for $n<\omega$. It is not hard to see that for $n=0$ and many $\alpha<\lambda_2$, the poset $\mathbb{C}_\omega^{+A_1\times A_1^+}$ induces a generic for the the poset $\mathrm{Add}(\nu^+,1)_{V[A_1\times A_1^+]}$ and hence collapses α. Similar arguments work for $n>0$. This finishes the proof of Lemma 3.1.

4. The tree property at $\aleph_n$ with $2\le n<\omega$

In this section we prove the following lemma.

Lemma 4.1. *The tree property holds at $\aleph_n$ with $2\le n\le\omega$ in W.*

By Claim 3.7, it is enough to show the following.

Claim 4.2. *The tree property holds at $\aleph_n$ with $2\le n<\omega$ in the model*

$$V[A][U][S][e][A_1^+].$$

We repeat Neeman's proof with the additional forcing to add A_1^+. For this we need lift the embeddings from Neeman's Section 4 to the additional extension by $\mathbb{A}_1^+$ and show that the forcing which adds the new embedding cannot add a branch through the appropriate kind of tree. Lifting to the additional forcing $\mathbb{A}_1^+$ poses no problem.

For embeddings with critical point κ_n for $n\ge 3$, we can do the quotient forcing $\pi(\mathbb{A}_1^+)/\mathbb{A}_1^+$ last since its chain condition is bounded below the critical point. So even though the height of the tree is collapsed, the $\kappa_2=\aleph_2$-cc of the quotient and its powers is preserved. Note that the quotient is isomorphic to $\mathrm{Add}(\kappa_1,\theta)_V$ for some θ.

Claim 4.3. *The forcing to add an embedding with domain $V[A][U][S][e]$ and critical point κ_{n+2} for some $n\ge 1$ preserves the κ_2-cc of the posets $\mathrm{Add}(\kappa_1,\theta)_V$.*

For $n = 1$ the conclusion follows directly from the argument at the end of Lemma 4.29 with $n = 1$. For $n > 1$ we use Claim 4.33 to see that the forcing to add the embedding retains the covering property for sets of size $< \kappa_1$ over V (no sets of size less than κ_1 are added). The claim follows from Lemma 1.3.

This observation provides all that we need to get the tree property at each $\aleph_{n+2}$ for $n \geq 1$. For definiteness we fix an elementary embedding π with critical point κ_{n+2} witnessing a large amount of supercompactness and an $\aleph_{n+2}$-tree T in $V[A][U][S][e][A_1^+]$. The argument requires us to repeat Neeman's proof of Lemma 4.29 with the extra poset $\mathbb{A}_1^+$. In particular we must see that the poset $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3 \times \mathrm{Add}(\kappa_n, \pi(\kappa_{n+2}))^V \times \mathrm{Add}(\kappa_{n+1}, \pi(\kappa_{n+3}))^V \times \pi(\mathbb{A}_1^+)/\mathbb{A}_1^+$ does not add a branch to T over $V[A][U][S][e][A_1^+]$. It is enough to see that the property of not adding branches is preserved when passing from $V[A][U][S][e]$ to $V[A][U][S][e][A_1^+]$. We prove a sequence of short claims.

Claim 4.4. *The square of* $\mathrm{Add}(\kappa_{n+1}, \pi(\kappa_{n+3}))$ *is* κ_{n+2}*-cc in* $V[A][U][S][e][A_1^+]$ *and does not add any* $< \kappa_{n+1}$*-sequences over this model.*

Proof. The square is κ_{n+2}-cc by Lemma 1.3. The covering property we need follows from Corollary 3.6. Further the addition of the generic $\hat{A}_{n+1}$ for $\mathrm{Add}(\kappa_{n+1}, \pi(\kappa_{n+3}))$ does not add any less than κ_{n+1}-sequences by Remark 3.4 and Lemma 1.5. □

Claim 4.5. $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$ *satisfies the hypotheses of Lemma 1.2 in the model*

$$V[A][U][S][e][A_1^+][\hat{A}_{n+1}]$$

This is straightforward using Claim 4.3 and the distributivity of $\mathrm{Add}(\kappa_{n+1}, \pi(\kappa_{n+3}))$ from the previous claim.

Claim 4.6. *The square of* $\mathrm{Add}(\kappa_n, \pi(\kappa_{n+2})$ *is* κ_{n+1}*-cc in the model*

$$V[A][U][S][e][A_1^+][\hat{A}_{n+1}][G_1 \times G_2 \times G_3]$$

where $G_1 \times G_2 \times G_3$ *is generic for* $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$.

This claim follows from Lemma 1.3. We have covering for sets of size less than κ_n, because we have argued that all $< \kappa_n$-sequences are in fact elements of $V[A][U][S][e][A_1^+]$ and so we can apply Corollary 3.6.

Lastly we need the following claim.

Claim 4.7. *The square of* $\pi(\mathbb{A}_1^+)/\mathbb{A}_1^+$ *is* κ_2*-cc in the model*

$$V[A][U][S][e][A_1^+][\hat{A}_{n+1}][G_1 \times G_2 \times G_3][\hat{A}_n].$$

Note that $\pi(\mathbb{A}_1^+)/\mathbb{A}_1^+$ is isomorphic to $\mathrm{Add}(\kappa_1, \pi(\lambda_2))$. The above claim is immediate from Claim 4.3 and facts about product forcing. This completes the proof that the tree property holds at $\aleph_{n+2}$ for $n \geq 1$. To complete the proof of Claim 4.2 and so Lemma 4.1 we prove the following claim.

Claim 4.8. *The tree property holds at* $\kappa_2 = \aleph_2$ *in* $V[A][U][S][e][A_1^+]$.

Note that the argument in Lemma 4.29 to lift the embedding with critical point κ_2 does not depend on adding only κ_3 subsets to κ_1. Since the forcing to add the embedding is a product we can reorganize so that $\hat{\mathbb{A}}_1$ includes the quotient $\pi(\mathbb{A}_1^+)/\mathbb{A}_1^+$. Now the argument to that the forcing to add the embedding does not add a branch is the Lemma 4.29 in the case $n = 0$. This finishes the proof of Lemma 4.1.

5. The tree property at $\aleph_{\omega+1}$

In this section we prove the following lemma.

Lemma 5.1. *There is a generic object for the main forcing which gives the tree property at $\aleph_{\omega+1}$ in W.*

Recall that the key issue in arguing that the tree property holds at $\aleph_{\omega+1}$ in any of the known models [4, 6, 5] is the careful selection of the cardinal that will become ω_1. We need to make a similar selection, but before we do so we must reduce the problem in much the same way that Neeman reduces the problem in Section 6 of his paper. For this we will need a preservation lemma.

5.1. **The preservation lemma.** We prove a generalization of the main preservation lemma in Magidor and Shelah's original paper [4] on the tree property at $\aleph_{\omega+1}$. The statement and proof of the lemma are very technical. To help clarify matters we state and prove a lemma with slightly different hypotheses, which is a natural strengthening of the Magidor and Shelah lemma. We will make use of some of the same definitions and claims as Magidor and Shelah, which will be included for completeness.

Definition 5.2. *A condition (p, r) is λ-wide at level α if there are λ-many extensions such that any two force distinct values for the branch on level α.*

As in Magidor and Shelah's paper we may assume that the level sets of T are in V and this makes the definition meaningful.

Lemma 5.3. *Let ν be a singular cardinal of cofinality ω. Suppose that $\mathbb{P}$ is χ-cc with $|\mathbb{P}|^{<\chi} < \nu$ and $\mathbb{R}$ is $< \chi$-closed for some $\chi < \nu$. If G is $\mathbb{P}$-generic, then forcing with $\mathbb{R}$ over $V[G]$ cannot add a branch through a ν^+-tree $T \in V[G]$.*

Proof. Let G, $\mathbb{P}$ and $\mathbb{R}$ be as in the lemma. Let $\dot{T}$ be a $\mathbb{P}$-name for T and $\dot{b}$ be a $\mathbb{P} \times \mathbb{R}$-name for a cofinal branch which is forced to not be a member of $V^{\mathbb{P}}$.

Claim 5.4. *Let λ be a cardinal with $|\mathbb{P}| < \lambda < \nu$.*

1. *Any condition (p, r) is λ-wide at some level γ.*
2. *If (p, r) is λ-wide at γ then it is λ-wide at γ' for all $\gamma' > \gamma$.*

The proof is exactly the same as Lemma 5.4 of [4]. Using this claim we can complete a construction which will require that there are ν^+ possibilities for $\dot{b}$ on some level, a contradiction.

Let λ_n for $n < \omega$ be increasing and cofinal in ν with $\lambda_0 > \chi^+$. Let $(p, r) \in \mathbb{P} \times \mathbb{R}$ and $n < \omega$. We describe a construction that produces a sequence $\langle r_i \mid i < \lambda_n \rangle$ of extensions of r, which will be used in a larger construction. For definiteness well order the conditions of $\mathbb{P}$ that are below p.

For each $i < \lambda_n$ we construct an maximal antichain A_i and an ordinal γ_i in addition to the condition r_i. Each r_i will be a lower bound on a sequence of conditions of length less than χ. We go by induction on ordinals less than χ and rely on the closure of $\mathbb{R}$ to continue the construction and the chain condition of $\mathbb{P}$ to ensure that the construction terminates.

Start by finding a level γ_0 such that (p, r) is λ_n-wide at γ_0. Then using width find extensions (p_i^0, r_i^0) deciding distinct values for the branch at level γ_0. Let $A_i^0 = \{p_i^0\}$.

For the successor step, assume that A_i^β and r_i^β have been defined for some β. For each $i < \lambda_n$, let p_i be the least condition in our well ordering incompatible with everything in A_i^β, if possible. If no such p_i exists then we are done with this coordinate of the construction. In this case we let $A_i = A_i^\beta$ and $r_i = r_i^\beta$.

Consider the conditions (p_i, r_i^β), for relevant i. There is a level $\gamma_{\beta+1} > \gamma_\beta$ where all of the conditions are λ_n-wide. Using width find conditions $(p_i^{\beta+1}, r_i^{\beta+1}) \leq (p_i, r_i^\beta)$ such that each decides a distinct value. Let $A_i^{\beta+1} = A_i^\beta \cup \{p_i^{\beta+1}\}$. This completes the successor step.

For a limit stage ζ, for each $i < \lambda_n$ we have constructed $\langle r_i^\delta : \delta < \zeta\rangle$ a decreasing sequence and $\langle A_i^\delta : \delta < \zeta\rangle$ an increasing sequence of antichains. We let r_i^ζ be a lower bound and $A_i^\zeta = \bigcup_{\delta<\zeta} A_i^\delta$.

For each $i < \lambda_n$ the construction must terminate at a successor step before stage χ by the chain condition of $\mathbb{P}$. Closure and chain condition together ensure that we can take the relevant lower bounds. Each A_i is a maximal antichain by construction. Let $\gamma = \sup \gamma_\beta$.

Use this construction as the inductive step of a construction of a tree $\langle r_s \mid s \in \bigcup_{n<\omega} \prod_{m<n} \lambda_n\rangle$ of conditions in $\mathbb{R}$ such that for all n and all s of length n, $\langle r_{s^\frown i} \mid i < \lambda_n\rangle$ are obtained from the above construction with (p, r_s) as input. We can assume that ordinals γ obtained in the construction increase with n. Let γ^* be the supremum of all γ's appearing in the construction. It follows that for $s \in \prod_{n<\omega} \lambda_n$ we have a condition $r_s \leq r_{s\restriction m}$ for all $m < \omega$. Each such r_s comes with an associated ω-sequence of maximal antichains in $\mathbb{P}$. Using the assumption that $|\mathbb{P}|^{<\chi} < \nu$ we can find a subset I of $\prod_{n<\omega} \lambda_n$ of size ν^+ such that the *order of construction* of the antichains is fixed for all $s \in I$.

Work in $V[G]$ where G is $\mathbb{P}$-generic. For each $s \in I$ extend r_s to r_s^* deciding the value of $\dot{b}$ at level γ^*. We claim that for all $s, t \in I$, r_s^*, r_t^* force different values for $\dot{b}$ at level γ^*. Let $n < \omega$ be the greatest such that $s \restriction n = t \restriction n$. Since G is generic and we can take $p \in G$, G intersects the antichains used in the construction of $r_{s\restriction n+1}$ and $r_{t\restriction n+1}$. By the choice of I not only are these antichains the same, but the order in which they were constructed is the same. So there is a condition $p^* \in G$ such that $(p^*, r_{s\restriction n+1})$ and $(p^*, r_{t\restriction n+1})$ force different values for $\dot{b}$ at some level below γ^*. It follows that r_s^* and r_t^* force different values at level γ^*. This is a contradiction, since level γ^* has size ν. □

This lemma is not a precise fit for our situation. In particular we will not be able to assume that the forcing in the $\mathbb{P}$ part of the lemma has the χ-cc. In order to repeat the proof of the above lemma in the desired context we need to make more specific assumptions about the posets in the $\mathbb{P}$ part. In what follows, $\mathbb{P}$ has been replaced by a product $\mathbb{P} \times \mathbb{Q} \times \mathbb{E}$.

Lemma 5.5. *Let $\mu < \nu$ be singular cardinals of cofinality ω, let κ be regular with $\kappa < \nu$ and let $\langle \mu_n \mid n < \omega\rangle$ be an increasing and cofinal sequence of regular cardinals less than μ. Fix posets as follows:*

- $\mathbb{P} = \mathrm{Add}(\omega, \kappa)$
- $\mathbb{Q}$ *is a poset which is subsumed by a poset $\mathbb{Q}'$ such that $\mathbb{Q}'$ is countably closed and for all $n < \omega$ $\mathbb{Q}' \simeq \mathbb{Q}_n \times \mathbb{Q}^n$ where $\mathbb{Q}_n$ is μ_n-cc and $\mathbb{Q}^n$ is $< \mu_n$-closed.*
- $\mathbb{E} = \mathrm{Coll}(\omega, \mu)$ *and*
- $\mathbb{R}$ *is μ-closed.*

Further assume that $|\mathbb{P}\times\mathbb{Q}\times\mathbb{E}|^\mu < \nu$*. If* $A\times G\times e$ *is* $\mathbb{P}\times\mathbb{Q}\times\mathbb{E}$*-generic and* $T\in V[A\times G\times e]$ *is a* ν^+*-tree, then forcing with* $\mathbb{R}$ *over* $V[A\times G\times e]$ *cannot add a branch through* T*.*

Proof. Let $\dot{T}$ be a $\mathbb{P}\times\mathbb{Q}\times\mathbb{E}$-name for a ν^+-tree and let $\dot{b}$ be a $\mathbb{P}\times\mathbb{Q}\times\mathbb{E}\times\mathbb{R}$-name for a cofinal branch through $\dot{T}$ which is not a member of the extension by $\mathbb{P}\times\mathbb{Q}\times\mathbb{E}$. We also fix an enumeration $\langle e_\alpha \mid \alpha<\mu\rangle$ of $\mathbb{E}$.

One remark is due on the use of $\mathbb{Q}$ and $\mathbb{Q}'$ in this argument. For the bulk of the argument we will work with $\mathbb{Q}'$, because of it's nice properties. Conditions in $\mathbb{Q}'$ deciding information about the extension by $\mathbb{Q}$ can always be projected into $\mathbb{Q}$ while deciding information about $\dot{T}$ and $\dot{b}$ in the same way.

As in the proof of the previous lemma we complete a construction which forms the inductive step of a further construction. Let $\lambda<\nu$ be a regular cardinal and $r_0\in\mathbb{R}$. We construct the following:

(1) $\langle q_i^\beta \mid \beta<\mu\rangle$ a decreasing sequence in $\mathbb{Q}'$.
(2) $\langle r_i^\beta \mid \beta<\mu\rangle$ a decreasing sequence in $\mathbb{R}$ and
(3) $\langle \gamma_\beta \mid \beta<\mu\rangle$ an increasing sequence of ordinals less than ν^+.

We work by induction on β. We let $r_i^0=r_0$, q_i^0 be the trivial condition in $\mathbb{Q}'$ and $\gamma_0=0$. For the successor step assume that we have constructed everything for all $\beta'\le\beta$ for some $\beta<\mu$. Let k be greatest such that $\mu_k\le\beta$. For ease of notation we let $e=e_\beta$.

We run another inductive construction on ordinals $\eta<\mu_{k+2}$ to construct:

(1) $\bar{q}_i^\beta(\eta)\in\mathbb{Q}_{k+2}$,
(2) $\hat{q}_i^\beta(\eta)\in\mathbb{Q}^{k+2}$,
(3) $p_i^\beta(\eta)\in\mathbb{P}$,
(4) $e_i^\beta(\eta)\in\mathbb{E}$ and
(5) $\gamma_\beta(\eta)<\nu^+$.

We do not mention the sequence lengths, because we do not know them in advance of the construction. The collection of $(p_i^\beta(\eta),\bar{q}_i^\beta(\eta))$ for relevant η will form a maximal antichain in $\mathbb{P}\times\mathbb{Q}_{k+2}$ below $(\emptyset,\bar{q}_i^\beta)$. The sequences of $\hat{q}_i^\beta(\eta)$ and $r_i^\beta(\eta)$ be decreasing in $\mathbb{Q}^{k+2}$ and $\mathbb{R}$ respectively. The sequence of $\gamma_\beta(\eta)$ will be an increasing sequence of ordinals.

Suppose that we have constructed everything relevant for all $\eta'\le\eta$. For $i<\lambda$ we let $(p_i,\bar{q}_i)\in\mathbb{P}\times\mathbb{Q}_{k+2}$ be incompatible with $(p_i^\beta(\eta'),\bar{q}_i^\beta(\eta'))$ for all $\eta'\le\eta$ if such a condition exists. If no condition exists, then we halt the construction on coordinate i. As in the proof of the previous lemma we can find an ordinal $\gamma_\beta(\eta+1)>\gamma_\beta(\eta)$ such that all conditions of the form $(p_i,\bar{q}_i,\hat{q}_i^\beta(\eta),e,r_i^\beta(\eta))$ are λ-wide at $\gamma_\beta(\eta+1)$. Still working as in the previous lemma we recursively construct extensions $(p_i^\beta(\eta+1),\bar{q}_i^\beta(\eta+1),\hat{q}_i^\beta(\eta+1),e_i^\beta(\eta+1),r_i^\beta(\eta+1))$ of these conditions which decide the value of $\dot{b}$ at level $\gamma_\beta(\eta+1)$ with the property that the i^{th} value is different from the j^{th} value for all $j<i$. This completes the successor step of the induction on η.

For the limit step we take lowerbounds for the decreasing sequences, leave the sequences from $\mathbb{P}$ and $\mathbb{Q}_{k+2}$ undefined and take the supremum of the $\gamma_\beta(\eta)$. This completes the induction on η. It is clear that each coordinate $i<\lambda$ halts at some stage less than μ_{k+2}. We let $r_i^{\beta+1}$ be a lowerbound for the sequence of $r_i^\beta(\eta)$, $q_i^{\beta+1}$

be the condition $\bar{q}_i^\beta$ joined with a lowerbound for $\hat{q}_i^\beta(\eta)$ and $\gamma_{\beta+1}$ be the supremum of the $\gamma_\beta(\eta)$. This completes the successor step of the induction on $\beta < \mu$.

The limit step of the induction on β is simple we just take a lowerbound on the sequences in $\mathbb{Q}'$ and $\mathbb{R}$. In the case of the lowerbound for the sequence in $\mathbb{Q}'$ we note that the sequence $\langle q_i^\beta \mid \mu_k \leq \beta < \mu_{k+1}\rangle$ is decreasing in $\mathbb{Q}_{k+2} \times \mathbb{Q}^{k+2}$, but without changing the $\mathbb{Q}_{k+2}$ coordinate. This completes the construction. In the end we let r_i^* be a lowerbound for the r_i^β and q_i^* be a lowerbound for the q_i^β. This is possible because $\mathbb{Q}'$ is countably closed and $\langle q_i^{\mu_k} \mid k < \omega\rangle$ is cofinal in the whole sequence.

Claim 5.6. *For all $i < \lambda$ let A_i be the set of all $(p_i^\beta(\eta), \bar{q}_i^\beta(\eta) \frown \hat{q}_i^*, e_i^\beta(\eta))$ such that $\beta < \mu$ and $\eta < \mu_{k+2}$ where k is greatest such that $\mu_k \leq \beta$ and $\hat{q}_i^*$ is the part of q_i^* in the poset $\mathbb{Q}^{k+2}$. A_i is predense in $\mathbb{P} \times \mathbb{Q} \times \mathbb{E}$ below the condition $(\emptyset, q_i^*, \emptyset)$.*

Proof. Let $(p, q, e) \leq (\emptyset, q_i^*, \emptyset)$. Let β be such that $e = e_\beta$. Then there is an $\eta < \mu_{k+2}$ such that $(p, \bar{q})$ is compatible with $(p_i^\beta(\eta), \bar{q}_i^\beta(\eta))$ where $\bar{q}$ is the part of q in $\mathbb{Q}_{k+2}$. Let $(p^*, \bar{q}^*)$ be a lowerbound for both conditions. It follows that $(p^*, \bar{q}^* \frown \hat{q}_i^*, e_i^\beta(\eta))$ is below both (p, q, e) and $(p_i^\beta(\eta), \bar{q}_i^\beta(\eta) \frown \hat{q}_i^*, e_i^\beta(\eta))$. □

As in the previous lemma we construct a tree with branches corresponding to elements of $\prod_{n<\omega} \lambda_n$ where $\langle \lambda_n \mid n < \omega\rangle$ is an increasing and cofinal sequence of regular cardinals less than ν. We let γ^* be the supremum of the levels appearing in the construction. For each $s \in \prod_{n<\omega} \lambda_n$ we have a condition (q_s, r_s) in $\mathbb{Q} \times \mathbb{R}$ which is a lowerbound for an appropriate decreasing sequence. We can find a set I of size ν^+ such that there is $q^* \in \mathbb{Q}$ such that for all $s \in I$, $q_s = q$.

Each $s \in \prod_{n<\omega} \lambda_n$ has a corresponding ω-sequence of predense sets given by the previous claim. Using the cardinal arithmetic assumption and making I smaller (still of size ν^+) we can assume that not only the sequences of predense sets, but their order of construction is fixed for all $s \in I$. Let A^n for $n < \omega$ be the common sequence of predense sets. For each A^n we choose a maximal antichain $B^n \subseteq A^n$.

Let $A \times G \times e$ be $\mathbb{P} \times \mathbb{Q} \times \mathbb{E}$-generic. Working in $V[A \times G \times e]$ we extend each r_s to r_s^* deciding the value of $\dot{b}$ at level γ^*. To complete the proof we argue that for $s, t \in I$, r_s^* and r_t^* decide different values for $\dot{b}$ at level γ^*.

Let n be greatest such that $s \restriction n = t \restriction n$. Let (p, q, e) be below the unique element (p', q', e') of $B^n \cap (A \times G \times e)$ and force that r_s^* and r_t^* decide the value of $\dot{b}$ at level γ^*. It follows from our inductive construction and the fact that we fixed the order of construction of A^n (and hence B^n) that there are conditions $r_1 \geq r_s^*$ and $r_2 \geq r_t^*$ such that (p', q', e') forces that r_1 and r_2 decide different values for $\dot{b}$ at some level below γ^*. So in $V[A \times G \times e]$ we must have that r_s^* and r_t^* decide different values for $\dot{b}$ at level γ^*. □

Remark 5.7. *The proof of the previous lemma can be strengthened in the following sense. We may assume that the poset $\mathbb{P} \times \mathbb{Q}$ is in fact a two step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ and there is a projection from $\mathbb{P} \times \mathbb{Q}'$ to $\mathbb{P} * \dot{\mathbb{Q}}$ where $\mathbb{Q}'$ is a poset in V with the same properties as $\mathbb{Q}'$ above.*

5.2. **Reduction of the problem.** In this section we reduce the problem of getting the tree property at $\aleph_{\omega+1}$ in W to the problem of getting it in some outer model which is easier to analyze. The analogous claim from Neeman's paper is Claim 6.5. We identify two posets. First we have a poset which is exactly the same as in [5], namely the forcing to refine both U and S to generics $B_{[1,\omega)}$ and $C_{[1,\omega)}$ for $\mathbb{B}^{+A_0*U_0}\restriction[\kappa_2,\nu)$ and $\mathbb{C}^{+A_0*U_0*B_{[1,\omega)}}\restriction[\kappa_2,\nu)$ respectively. This poset is μ-closed in $V[A][U][S]$ and by the proofs of Claims 3.7 and 3.3 it is still μ-closed in $V[A][U][S][A_1^+][A^+][U^+][S^+]$. The second poset is analogous to the previous one, but for the forcing above ν. In particular it is the forcing to refine U^+ and S^+ to generics $B^+_{[\omega,\omega\cdot 2)}$ and $C^+_{[\omega,\omega\cdot 2)}$ for $\mathbb{B}\restriction[\lambda_1,\iota)$ and $\mathbb{C}^{+B^+_{[\omega,\omega\cdot 2)}}\restriction[\lambda_1,\iota)$ respectively. It is clear that this forcing is μ-closed in the model $V[A_1\times A_1^+\times A^+][U^+][S^+]$. We reduce the problem to showing that the tree property holds in the outer model obtained by the product of these two forcings. For each $\mu\in\text{Index}$, we define $\mathbb{L}(\mu)$ to be the poset $\mathbb{A}_1\times\mathbb{A}_1^+\times\mathbb{C}_0^{+A_0*U_0}(\mu^+)\times\text{Coll}(\omega,\mu)$.

Lemma 5.8. *To prove Lemma 5.1, it is enough to show that there is a μ such that the tree property holds at ν^+ in the extension of*

$$V[A_{[2,\omega)}][A_0*U_0][B_{[1,\omega)}][C_{[1,\omega)}][A^+][B^+_{[\omega,\omega\cdot 2)}][C^+_{[\omega,\omega\cdot 2)}]$$

by $\mathbb{L}(\mu)$.

We note that the generic extension in the above lemma is the extension of W by the product of the two posets mentioned above. We prove the lemma in two steps.

Claim 5.9. *If the tree property holds at ν^+ in*

$$V[A_{[2,\omega)}][A_0*U_0][B_{[1,\omega)}][C_{[1,\omega)}][A_1\times A_1^+][A^+][U^+][S^+][S_0][e],$$

then it holds in W.

Proof. We apply Lemma 5.3 or even the original lemma of Magidor and Shelah in the model $V[A][U][S][A_1^+][A^+][U^+][S^+]$ with $\mathbb{P}=\text{Coll}(\omega,\mu)$ and $\mathbb{R}$ as the poset which refines U and S to $B_{[1,\omega)}$ and $C_{[1,\omega)}$. The poset $\mathbb{R}$ is closed in the relevant model by remarks above. □

Claim 5.10. *If the tree property holds at ν^+ in the model*

$$V[A_{[2,\omega)}][A_0*U_0][B_{[1,\omega)}][C_{[1,\omega)}][A^+][B^+_{[\omega,\omega\cdot 2)}][C^+_{[\omega,\omega\cdot 2)}][A_1\times A_1^+][S_0][e],$$

then it holds at ν^+ in the model

$$V[A_{[2,\omega)}][A_0*U_0][B_{[1,\omega)}][C_{[1,\omega)}][A_1\times A_1^+][A^+][U^+][S^+][S_0][e].$$

Proof. We apply Lemma 5.5. Let T be a ν-tree in the inner model. It is enough to show that T obtains a branch in

$$V[A_{[2,\omega)}][A_0*(U_0\restriction\mu)][B_0\restriction[\mu,\kappa_2)][\hat{B}_{[1,\omega)}][\hat{C}_{[1,\omega)}][A_1\times A_1^+][A^+][U^+][S^+][\hat{S}_0][e].$$

where

- $\hat{B}_{[1,\omega)}$ refines $B_{[1,\omega)}$ to a generic for $\mathbb{B}\restriction[\kappa_2,\nu)$,
- $\hat{C}_{[1,\omega)}$ refines $C_{[1,\omega)}$ to a generic for $\mathbb{C}^{+\hat{B}_{[1,\omega)}}\restriction[\kappa_2,\nu)$
- $\hat{S}_0$ refines S_0 to a generic for $\mathbb{C}_0$ and
- $B_0\restriction[\mu,\kappa_2)$ refines a restriction of U_0 to a generic for $\mathbb{B}_0\restriction[\mu,\kappa_2)$.

If T has a cofinal branch in this model, then it will have a cofinal branch in two mutually generic extensions the desired model. This is enough to finish the proof.

In the model where we have refined $B_{[1,\omega)}$, $C_{[1,\omega)}$ and S_0 to their $\hat{}$-versions we show that ν and ν^+ are preserved. By Remark 3.5 we have that ν and ν^+ are preserved in the model

$$V[A_{[2,\omega)}][A_0*(U_0\restriction\mu)][B_0\restriction[\mu,\kappa_2)][\hat{B}_{[1,\omega)}][\hat{C}_{[1,\omega)}][A_1][\hat{S}_0]$$

since it is an inner model of the forcing extension mentioned in the remark. Extending this idea further using the proof of Claim 3.10 we see that ν and ν^+ must be preserved when we add the further generics A_1^+, A^+, U^+ and S^+.

We can now apply Lemma 5.5 and Remark 5.7 in the model

$$V[A_{[2,\omega)}][B_0 \upharpoonright [\mu, \kappa_2)][\hat{B}_{[1,\omega)}][\hat{C}_{[1,\omega)}][A_1 \times A_1^+][A^+][U^+][S^+][\hat{S}_0]$$

with $\mathbb{P} * \dot{\mathbb{Q}} = \mathbb{A}_0 * (\mathbb{U} \upharpoonright \mu)$, $\mathbb{E} = \mathrm{Coll}(\omega, \mu)$ and $\mathbb{R}$ as the poset to refines U^+ and S^+ to $B^+_{[\omega,\omega\cdot 2)}$ and $C^+_{[\omega,\omega\cdot 2)}$. For $\mu \in$ Index there is a largest point λ in the domain of ϕ below μ. We can assume that $\phi(\lambda)$ returns an $\mathbb{A}_0 \upharpoonright \lambda * \mathbb{U}_0 \upharpoonright \lambda$-name for a reflection of the forcing

$$\mathbb{B}^{+A_0 * U_0} \upharpoonright [\kappa_2, \nu) * \mathbb{C}^{+A_0 * U_0 * B_{[1,\omega)}} \upharpoonright [\kappa_2, \nu).$$

Along the lines of the proofs Lemma 3.3 and Remark 3.5, it is not hard to see that $\mathbb{A}_0 * (\mathbb{U} \upharpoonright \mu)$ is the projection of $\mathbb{A}_0 \times \mathbb{B} \upharpoonright \mu$ and $\mathbb{B} \upharpoonright \mu$ has the properties needed to play the role of $\mathbb{Q}'$ in Lemma 5.5. We note that the desired sequence $\langle \mu_k \mid k < \omega \rangle$ is a reflection of the sequence $\langle \kappa_k \mid k \geq 2 \rangle$ in the ultrapower by some supercompactness measure on κ_2.

Further we have $|\mathbb{P} * \mathbb{Q} \times \mathbb{E}|^\mu < \nu$, since this poset has size κ_2 and in the model where we apply the preservation all μ-sequences come from V. It is also clear that $\mathbb{R}$ is μ-closed in this model since it is μ-closed where it is defined and the further forcing to get to the current model does not add any μ-sequences. This completes the proof. □

5.3. **The selection of ω_1.** In this section we complete the proof of Lemma 5.8. The proof is essentially the same as in [5]. We note that the arguments given in Lemmas 5.6 and 5.7 of [5] can be repeated with the extra generics $B^+_{[\omega,\omega\cdot 2)}$ and $C^+_{[\omega,\omega\cdot 2)}$ which are both $< \nu^+$-directed closed in V. Each proof begins with an application of Laver indestructibility and so we can incorporate the extra generic objects at this stage. We also note that the change in definition of $\mathbb{L}$ does not change the proof of Lemma 5.8 of [5]. With the analogs of Lemmas 5.6, 5,7 and 5.8, we have the requirements of Lemma 3.10 of [5]. This is exactly what we need to finish the proof of Lemma 5.8.

6. The tree property at $\aleph_{\omega+2}$

In this section we prove the following lemma.

Lemma 6.1. *The tree property holds at $\aleph_{\omega+2}$ in the W.*

The proof for $\aleph_{\omega+2}$ is slightly different than the one for $\aleph_{\omega+n}$ for $n \geq 3$ and so it must be handled separately.

Proof. To start the proof we identify two posets needed to lift the relevant elementary embedding. We refine U^+ and S^+ to obtain generics $B_{[\omega+1,\omega\cdot 2)}$ and $C_{[\omega+1,\omega\cdot 2)}$ for the posets $\mathbb{B}^{+F} \upharpoonright [\lambda_2, \iota)$ and $\mathbb{C}^{+F} \upharpoonright [\lambda_2, \iota)$ where $F = A_1 \times A_1^+ * U \upharpoonright [\lambda_1, \lambda_2)$. Call these posets $\mathbb{P}_1$ and $\mathbb{P}_2$ respectively.

Claim 6.2. *$\mathbb{P}_1$ and $\mathbb{P}_2$ are $< \nu^+$ closed in $V[A_1 \times A_1^+][A^+][U^+][S^+]$.*

Proof. The proof is similar to Claim 4.30 of [5]. By easy modifications of Claims 4.9 and 4.16 of [5], we see that any decreasing sequence of length $< \lambda_2$ in either $\mathbb{P}_1$ or $\mathbb{P}_2$ which belongs to $V[F]$ has a lowerbound. It remains to show that every $< \nu^+$-sequence from $V[A_1 \times \mathbb{A}_1^+][A^+][U^+][S^+]$ is a member of $V[F]$. This is clear from the proof of Claim 3.10. The addition of A^+, U^+ and S^+ does not add any $<\nu^+$-sequences to $V[A_1 \times A_1^+]$. $\square$

Following Neeman, we let π be an elementary embedding with critical point λ_2 witnessing the γ-supercompactness of λ_2 with the following properties:

- $\pi : V[A_{[\omega+2,\omega\cdot 2)})] \to V^*[A^*_{[\omega+2,\omega\cdot 2)}]$ (Note that V^* is not definable in V, since we applied the indestructibility of λ_2 in V to obtain this embedding.)
- $\pi(\phi)(\lambda_2)$ returns an $\mathbb{A}_1 \times \mathbb{A}_1^+ * \mathbb{U} \restriction [\lambda_1, \lambda_2)$-name for $\mathbb{B}^{+F} \restriction [\lambda_2, \iota) \times \mathbb{C}^{+F} \restriction [\lambda_2, \iota)$
- If $G_1 \times G_2$ is generic for the interpretation of $\pi(\phi)(\lambda_2)$, then π extends to

$$\pi : V[A_1 \times A_1^+][A_{[\omega+1,\omega\cdot 2)}][U \restriction [\lambda_1, \lambda_2)][G_1][G_2] \to V^*[A_1 \times (A_1^+)^*][A^*_{[\omega+1,\omega\cdot 2)}][G_1^*][G_2^*]$$

  using a variation on Lemma 4.12 of [5].
- Moreover the forcing to add this embedding is

$$\mathrm{Add}(\kappa_1, \pi(\lambda_2))_V \times \mathrm{Add}(\lambda_1, \pi(\lambda_3))_V.$$

  We let $\hat{A}_1^+ \times \hat{A}_{\omega+1}$ be the generic objects.
- π restricts to an elementary embedding

$$\pi : V[A_1 \times A_1^+][A_{[\omega+1,\omega\cdot 2)}][U_{[\omega,\omega\cdot 2)}][S_{[\omega+1,\omega\cdot 2)}] \to$$
$$V^*[A_1 \times (A_1^+)^*][A^*_{[\omega+1,\omega\cdot 2)}][U^*_{[\omega,\omega\cdot 2)}][S^*_{[\omega+1,\omega\cdot 2)}]$$

Next by standard arguments we can lift this last embedding further to include $S \restriction [\lambda_1, \lambda_2)$ by forcing with the poset $\mathbb{P}_3 = \pi(\mathbb{C}_\omega)^{+A_1 \times A_1^+} \restriction [\lambda_2, \pi(\lambda_2))$. Let G_3 be the generic object. Finally we can lift the embedding to the further extension by A, U, S and e, since this forcing is all small relative the critical point of π.

The issue now is to repeat Neeman's proof in this slightly different context. The main problem is to deal with the extra forcing of size ν^+ which adds A, U, S and e. Over the final model, the generic objects needed to add the embedding are $\hat{A}_{\omega+1}$, G_1, G_2, G_3 and $\hat{A}_1^+$. We add the generics in the order that they are listed. If T is a λ_2-tree in W, then by standard arguments it obtains a branch in the extension by the 5 generic objects above. It remains to see that these generics could not have added the branch. We work through the posets in order.

Claim 6.3. *The forcing to add $\hat{A}_{\omega+1}$ over the final model is $<\nu^+$ distributive and its square is λ_2-cc.*

Proof. The proof for distributivity is similar to Claim 3.7. The proof of chain condition follows from Lemma 1.3 and Corollary 3.8. $\square$

By Lemma 1.1 it follows that $\hat{A}_{\omega+1}$ cannot add a branch through T and that λ_2 is preserved in the extension. Next we wish to apply Lemma 1.2 simultaneously to the forcing to add $G_1 \times G_2 \times G_3$. For this we need a few more preparations.

Claim 6.4. *$\mathbb{P}_3$ is $<\nu^+$-closed in $V[A_1][A^+][U^+][S^+ \restriction [\lambda_2, \iota)]$.*

Proof. An analog of Claim 4.15 of Neeman together with the resemblence between V and V^* imply that $\mathbb{P}_3$ is $<\nu^+$-closed in $V[A_1 \times A_1^+]$. The claim follows since the remaining forcing does not add any ν-sequences by Claim 3.7. $\square$

At this point we recall that in the final model $2^{\omega_1} = \aleph_{\omega+2}$, so to apply Lemma 1.2 it is enough to set $\tau = \omega_1$. However it is not clear that in the current model there is an inner model given by τ^+-cc forcing in which $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$ is defined. We solve this problem by applying the lemma in a mutually generic extension which does have such an inner model. It follows that $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$ could not have added a branch through T since any such branch would exist in two mutually generic extensions of $W[\hat{A}_{\omega+1}]$.

In $W[\hat{A}_{\omega+1}]$ we force to remove the dependence of U and S on $A_0 * U_0$ by adding generics $\bar{B}$ and $\bar{C}$ for $\mathbb{B} \restriction [\kappa_2, \nu)$ and $\mathbb{C} \restriction [\kappa_2, \nu)$. It is not hard to see that this extension preserves cardinals. By rearranging the forcing we see that in the model

$$V[A_{[2,\omega)}][\bar{B}][\bar{C}][A_1 \times A_1^+][A^+][U^+][S^+][\hat{A}_{\omega+1}]$$

the forcing to add $A_0 * U_0 * S_0$ is κ_2-cc, the forcing $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$ is $< \kappa_2$-closed by Claims 6.2 and 6.4, and $2^{\kappa_1} = \lambda_{\omega+2}$. So by an application of Lemma 1.2 and mutual genericity, we have that forcing with $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$ cannot add a branch through T over the model $W[\hat{A}_{\omega+1}]$.

It remains to show that the forcing $\pi(\mathbb{A}_1^+)/\mathbb{A}_1^+$ to add $\hat{A}_1^+$ cannot add a branch through T over $W[\hat{A}_{\omega+1}][G_1 \times G_2 \times G_3]$. By standard arguments the height of T has been a collapsed to cofinality ν^+ in this model.

We note that the quotient forcing isomorphic to $\mathrm{Add}(\kappa_1, \pi(\lambda_2))$. Since V has the κ_2 covering property in W and the forcing to obtain $W[\hat{A}_{\omega+1}][G_1 \times G_2 \times G_3]$ does not add any κ_2-sequences, V still has the κ_2 covering property in this further extension. It follows that the square of $\mathrm{Add}(\kappa_1, \pi(\lambda_2))_V$ has the κ_2-cc in $W[\hat{A}_{\omega+1}][G_1 \times G_2 \times G_3]$. So we are finished by an application of Lemma 1.1. □

7. The tree property at $\aleph_{\omega+n}$ with $3 \leq n < \omega$

In this section we sketch the proof of the following lemma.

Lemma 7.1. *The tree property holds at $\aleph_{\omega+n}$ for all $n \geq 3$ in W.*

The proof is an easy adaptation of arguments of Neeman. A direct application of arguments of Neeman yields:

Lemma 7.2. *The tree property holds at λ_n for $n \geq 3$ in $V[A_1 \times A_1^+][A^+][U^+][S^+]$.*

However we note that Neeman's argument yields the slightly stronger statement that the tree property at each λ_n is indestructible under forcing of size less than λ_{n-1} from the ground model. It is worth noting that the proof of this fact is easier than the proof of Claim 4.2. We apply this indestructibility under small forcing to add A, U, S and e which has size $\nu^+ = \lambda_1$ in V while preserving the tree property at each λ_n for $n \geq 3$. This completes the proof of Lemma 7.1 and with it the proof of the main theorem.

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