STATIONARY REFLECTION AND THE FAILURE OF SCH

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Abstract. In this paper we prove that from large cardinals it is consistent that there is a singular strong limit cardinal ν such that the singular cardinal hypothesis fails at ν and every collection of fewer than $\mathrm{cf}(\nu)$ stationary subsets of ν^+ reflects simultaneously. For $\mathrm{cf}(\nu) > \omega$, this situation was not previously known to be consistent. Using different methods, we reduce the upperbound on the consistency strength of this situation for $\mathrm{cf}(\nu) = \omega$ to below a single partially supercompact cardinal. The previous upper bound of infinitely many supercompact cardinals was due to Sharon.

1. Introduction

We study stationary reflection at successors of singular cardinals and its connection with cardinal arithmetic. We start by recalling some basic definitions. For an ordinal δ and a set $S \subseteq \delta$, we say that S is *stationary* if it meets every closed and unbounded subset of δ. If $\{S_i \mid i \in I\}$ is a collection of stationary subsets of a regular cardinal κ, then we say that $\{S_i \mid i \in I\}$ reflects simultaneously if there is an ordinal δ such that $S_i \cap \delta$ is stationary for all $i \in I$.

The consistency of stationary reflection at the successor of singular cardinal is already complex in the context of the generalized continuum hypothesis (GCH). A theorem of Magidor [12] shows that it is consistent relative to the existence of infinitely many supercompact cardinals that every finite collection of stationary subsets of $\aleph_{\omega+1}$ reflects. Recently, the second and third author [11] were able show the same result from an assumption below the existence of a cardinal κ which is κ^+-supercompact. Both of these models satisfy GCH. Combining stationary reflection at the successor of a singular cardinal with the failure of SCH presents additional difficulties.

For a singular cardinal ν, the singular cardinal hypothesis (SCH) at ν is the assertion that if ν is strong limit, then $2^\nu = \nu^+$. The failure of the singular cardinal hypothesis at a singular cardinal ν is known to imply the existence of many nonreflecting objects. For instance, Foreman and Todorcevic [4] have shown that the failure of the singular cardinal hypothesis at

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ν implies that there are two stationary subsets of $[\nu^+]^\omega$ which do not reflect simultaneously. This was improved by Shelah [17] to obtain a single stationary subset of $[\nu^+]^\omega$ which does not reflect. Reflection for stationary subsets of $[\nu^+]^\omega$ has a different character than reflection for stationary subsets of ordinals.

In his PhD thesis from 2005, Sharon [16] proved that relative to the existence of infinitely many supercompact cardinals it is consistent that there is a singular cardinal ν of cofinality ω such that SCH fails at ν and every stationary subset of ν^+ reflects. Sharon's method is a tour de force construction, which builds on Gitik's long extenders forcing [7] for ω sequences of hypermeasurable cardinals. As such, the construction does not extend to singular cardinals of uncountable cofinalities, and the question of whether the failure of SCH at singular κ of uncountable cofinality together with stationary reflection at κ^+ is at all consistent.

This paper follows a study by the authors on stationary reflection at successors of singular cardinals at which SCH fails. This study was prompted by two other recent studies. First, the work of second and third authors in [11] on stationary reflection in Prikry forcing extensions from subcompactness assumptions. The arguments of [11] show how to examine the stationary reflection in the extension by Prikry type forcing by studying suitable iterated ultrapowers of V. This approach and method has been highly effective in our situation and we follow it here. The second study is Gitik's recent work [5] for blowing up the power of a singular cardinal using a Mitchell order increasing sequence of overlapping extenders. The new forcing machinery of [5] gives new models combining the failure of SCH with reflection properties at successors of singulars. Moreover the arguments are uniform in the choice of cofinality. For example, Gitik has shown that in a related model the tree property holds at κ^+ [9] and that for all $\delta < \kappa$ there is a stationary subset of κ^+ of ordinals of cofinality greater than δ which is not a member of $I[\kappa^+]$ [8].

Our first theorem provides a model for stationary reflection the successor of a singular cardinal ν where SCH fails and the cofinality of ν can be some arbitrary cardinal chosen in advance.

Theorem 1.1. *Let $\eta < \lambda$ be regular cardinals. Suppose that there is an increasing sequence of cardinals $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ with*

(1) $\eta < \kappa_0$,
(2) *for each $\alpha < \eta$, κ_α carries a (κ_α, λ)-extender E_α and there is a supercompact cardinal between $\sup_{\beta<\alpha} \kappa_\beta$ and κ_α, and*
(3) *the sequence $\langle E_\alpha \mid \alpha < \eta \rangle$ is Mitchell order increasing and coherent.*

There is a cardinal and cofinality preserving extension in which, setting $\kappa = \sup_{\alpha<\eta} \kappa_\alpha$, $2^\kappa = \lambda$ and every collection of fewer than η stationary subsets of κ^+ reflects.

We have the following improvement of Gitik's result about $I[\kappa^+]$.

Theorem 1.2. *Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be as in Theorem 1.1 with the exception that we only assume there is a single supercompact cardinal $\theta < \kappa_0$. There is a cardinal and cofinality preserving extension in which setting $\kappa = \sup_{\alpha<\eta} \kappa_\alpha$, we have that κ is strong limit, $2^\kappa = \lambda$ and there is a scale of length κ^+ such that the set of nongood points of cofinality less than θ is stationary.*

After a suitable preparation, the extension is obtained using the same forcing as in Theorem 1.1. Standard arguments show that in this model, the set of nongood points is stationary in cofinalities that are arbitrarily high below θ. Further, the same argument shows that we can take θ between $\sup_{\alpha<\beta} \kappa_\alpha$ and κ_β and reach the same conclusion.

These two results continue a long line of research about the interaction between the failure of the singular cardinal hypothesis and weak square-like principles, [10, 15, 19, 20, 18, 21].

We do not know whether these results can be also obtained at small singular cardinals of uncountable cofinalites. For concreteness we suggest the following question.

Question 1.3. Is it consistent that SCH fails $\aleph_{\omega_1}$ and every stationary subset of $\aleph_{\omega_1+1}$ reflects?

We also give another model for stationary reflection at ν^+ where ν is a singular cardinal of cofinality ω where the singular cardinal hypothesis fails. This construction replaces the supercompactness assumption in Sharon's result and Theorem 1.1 with the weaker one of subcompactness together with hypermeasurability.

Theorem 1.4. *Suppose that κ is κ^+-Π^1_1-subcompact and carries a (κ, κ^{++})-extender. There is a forcing extension in which κ is singular strong limit of cofinality ω, $2^\kappa = \kappa^{++}$ and every finite collection of stationary subsets of κ^+ reflects simultaneously.*

The construction and proof follows the lines of the second and third authors‘ paper [11], and the work of Merimovich [13] on generating generics for extender based forcing over iterated ultrapowers. The large cardinal assumption in the theorem is the natural combination of the assumption from [11] and an assumption sufficient to get the failure of SCH by extender based forcing. Unfortunately, we are unable to adapt the argument from the previous theorem to a singular cardinal of uncountable cofinality. We ask

Question 1.5. It is possible to obtain the result of Theorem 1.1 without any supercompactness assumptions?

The paper is organized as follows. In Section 2, we define Gitik's forcing from [5], which will be used in our main theorem. In Section 3, we prove that in mild generic extensions of V we can find a generic for Gitik's forcing over a suitable iterated ultrapower. In Section 4, we give the proof of the main theorem by arguing that stationary reflection holds in the generic extension of the iterated ultrapower constructed in the previous section. In Section 5, we prove Theorem 1.2. In Section 6, we give the proof of Theorem 1.4.

2. Gitik's forcing

In this section we give a presentation of Gitik's forcing [5] for blowing up the power of a singular cardinal with a Mitchell order increasing sequence of extenders.

Let us start with the following definitions:

Definition 2.1. Let E_0, E_1 be (κ_0, λ_0) and (κ_1, λ_1)-extenders respectively. We say that E_0 is less than E_1 in the Mitchell order, or $E_0 \trianglelefteq E_1$, if $E_0 \in \mathrm{Ult}(V, E_1)$. We say that E_0 coheres with E_1 if $j_{E_1}(E_0) \restriction \lambda_0 = E_0$.

The existence of a long sequence of extenders E_i, where E_i is (κ_i, λ)-extender, they are Mitchell increasing and pairwise coherent follows from the existence of superstrong cardinal or even weaker large cardinal axioms.

Before we begin with the definition of the forcing, let us show a few basic facts about extenders and Mitchell order. We recall the notion of width from [2].

Definition 2.2. Let $k : M \to N$ be an elementary embedding between transitive models of set theory and let μ be an ordinal. We say the embedding k has width $\leq \mu$ if every element of N is of the form $k(f)(a)$ for some $f \in M$ and $a \in N$ such that $M \models |\operatorname{dom}(f)| \leq \mu$.

Lemma 2.3. *Let E be a (κ, λ)-extender and let $\leq_E$ be the Rudin-Keisler order of the extender E. Let $k\colon V \to M$ be an elementary embedding with width $\leq \kappa$. Then the set k“$\operatorname{dom} E$ is $\leq_{k(E)}$-cofinal in $\operatorname{dom} k(E)$.*

Proof. By [7], the Rudin-Keisler order $\leq_E$ is κ^+-directed.

Let $a \in \operatorname{dom} k(E)$. Then, by the definition of width, there is a function $f\colon \kappa \to \operatorname{dom} E$ and some generator b such that $k(f)(b) = a$. Let a' be a $\leq_E$ upper bound of $\operatorname{im} f$. Then, clearly, $k(a')$ is $\leq_{k(E)}$ above a. $\square$

Lemma 2.4. *Let E_0 be a (κ_0, λ) extender and let E_1 be a (κ_1, λ)-extender, where $\kappa_0 \leq \kappa_1$. Let us assume that $E_0 \trianglelefteq E_1$. Then the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "E_0"] \arrow[d, "E_1"] & M_0 \arrow[d, "j_{E_0}(E_1)"] \\ M_1 \arrow[r, "E_1"] & N \end{tikzcd}$$

where each arrow represents the ultrapower map using the indicated extender.

Proof. First, since $E_0 \in M_1$, all maps are internally defined and in particular, all models are well founded.

Moreover,

$$(j_{E_0})^V \restriction M_1 = (j_{E_0})^{M_1}.$$

In order to verify this, it is sufficient to show that those maps are the same on ordinals. Indeed, let us consider the class:

$$\{(f, a) \mid f\colon \kappa_0^{<\omega} \to \mathrm{Ord}\},$$

ordered using the extender E_0:

$$(f_0, a_0) \leq (f_1, a_1) \iff j^V_{E_0}(f_0)(a_0) \leq j^V_{E_0}(f_1)(a_1).$$

Using the combinatorial definition of the extender ultrapower, we conclude that M_1 can compute this order correctly, and in particular, it computes correctly the ordertype of the elements below any constant function.

Now, let us consider an element of $\mathrm{Ult}(M_1, E_0)$. By the definition, it has the form: $x = j_{E_0}(g)(a_0)$ where $g\colon \kappa_0^{<\omega} \to M_1$. Going backwards, we can find a function in V, f, and a generator a_1 such that:

$$x = j_{E_0}\left(j_{E_1}(f(-, a_1))\right)(a_0) = j_{j_{E_0}(E_1)}\big(j_{E_0}(f)\big)\left(a_0, j_{E_0}(a_1)\right).$$

This element in obviously in $\mathrm{Ult}(M_0, j_{E_0}(E_1))$.

On the other hand, if $y \in \mathrm{Ult}(M_0, j_{E_0}(E_1))$ then there is some generator $a_1' \in \mathrm{dom}\, j_{E_0}(E_1)$ and a function g such that $y = j_{j_{E_0}(E_1)}(g)(a_1')$. By Lemma 2.3, we may assume that $a_1' = j_{E_0}(a_1)$. Let f be a function in V and a_0 be some generator such that $g = j_{E_0}(f)(a_0)$, then we have:

$$\begin{aligned} y &= j_{j_{E_0}(E_1)}\left(j_{E_0}(f)(a_0, -)\right)\left(j_{E_0}(a_1)\right) \\ &= j_{j_{E_0}(E_1)}\left(j_{E_0}\left(f(-, a_1)\right)(a_0)\right) \\ &= j_{E_0}\big(j_{E_1}(f)(-, a_1)\big)(a_0), \end{aligned}$$

as wanted. □

Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be a sequence of cardinals as in Theorem 1.1. Following work of Merimovich (see for example [13]), we can assume that the extenders E_α are of the form $\langle E_\alpha(d) \mid d \in [\lambda]^\kappa \rangle$ where for $X \subseteq \kappa_\alpha$, $X \in E_\alpha(d)$ if and only if $\{(j_{E_\alpha}(\xi), \xi) \mid \xi \in d\} \in j_{E_\alpha}(X)$. For $d \in [\lambda]^\kappa$ with $\kappa \in d$, it is easy to see that the measure $E_\alpha(d)$ concentrates on a set of order preserving functions ν with $\kappa \in \mathrm{dom}(\nu)$. So we assume that every measure one set mentioned below is of this form. We also fix functions $\langle \ell_\alpha \mid \alpha < \eta \rangle$ so that for every $\alpha < \eta$, $j_{E_\alpha}(\ell_\alpha)(\kappa_\alpha) = \lambda$. The existence of such functions can always be arranged by a simple preliminary forcing.

Using the set $d \in [\lambda]^\kappa$ to index the extenders E_α has the advantage that the projection maps from $E_\alpha(d')$ and $E_\alpha(d)$ for $d \subseteq d'$ can be made very explicit. The measure $E_\alpha(d')$ concentrates on partial functions from d' to κ with domain smaller than κ. Thus, the map $\nu \to \nu \restriction d$ is a projection from $E_\alpha(d')$ to $E_\alpha(d)$.

For every two cardinals $\kappa < \lambda$, let $\mathcal{A}(\lambda, \kappa)$ be the poset consisting of partial functions $f : \lambda \to \kappa$ with $|f| \leq \kappa$ and $\kappa \in \mathrm{dom}(f)$. Therefore $\mathcal{A}(\lambda, \kappa)$ is isomorphic to Cohen forcing for adding λ many subsets of κ^+.

We let $\mathbb{P}$ be Gitik's forcing from [5] defined from the sequence of extenders $\langle E_\alpha \mid \alpha < \eta \rangle$. We give a compact description of the forcing. A condition $p \in \mathbb{P}$ is a sequence $\langle p_\alpha \mid \alpha < \eta \rangle$ such that there is a finite set $s^p \subseteq \eta$ such that for each $\alpha < \eta$, $p_\alpha = (f_\alpha, \lambda_\alpha)$ if $\alpha \in s^p$, and $p_\alpha = (f_\alpha, A_\alpha)$ otherwise, and the following conditions hold.

(1) $f_\alpha \in \mathcal{A}(\lambda_{\alpha^*}, \kappa_\alpha)$ where α^* is the next element of s^p above α if it exists and $f_\alpha \in \mathcal{A}(\lambda, \kappa_\alpha)$ otherwise.
(2) For all $\alpha \in s^p$, λ_α is a cardinal and $\sup_{\beta<\alpha} \kappa_\beta < \lambda_\alpha < \kappa_\alpha$.
(3) For all $\alpha \in \eta \setminus s^p$, if $\alpha > \max(s^p)$, then $A_\alpha \in E_\alpha(\mathrm{dom}(f_\alpha))$, otherwise if α^* is the least element of s^p above α, then $A_\alpha \in E_\alpha(\mathrm{dom}(f_\alpha))$.
(4) For $\alpha \notin s^p$, $f^p_\alpha(\kappa_\alpha) = 0$ (This gives a clean way to distinguish between Cohen functions associated to members of s^p and those which are not.)
(5) The sequence $\langle \mathrm{dom}(f_\alpha) \mid \alpha < \eta \rangle$ is increasing.

We adopt the convention of adding a superscript f^p_α, A^p_α, etc. to indicate that each component belongs to p. When the value of λ_{α^*} might behave non-trivially, we will add it as a third coordinate to the pairs p_α, where $\alpha \notin s^p$. We call $\eta \setminus s^p$ the *pure part* of p and s^p the *non-pure part* of p.

We briefly sketch the notion of extension. p is a direct extension of q if $s^p = s^q$ and for all α f^p_α is stronger than f^q_α and A^p_α projects to a subset of A^q_α using the natural Rudin-Keisler projection from $E_\alpha(\mathrm{dom}\, f^p_\alpha)$ to $E_\alpha(\mathrm{dom}\, f^q_\alpha)$.

Let us describe now the one point extension. Suppose that $\nu \in A^p_\beta$. We let $q = p \frown \nu$ be the condition with $s^q = s^p \cup \{\beta\}$ and the following.

(1) $f^q_\beta = f^p_\beta{}^\frown \nu$ is the overwriting of f^p_β by ν, that is
$$(f^p_\beta{}^\frown \nu)(\tau) = \begin{cases} \nu(\tau) \text{ if } \tau \in \mathrm{dom}(\nu) \\ f^p_\beta(\tau) \text{ otherwise.} \end{cases}$$
(2) $\lambda^q_\beta = \ell_\beta(\nu(\kappa_\beta))$.
(3) For $\alpha \in [\max(s^p) \cap \beta, \beta)$, $f^q_\alpha = f^p_\alpha \circ \nu^{-1}$ and $A^q_\alpha = \{\xi \circ \nu^{-1} \mid \xi \in A^p_\alpha\}$ if applicable.

The following analysis of dense open sets of $\mathbb{P}$ was established in [5].

Lemma 2.5. *For every condition $p \in \mathbb{P}$ and dense open set $D \subseteq \mathbb{P}$, there are $p^* \geq^* p$ and a finite subset $\{\alpha_0, \ldots, \alpha_{m-1}\}$ of the pure part of p, such that for every sequence $\vec{\nu} = \langle \nu_{\alpha_0}, \ldots, \nu_{\alpha_{m-1}} \rangle \in \prod_{i<m} A^{p^*}_{\alpha_i}$, $p^*{}^\frown \vec{\nu} \in D$.*

For limit $\delta \leq \eta$, we define $\bar{\kappa}_\delta = \sup_{\alpha<\delta} \kappa_\alpha$. Gitik used the previous lemma to prove:

Theorem 2.6. *In the generic extension by $\mathbb{P}$, cardinals and cofinalities are preserved and for every limit $\delta \leq \eta$, the singular cardinal hypothesis fails at $\bar{\kappa}_\delta$.*

For use later, we define $\vec{\mathcal{A}}$ to be the full-support product $\prod_{\alpha<\eta} \mathcal{A}(\lambda, \kappa_\alpha)$, and similarly, for each $\beta < \eta$, $\vec{\mathcal{A}}_{\geq\beta} = \prod_{\beta\leq\alpha<\eta} \mathcal{A}(\lambda, \kappa_\alpha)$. We aim to show that if $\vec{H} = \langle H(\alpha) \mid \alpha < \eta \rangle$ is $\vec{\mathcal{A}}$ generic over V, then there is a generic for the image of $\mathbb{P}$ in a suitable iterated ultrapower.

3. The iterated ultrapower M_η and the generic filter G^*

We consider the following iterated ultrapower

$$\langle M_\alpha, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \eta \rangle$$

by the extenders in $\vec{E}$. The iteration is defined by induction of α. Let $M_0 = V$. For every $\alpha < \eta$, given that $j_{0,\alpha} : M_0 \to M_\alpha$ has been defined, we take $E^\alpha_\alpha = j_{0,\alpha}(E_\alpha)$, and set $j_{\alpha,\alpha+1} : M_\alpha \to M_{\alpha+1} \cong \mathrm{Ult}(M_\alpha, E^\alpha_\alpha)$. At limit stages $\delta \leq \eta$ we take M_δ to be the direct limit of the system $\langle M_\alpha, j_{\alpha,\beta} : \alpha \leq \beta < \delta\rangle$, and $j_{\alpha,\delta}$ to be the limit maps.

For every $\beta \leq \eta$, we shall abbreviate and write j_β for $j_{0,\beta}$. For a given $\beta \leq \eta$ we shall denote $j_\beta(\kappa_\alpha), j_\beta(E_\alpha)$, $j_\beta(\lambda)$ by κ^β_α, E^β_α, and λ^β respectively. Similarly, we will denote $j_\beta(x) = x^\beta$ for objects $x \in V$ whose images along the iteration will be significant for our construction.

Lemma 3.1. *Let $\beta \leq \eta$. For every $x \in M_\beta$ there are $\beta_0 < \beta_1 < \cdots < \beta_{l-1}$ below β, ordinals $\tau_0, \ldots, \tau_{l-1}$ below λ, and $f : \prod_{i<l} \kappa_{\beta_i} \to V$, such that*

$$x = j_\beta(f)\left(j_{\beta_0}(\tau_0), j_{\beta_1}(\tau_1), \ldots, j_{\beta_{l-1}}(\tau_{l-1})\right).$$

Proof. It is immediate from the definition of the iteration that for every $x \in M_\beta$ there are $\beta_0 < \cdots < \beta_{l-1}$ below β, finite subsets of ordinals $a_0, \ldots, a_{l-1}$, with each $a_i \in [\lambda^{\beta_i}]^{<\omega}$ a generator of $E^{\beta_i}_{\beta_i}$, and a function $g : \prod_{i<l} \kappa^{<\omega}_{\beta_i} \to V$ so that

$$x = j_{0,\beta}(g)(j_{\beta_0+1,\beta}(a_0), \ldots, j_{\beta_{l-1}+1,\beta}(a_{l-1})).$$

Note that for each $i < l$, $j_{\beta_i+1,\beta}(a_i) = a_i$, because

$$\mathrm{crit}(j_{\beta_i+1,\beta}) = \kappa^{\beta_i+1}_{\beta_i+1} > \lambda^{\beta_i}.$$

It follows that

$$x = j_\beta(g)(a_0, a_1, \ldots, a_{l-1}).$$

We may further assume that each $a_i < \lambda^{\beta_i}$ is an ordinal. Finally, we claim that we may replace each a_i with an ordinal of the form $j_{\beta_i}(\tau_i)$, for some $\tau_i < \lambda$. By Lemma 2.3, in M_{β_i}, the extender $E^{\beta_i}_{\beta_i} = j_{\beta_i}(E_{\beta_i})$ is generated by a subset of its measures, which are of the form $j_{\beta_i}(E_{\beta_i}(\tau_i)) = E^{\beta_i}_{\beta_i}(j_{\beta_i}(\tau_i))$. Namely, for every $a_i \in \lambda^{\beta_i}$ there exists some $\tau_i \in \lambda$ such that $a_i \leq^{RK}_{E^{\beta_i}_{\beta_i}} j_{\beta_i}(\tau_i)$.

In particular

$$a_i = j_{\beta_i}(h)(b_1, \ldots, b_k) \leq^{RK}_{E^{\beta_i}_{\beta_i}} j_{\beta_i}(\tau_i).$$

The claim follows. $\square$

Having defined the iterated ultrapower M_η we proceed to introduce relevant conditions in $j_\eta(\mathbb{P})$. Let $p = \langle p_\alpha \mid \alpha < \eta\rangle$ be a pure condition in $\mathbb{P}$, that is, a condition with $s^p = \emptyset$. For every $\alpha < \eta$, we set $p_\alpha = \langle f_\alpha, A_\alpha, \lambda\rangle$, where $f_\alpha \in A(\lambda, \kappa_\alpha)$ and $A_\alpha \in E_\alpha(d_\alpha)$ where $d_\alpha = \mathrm{dom}(f_\alpha) \in [\lambda]^{\leq\kappa_\alpha}$.

Since $j_{\alpha,\alpha+1} = j^{M_\alpha}_{E^\alpha_\alpha}$ it follows that the function

$$\nu^{p^\alpha}_{\alpha,\alpha+1} := (j_{\alpha,\alpha+1} \restriction d^\alpha_\alpha)^{-1}$$

belongs to $j_{\alpha,\alpha+1}(A^\alpha_\alpha) = A^{\alpha+1}_\alpha$, which is the measure one set of the α-th component of $j_{\alpha+1}(p)$. By applying $j_{\alpha+1,\eta}$ to $j_{\alpha+1}(p)$, we conclude that the function

$$\nu^{p^\alpha}_{\alpha,\eta} := j_{\alpha+1,\eta}(\nu^{p^\alpha}_{\alpha,\alpha+1}) = (j_{\alpha,\eta} \restriction d^\alpha_\alpha)^{-1}$$

belongs to $j_\eta(A_\alpha)$, and thus $j_\eta(p)$ has an one point extension at the α-th coordinate of the form

$$j_\eta(p)^\frown\langle \nu^{p^\alpha}_{\alpha,\eta}\rangle.$$

It is clear that for every finite sequence $\alpha_0 < \cdots < \alpha_{m-1}$ of ordinals below η, and pure condition $p \in \mathbb{P}$, we have $j_\eta(p)^\frown\langle \nu^{p^{\alpha_0}}_{\alpha_0,\eta}, \nu^{p^{\alpha_1}}_{\alpha_1,\eta}, \dots, \nu^{p^{\alpha_{m-1}}}_{\alpha_{m-1},\eta}\rangle$ is an extension of $j_\eta(p)$ in $j_\eta(\mathbb{P})$. It is also clear that if $p, q \in \mathbb{P}$ are two pure conditions so that for all $\alpha < \eta$, f^p_α, f^q_α are compatible, then for every two sequences $\alpha_0 < \cdots < \alpha_{m-1}$ and $\beta_0 < \cdots < \beta_{l-1}$ of ordinals below η, the conditions

$$j_\eta(p)^\frown\langle \nu^{p^{\alpha_0}}_{\alpha_0,\eta}, \nu^{p^{\alpha_1}}_{\alpha_1,\eta}, \dots, \nu^{p^{\alpha_{m-1}}}_{\alpha_{m-1},\eta}\rangle$$

and

$$j_\eta(q)^\frown\langle \nu^{q^{\beta_0}}_{\beta_0,\eta}, \nu^{q^{\beta_1}}_{\beta_1,\eta}, \dots, \nu^{q^{\beta_{l-1}}}_{\beta_{l-1},\eta}\rangle$$

are compatible in $j_\eta(\mathbb{P})$. We remark that in the assumption that the sequence of extenders $\langle E_\alpha \mid \alpha < \eta\rangle$ is Mitchell order increasing is used in order to be able to permute the order in which the extensions are done, using a sequence of applications of Lemma 2.4.

Let $\vec{H} = \langle H(\alpha) \mid \alpha < \eta\rangle$ be V-generic for $\vec{\mathcal{A}} = \prod_{\alpha<\eta} \mathcal{A}(\lambda, \kappa_\alpha)$.

Definition 3.2. Define $G^* \subset j_\eta(\mathbb{P})$ to be the filter generated by all conditions

$$j_\eta(p)^\frown\langle \nu^{p^{\alpha_0}}_{\alpha_0,\eta}, \nu^{p^{\alpha_1}}_{\alpha_1,\eta}, \dots, \nu^{p^{\alpha_{m-1}}}_{\alpha_{m-1},\eta}\rangle,$$

where $p \in \mathbb{P}$ satisfy that $f^p_\alpha \in H(\alpha)$ for all $\alpha < \eta$, $m < \omega$, and $\alpha_0 < \cdots < \alpha_{m-1}$ are ordinals below η.

Our first goal is to prove that G^* generates a $j_\eta(\mathbb{P})$ generic filter over M_η.

Proposition 3.3. *G^*_η is $j_\eta(\mathbb{P})$ generic over M_η.*

Before moving to the proof of the proposition, we discuss finite subiterates of M_η.

3.1. **Finite sub-iterations N^F.** The model M_η can be seen as a directed limit of all its finite subiterates, N^F, $F \in [\eta]^{<\omega}$. Given a finite set $F = \{\beta_0, \dots, \beta_{l-1}\} \in [\eta]^{<\omega}$, we define its associated iteration $\langle N^F_i, i^F_{m,n} \mid m \leq n \leq l\rangle$ by $N^F_0 = V$, and $i^F_{n,n+1} : N^F_n \to N^F_{n+1} \cong \mathrm{Ult}(N^F_n, i^F_n(E_{\beta_n}))$. Since for the moment we handle a single finite set at a time, we will suppress the mention of the finite set F and refer only to the iteration as $i_{m,n} : N_m \to N_n$ for $m \leq n \leq l$ and as usual we set $i_n = i_{0,n}$ for $n \leq l$.

The proof of Lemma 3.1 shows that the elements of N_l are of the form $i_l(f)\,(\tau_0, i_1(\tau_1), \dots, i_{l-1}(\tau_{l-1}))$. Let $k : N_l \to M_\eta$ be the usual factor map defined by

$$k(i_l(f)\,(\tau_0, i_1(\tau_1), \dots, i_{l-1}(\tau_{l-1}))) = j_\eta(f)\left(j_{\beta_0}(\tau_0), j_{\beta_1}(\tau_1), \dots, j_{\beta_{l-1}}(\tau_{l-1})\right)$$

It is routine to verify that k is well defined, elementary and $j_\eta = k \circ i_l$.

Moreover, following this explicit description of k, it is straightforward to verify that $k : N_l \to M_\eta$ is the resulting iterated ultrapower limit embedding, associated to the iteration of N_l by the sequence

$$\langle i_{0,n_\alpha}(E_\alpha) \mid \alpha \in \eta \setminus \{\beta_0, \dots \beta_{l-1}\}\rangle,$$

where n_α is the minimal $n < l$ for which $\beta_n \geq \alpha$, if exists, and $n_\alpha = l$ otherwise. This again uses the Mitchell order assumption of the sequence of extenders in order to get the desired commutativity.

Next, we observe that our assignment of generators $\nu^{p^\alpha}_{\alpha,\eta}$, $\alpha < \eta$, to $j_\eta(p)$ of conditions $p \in \mathbb{P}$ can be defined at the level of the finite subiterates.

Indeed, for a condition $p \in \mathbb{P}$, $n < l$, we temporarily define

$$\nu^{i_n(p)}_{n,n+1} = (i_{n,n+1} \restriction d)^{-1} = \{(i_{n,n+1}(\tau), \tau) \mid \tau \in d\},$$

where $d = \operatorname{dom}(f^{i_n(p)})_{\beta_n}$. We let $\nu^{i_n(p)}_{n,l} = i_{n+1,l}(\nu^{i_n(p)}_{n,n+1})$ be the natural push forward to N_l. As with the conditions $j_\eta(p)$, we have that

$$i_l(p)^\frown\langle \nu^{i_0(p)}_{0,l}, \dots, \nu^{i_{l-1}(p)}_{l-1,l}\rangle$$

is a valid extension of $i_l(p)$ in $i_l(\mathbb{P})$.

To record this definition (which depends on F), we make a few permanent definitions which explicitly mention F. First, we record the embeddings $i^F = i_l$ and $k^F = k$ and the model $N^F = N_l$. We denote the condition defined above by $i^F(p)^\frown\vec{\nu}^{p,F}$ and refer to it as the *natural non-pure extension of* $i^F(p)$.

Using the description of $k^F : N^F \to M_\eta$, it is straightforward exercise in applying Łoś's Theorem to show that $k^F(\vec{\nu}^{p,F}) = \langle \nu^{p^{\beta_n}}_{\beta_n,\eta} \mid n < l\rangle$, and conclude that

$$k^F\left(i^F(p)^\frown\vec{\nu}^{p,F}\right) = j_\eta(p)^\frown\langle \nu^{p^{\beta_0}}_{\beta_0,\eta}, \dots, \nu^{p^{\beta_{l-1}}}_{\beta_{l-1},\eta}\rangle.$$

Finally, we note that the forcing $\vec{\mathcal{A}} = \prod_{\alpha<\eta} \mathcal{A}(\lambda, \kappa_\alpha)$ has a natural factorization, associated with F. Setting $\beta_{-1} = 0$ and $\beta_l = \eta$, we have

$$\vec{\mathcal{A}} = \prod_{n \leq l} \vec{\mathcal{A}} \restriction [\beta_{n-1}, \beta_n),$$

where for each $n \leq l$,

$$\vec{\mathcal{A}} \restriction [\beta_{n-1}, \beta_n) = \prod_{\beta_{n-1} \leq \alpha < \beta_n} \mathcal{A}(\lambda, \kappa_\alpha).$$

For each $0 \leq n \leq l$, we define

$$\vec{\mathcal{A}}^F_n = i_n(\vec{\mathcal{A}} \restriction [\beta_{n-1}, \beta_n))$$

and denote the resulting product by $\vec{\mathcal{A}}^F = \prod_{n \leq l} \vec{\mathcal{A}}^F_n$. Suppose that $\vec{H} \subseteq \vec{\mathcal{A}}$ is a V-generic filter. For each $n \leq l$, the fact $\vec{\mathcal{A}} \restriction [\beta_{n-1}, \beta_n)$ is a $\kappa^+_{\beta_{n-1}}$-closed forcing guarantees that $i_n``\vec{H} \restriction [\beta_{n-1}, \beta_n)) \subseteq \vec{\mathcal{A}}^F_n$ forms a generic filter over N_n, hence also $N_l = N^F$. For each n, we denote the resulting N_l generic for $\vec{\mathcal{A}}^F_n$ by $\vec{J}^F_n$.

It is clear that the product $\prod_{0 \leq n \leq l} \vec{J}^F_n$ is $\vec{\mathcal{A}}^F$ generic over N_l. We note that for each pure $p \in \mathbb{P}$ with $\vec{f}^p \in \vec{H}$, if $\vec{f}$ is the Cohen part of the natural non-pure extension of $i^F(p)$, then $\vec{f} \in \vec{\mathcal{A}}^F$. Moreover the collection of such $\vec{f}$ form a $\vec{\mathcal{A}}^F$-generic filter over N^F which is obtained by modifying $\prod_{0 \leq n \leq l} \vec{J}^F_n$ on the coordinates β_n for $n < l$ using the overwriting procedure used in the definition of the natural non-pure extension of $i^F(p)$. The individual genericity of the modified filters is immediate from Woodin's surgery argument [6]. The fact that the product remains generic follows from several straightforward applications of Easton's lemma. We denote by $\vec{H}^F$ the resulting N^F-generic filter over $\vec{\mathcal{A}}^F$.

Remark 3.4. G^* can be constructed in $N^F[\vec{H}^F]$ in the same way that it was constructed in V using the fact that M_η can be described as an iterated ultrapower of N^F and starting with conditions q such that $s^q = F$ and $\vec{f}^q \in \vec{H}^F$.

We turn to the proof of Proposition 3.3.

Proof. It is clear that $G^* \subseteq j_\eta(\mathbb{P})$ is a filter. We verify that G^* meets every dense open set $D \subseteq j_\eta(\mathbb{P})$ in M_η. Since M_η is the direct limit of its finite subiterates. There are $F = \{\beta_0, \dots \beta_{l-1}\} \in [\eta]^{<\omega}$ and $\bar{D}$ such that $k^F(\bar{D}) = D$.

Let p' be the natural non-pure extension of $i^F(p)$ for some p with $\vec{f}^p \in \vec{H}$. Now by its definition $\vec{f}^{p'} \in \vec{H}^F$. Appealing to Lemma 2.5 and the genericity of $\vec{H}^F$, we conclude that there exists a direct extension $p^* \geq^* p'$, with $\langle f^{p^*}_\alpha \mid \alpha < \eta \rangle \in \vec{H}^F$ and a finite set $\{\alpha_0, \dots, \alpha_{m-1}\} \subset (\eta \setminus F)$, such that $p^{*\frown}\vec{\nu} \in \bar{D}$ for every $\vec{\nu} = \langle \nu_{\alpha_0}, \dots, \nu_{\alpha_{m-1}} \rangle \in \prod_{i<m} A^{p^*}_{\alpha_i}$. By the elementarity of k^F, $k^F(p^*)^\frown \vec{\nu} \in D$ for every $\vec{\nu} \in \prod_{i<m} k^F(A^{p^*})_{\alpha_i}$.

In particular,

$$k^F(p^*)^\frown \langle \nu^{(p^*)^{\alpha_0}}_{\alpha_0,\eta}, \dots, \nu^{(p^*)^{\alpha_{m-1}}}_{\alpha_{m-1},\eta} \rangle \in \mathcal{D}.$$

It remains to verify that the last condition belongs to G^*. This is immediate from Remark 3.4 and the fact that $\vec{f}^{p^*} \in \vec{H}^F$. □

Proposition 3.5. *$M_\eta[G^*]$ is closed under $\kappa_0 = \mathrm{crit}(j_\eta)$ sequences of its elements in $V[\vec{H}]$.*

Proof. Let $\langle x_\mu \mid \mu < \kappa_0\rangle$ be a sequence of elements in $M_\eta[G^*]$. Since $M_\eta[G^*]$ is a model of ZFC, we may assume that all x_μ are ordinals. By Lemma 3.1, for each x_μ is of the form

$$x_\mu = j_\eta(g_\mu)(j_{\beta^\mu_0}(\tau^\mu_0), \dots, j_{\beta^\mu_{l^\mu-1}}(\tau^\mu_{l^\mu-1}))$$

for some finite sequences $\beta^\mu_0 < \dots, \beta^\mu_{l^\mu-1} < \eta$ and $\tau^\mu_0, \dots, \tau^\mu_{l^\mu-1} < \lambda$. Moreover, since the Rudin-Keisler order $\leq^{RK}_{E_{\beta^\mu_i}}$ is κ_0^+-directed, we may assume that there exists some $\tau^* < \lambda$ such that $\tau^\mu_i = \tau^*$ for all $\mu < \kappa_0$ and $i < l^\mu$. Hence for $\langle x_\mu \mid \mu < \kappa_0\rangle$ to be a member of $M_\eta[G^*]$, it suffices to verify that the sequences $\langle j_\eta(g_\mu) \mid \mu < \kappa_0\rangle$ and $\langle j_\alpha(\tau^*) \mid \alpha < \eta\rangle$ belong to $M_\eta[G^*]$.

The first sequence already belongs to M_η as $\mathrm{crit}(j_\eta) = \kappa_0$ implies that it is just $j_\eta(\vec{g}) \restriction \kappa_0$. The latter sequence $\langle j_\alpha(\tau^*) \mid \alpha < \eta\rangle$ can be recovered from G^* as follows. It follows from a simple density argument that for every $\alpha < \eta$, there exists some $q \in G^*$ such that

(1) α is a non-pure coordinate of q, and
(2) $j_\eta(\tau^*) \in \mathrm{dom}(f^q_\alpha)$.

It is also clear that two conditions $q, q' \in G^*$ of this form must satisfy $f^q_\alpha(j_\eta(\tau^*)) = f^{q'}_\alpha(j_\eta(\tau^*))$. We may therefore define in $M_\eta[G^*]$ a function $t_{j_\eta(\tau^*)} : \eta \to \lambda^\eta$ by $t_{j_\eta(\tau^*)}(\alpha) = f^q_\alpha(j_\eta(\tau^*))$ for some condition $q \in G^*$ as above. we claim that $t_{j_\eta(\tau^*)}(\alpha) = j_\alpha(\tau^*)$ for all $\alpha < \eta$. Indeed, for every $\alpha < \eta$, there exists a condition $p \in \mathbb{P}$ with $\langle f^p_\alpha \mid \alpha < \eta\rangle \in \vec{H}$ so that $\tau^* \in \mathrm{dom}(f^p_\alpha)$, and clearly, the condition $q = j_\eta(p)^\frown\langle \nu^{p^\alpha}_{\alpha,\eta}\rangle \in G^*$ has $j_\eta(\tau^*) \in \mathrm{dom}(f^q_\alpha)$. But $j_\eta(\tau^*) \in \mathrm{dom}(\nu^{p^\alpha}_{\alpha,\eta}) = j_{\alpha+1,\eta}``j_\alpha(\mathrm{dom}(f^p_\alpha))$ and $\nu^{p^\alpha}_{\alpha,\eta} = (j_{\alpha,\eta} \restriction d^\alpha_\alpha)^{-1}$, thus, it follows that

$$f^q_\alpha(j_\eta(\tau^*)) = \nu^{p^\alpha}_{\alpha,\eta}(j_\eta(\tau^*)) = j_\alpha(\tau^*).$$

□

Fix some ordinal $\beta < \eta$. The forcing $\vec{\mathcal{A}}$ naturally breaks into the product $\vec{\mathcal{A}} \restriction \beta \times \vec{\mathcal{A}}_{\geq\beta}$, and we observe that

(1) The latter part $\vec{\mathcal{A}}_{\geq\beta}$ is κ^+_β-closed. Therefore, if $\vec{H}_{\geq\beta} \subset \vec{\mathcal{A}}_{\geq\beta}$ is V-generic, then its pointwise image $j_\beta``\vec{H}_{\geq\beta}$ generates an M_β generic filter for the forcing
$$j_\beta(\vec{\mathcal{A}}_{\geq\beta}) = \prod_{\beta\leq\alpha<\eta} \mathcal{A}(\lambda^\beta, \kappa^\beta_\alpha)^{M_\beta}.$$
We denote this generic by $\vec{H}^\beta_{\geq\beta}$.
(2) Using the same arguments as above, a V-generic filter $\vec{H}_\beta \subset \vec{\mathcal{A}} \restriction \beta$ generates an M_β-generic filter G^*_β for the $j_\beta(\mathbb{P}_{\vec{E}\restriction\beta})$.

We conclude that in the model $V[\vec{H}]$, we can form the generic extension $M_\beta[G^*_\beta \times \vec{H}^\beta_{\geq\beta}]$ of M_β, with respect to the product $j_\beta(\mathbb{P}_{\vec{E}\restriction\beta}) \times j_\beta(\vec{\mathcal{A}}_{\geq\beta})$.
We have the following.

Proposition 3.6. *For each $\beta < \eta$, the model $M_\beta[G^*_\beta \times \vec{H}^\beta_{\geq\beta}]$ can compute $M_\eta[G^*_\eta]$. In fact,*

$$\bigcap_{\beta<\eta} M_\beta[G^*_\beta \times \vec{H}^\beta_{\geq\beta}] = M_\eta[G^*].$$

4. Stationary reflection in $M_\eta[G^*_\eta]$

We assume that for every $\alpha < \eta$, there is an indestructible supercompact cardinal θ_α such that $\sup_{\beta<\alpha} \kappa_\beta < \theta_\alpha < \kappa_\alpha$.

We prove the following which completes the proof of Theorem 1.1.

Theorem 4.1. *In $M_\eta[G^*]$, every collection of fewer than η many stationary subsets of $j_\eta(\bar\kappa^+_\eta)$ reflects.*

We start by proving a stationary reflection fact that will be used as an intermediate step in the proof.

Claim 4.2. *For every $\alpha < \eta$, every collection of fewer than η many stationary subsets of $j_\alpha(\bar\kappa^+_\eta)$ with cofinalities bounded by $\sup_{\beta<\alpha} \kappa^\alpha_\beta$ reflects in $M_\alpha[G^*_\alpha \times \vec{H}^\alpha_{\geq\alpha}]$.*

Proof. Let T_i for $i < \mu$ be such a collection of stationary sets. By elementarity, $j_\alpha(\theta_\alpha)$ is an indestructible supercompact cardinal between $\sup_{\beta<\alpha} \kappa^\alpha_\beta$ and κ^α_α. Recall that $\vec{H}^\alpha_{\geq\alpha}$ is generic for $(\kappa^\alpha_\alpha)^+$-directed closed forcing. By the indestructibility of $j_\alpha(\theta_\alpha)$, there is a $j_\alpha(\bar\kappa^+_\eta)$-supercompact embedding $k : M_\alpha[\vec{H}^\alpha_{\geq\alpha}] \to N$. Further, G^*_α is generic for $j_\alpha(\bar\kappa^{++}_\alpha)$-cc forcing and $j_\alpha(\bar\kappa^{++}_\alpha) < j_\alpha(\theta_\alpha)$. By standard arguments, we can extend k to include $M_\alpha[G^*_\alpha \times \vec{H}^\alpha_{\geq\alpha}]$ in the extension by a $j_\alpha(\bar\kappa^{++}_\alpha)$-cc forcing. Each T_i remains stationary in this extension, so $\{k(T_i) \mid i < \mu\}$ reflects at $\sup k``j_\alpha(\bar\kappa^+_\eta)$. It follows that $\{T_i \mid i < \mu\}$ reflects in $M_\alpha[G^*_\alpha \times \vec{H}^\alpha_{\geq\alpha}]$. $\square$

We begin the proof of Theorem 4.1. Suppose that for each $i < \mu$, $S_i \in M_\eta[G^*]$ is a stationary subset of $j_\eta(\bar\kappa^+_\eta)$. We can assume that for all i, the cofinality of each ordinal in S_i is some fixed γ_i. It follows that there is some $\bar\alpha < \eta$ such that $\sup_{i<\mu} \gamma_i < \sup_{\beta<\bar\alpha} \kappa^{\bar\alpha}_\beta$. For each α in the interval $[\bar\alpha, \eta)$, let $T^\alpha_i = \{\delta < j_\alpha(\bar\kappa^+_\eta) \mid j_{\alpha,\eta}(\delta) \in S_i\}$.

Claim 4.3. *For $\alpha \geq \bar\alpha$, if $\{T^\alpha_i \mid i < \mu\}$ reflects at an ordinal of cofinality less than κ^α_α in $M_\alpha[G^*_\alpha \times \vec{H}^\alpha_{\geq\alpha}]$, then $\{S_i \mid i < \mu\}$ reflects in $M_\eta[G^*]$.*

Proof. Let $\delta < j_\alpha(\bar\kappa^+_\eta)$ with $\mathrm{cf}(\delta) < \kappa^\alpha_\alpha$ be a common reflection point of the collection $\{T^\alpha_i \mid i < \mu\}$. We claim that each S_i reflects at $j_{\alpha,\eta}(\delta)$. Let $D \subseteq j_{\alpha,\eta}(\delta)$ be club in $M_\eta[G^*]$, of order type $\mathrm{cf}(\delta) = \mathrm{cf}(j_{\alpha,\eta}(\delta))$.

Since $\mathrm{crit}\, j_{\alpha,\eta} > \mathrm{cf}(\delta)$, $j_{\alpha,\eta}$ is continuous at δ and $E = \{\gamma < \delta \mid j_{\alpha,\eta}(\gamma) \in D\}$ is a club in δ. Note that $E \in M_\alpha[G^*_\alpha \times \vec{H}^\alpha_{\geq\alpha}]$. Since T^α_i reflects at δ, $T^\alpha_i \cap E \neq \emptyset$ and hence $S_i \cap D \neq \emptyset$. $\square$

Combining the previous two claims if for some $\alpha \geq \bar{\alpha}$, $\{T^\alpha_i \mid i < \mu\}$ consists of stationary sets, then $\{S_i \mid i < \mu\}$ reflects. So we assume for a contradiction that for each $\alpha \geq \bar{\alpha}$, there are $i_\alpha < \mu$ and a club $C_\alpha \in M_\alpha[G^*_\alpha \times \vec{H}^\alpha_{\geq \alpha}]$ such that $T^\alpha_{i_\alpha} \cap C_\alpha = \emptyset$. We fix $J \subseteq \eta$ unbounded and $i^* < \mu$ such that for all $\alpha \in I$, $i_\alpha = i^*$.

Let I_η be the ideal of bounded subsets of η. For $\alpha \leq \eta$, let $\vec{H}^\alpha/I_\eta$ be the generic for $j_\alpha(\vec{\mathbb{A}})/I_\eta$ derived from j_α"$\vec{H}$. Recall that $\vec{H}$ is generic for $\vec{\mathbb{A}}$ which is a product of Cohen posets, so this makes sense.

Claim 4.4. *For $\alpha \geq \bar{\alpha}$, there is a club subset of C_α in $M_\alpha[\vec{H}^\alpha/I_\eta]$.*

Proof. We start by showing that for all $\beta > \alpha$ below η there is a club subset of C_α in $M_\alpha[\vec{H}^\alpha]$.

To see this, note that $M_\alpha[G^*_\alpha \times \vec{H}^\alpha_{\geq \alpha}]$ is $(\bar{\kappa}^\alpha_\beta)^{++}$-cc extension of $M_\alpha[\vec{H}^\alpha \restriction [\beta, \eta)]$ and $\bar{\kappa}^\alpha_\beta < j_\alpha(\bar{\kappa}_\eta)$. For the moment we let $\mathbb{Q}_\beta$ denote the $(\bar{\kappa}^\alpha_\beta)^{++}$-cc poset used in this extension.

We set take $C_{\alpha,\beta}$ to be the set of closure points of the function assigning each γ to the supremum of the set

$$\{\gamma^* \mid \exists q \in \mathbb{Q}_\beta,\ q \Vdash \check{\gamma}^* = \min(\dot{C}_\alpha \setminus (\check{\gamma} + 1))\},$$

as computed in the model $M_\alpha[\vec{H}^\alpha \restriction [\beta, \eta)]$. Clearly this is a club. Further if $\beta < \beta'$, then $C_{\alpha,\beta'} \subseteq C_{\alpha,\beta}$.

Since the clubs are decreasing, $\bigcap_{\beta > \alpha} C_{\alpha,\beta}$ is definable in $M_\alpha[\vec{H}^\alpha/I_\eta]$ as the set of ordinals γ such that for some condition $\vec{a} \in j_\alpha(\vec{\mathbb{A}})$, $\vec{a}/I_\eta \in \vec{H}^\alpha/I_\eta$ and for all sufficiently large β, $\vec{a} \restriction [\beta, \eta)$ forces $\gamma \in C_{\alpha,\beta}$. So $\bigcap_{\beta > \alpha} C_{\alpha,\beta}$ is as required for the claim. $\square$

Let $\dot{D}_\alpha$ be a $\vec{H}^\alpha/I_\eta$-name for the club from the previous lemma.

Claim 4.5. $\langle j_{\alpha,\omega}(\dot{D}_\alpha)_{\vec{H}^\eta/I_\eta} \mid \bar{\alpha} \leq \alpha < \eta \rangle \in M_\eta[G^*]$.

Proof. It is clear from the definition of G^* that $\vec{H}^\eta/I_\eta \in M_\eta[G^*]$. Further by Proposition 3.5, $M_\eta[G^*]$ is closed under η-sequences. So the sequence of names for clubs $\langle j_{\alpha,\eta}(\dot{D}_\alpha) \mid \alpha < \eta \rangle \in M_\eta[G^*]$. $\square$

To get a contradiction it is enough to show that $\bigcap_{\bar{\alpha} \leq \alpha < \eta} j_\alpha(\dot{D}_\alpha)_{\vec{H}^\eta/I_\eta} \cap S_{i^*} = \emptyset$. Suppose that there is some δ in the intersection. We can find some $\alpha \in J$ and $\bar{\delta}$ such that $j_{\alpha,\eta}(\bar{\delta}) = \delta$. However, by the definitions of D_α and $T^\alpha_{i^*}$, we must have that $\bar{\delta} \in C_\alpha \cap T^\alpha_{i^*}$, a contradiction.

This completes the proof of Theorem 1.1.

5. Bad scales

In this section we give the proof of Theorem 1.2. For this theorem we work with the forcing $\mathbb{P}$ as before and assume that there is an indestructibly supercompact cardinal $\theta < \kappa_0$. Working in V, let $\vec{f}$ be a scale of length $\bar{\kappa}^+_\eta$

in $\prod_{\alpha<\eta}\kappa_\alpha^+$. As before let $\vec{H}$ be generic for $\vec{\mathcal{A}}$ over V and let G^* be the $j_\eta(\mathbb{P})$ generic over M_η defined above.

Lemma 5.1. *In $M_\eta[G^*]$, $j_\eta(\vec{f})$ is a scale of length $j_\eta(\bar{\kappa}_\eta^+)$ in $\prod_{\alpha<\eta} j(\kappa_\alpha)^+$.*

Proof. Let $g \in \left(\prod_{\alpha<\eta} j(\kappa_\alpha)^+\right) \cap M_\eta[G^*]$. Clearly $g \in V[\vec{H}]$. Since each κ_α^+ is a continuity point of j_η, we can find an ordinal $\gamma_\alpha < \kappa_\alpha^+$ such that $j_\eta(\gamma_\alpha) > g(\alpha)$.

By the distributivity of $\vec{\mathbb{A}}$, the sequence $\tilde{g} = \langle \gamma_\alpha \mid \alpha < \eta\rangle$ belongs to V. Pick (in V) an ordinal ζ such that f_ζ dominates $\tilde{g}$. Then, $j_\eta(f_\zeta) = j_\eta(f)_{j_\eta(\zeta)}$ dominates g. □

Lemma 5.2. *For $\delta < \kappa_\eta^+$ with $\eta < \mathrm{cf}(\delta) < \kappa_0$, if $j_\eta(\delta)$ is a good point for $j(\vec{f})$ in $M_\eta[G^*]$, then δ is a good point for $\vec{f}$ in $V[\vec{H}]$.*

Proof. Suppose that $A \subseteq j_\eta(\delta)$ and $\alpha^* < \eta$ witness that $j_\eta(\delta)$ is good for $\vec{f}$ in $M_\eta[G^*]$. Note that j_η“δ is cofinal in $j(\delta)$. By thinning A if necessary we let $B \subseteq j_\eta$“δ be an unbounded such that each element $\gamma \in B$ has a greatest element of A less than or equal to it. For each γ in B, let α_γ be such that for all $\alpha \geq \alpha_\gamma$,

$$j(\vec{f})_{\max(A\cap(\gamma+1))}(\alpha) \leq j(\vec{f})_\gamma(\alpha) < j(\vec{f})_{\min(A\setminus(\gamma+1))}(\alpha).$$

Let B' be an unbounded subset of B on which α_γ is fixed. Then B' witnesses that $j_\eta(\delta)$ is good. Since $B' \subseteq j_\eta$“δ, we have that $\{\gamma < \delta \mid j_\eta(\gamma) \in B'\}$ witnesses that δ is good for $\vec{f}$ in $V[\vec{H}]$. □

In $V[\vec{H}]$, let $S = \{\delta < \bar{\kappa}_\eta^+ \mid \delta \text{ is nongood for } \vec{f} \text{ and } \eta < \mathrm{cf}(\delta) < \theta\}$. We claim that S is stationary. Let $k : V[\vec{H}] \to N$ be an elementary embedding witnessing that θ is $\bar{\kappa}_\eta^+$-supercompact in $V[\vec{H}]$. Standard arguments show that $\sup k$“$\bar{\kappa}_\eta^+$ is a nongood point for $k(\vec{f})$. It follows that S is stationary, since $\sup k$“$\bar{\kappa}_\eta^+ \in k(C)$ every club $C \subseteq \bar{\kappa}_\eta^+$ in $V[\vec{H}]$.

Now suppose that in $M_\eta[G^*]$, there is a club D of good points for $j_\eta(\vec{f})$. In $V[\vec{H}]$, let $C = \{\delta < \bar{\kappa}_\eta^+ \mid j_\eta(\delta) \in D\}$. By the previous lemma, C is a $< \kappa_0$-club consisting of good points for $\vec{f}$. However, $S \cap C$ is nonempty, a contradiction.

6. Countable cofinality

Our goal is to show that given the assumptions of Theorem 1.4, there exists forcing extension which adds generic for the extender based Prikry forcing by a (κ, κ^{++})-extender (in particular, forces $\mathrm{cf}(\kappa) = \omega$ and $2^\kappa = \kappa^{++}$) and satisfies that every stationary subset of κ^+ reflects.

A necessary step for obtaining the latter is to add a club $D \subseteq \kappa^+$ disjoint from $S^{\kappa^+}_\kappa$, the set of ordinals $\alpha < \kappa^+$ of cofinality κ in the ground model. In [11], the second and third authors address the situation for adding a single

Prikry sequence to κ. It is shown that under the subcompactness assumption of κ, there is a Prikry-type forcing which both singularizes κ and adds a club D as above, without generating new nonreflecting stationary sets.

An additional remarkable aspect of the argument of [11] is that it presents a fertile framework in which the arguments address an extension $N_\omega[\mathcal{H}]$ of an iterated ultrapower N_ω of V, and assert directly that stationary reflection holds in $N_\omega[\mathcal{H}]$ without having to specify the poset by which $\mathcal{H}$ is added.

Let us briefly describe the key ingredients of the construction of [11], to serve as a reference for our arguments in the context of the failure of SCH. In [11], one starts from a normal measure U on κ in V, and consider the ω-iterated ultrapower by U, given by

$$N_0 = V,\, i_{n,n+1} : N_n \to N_{n+1} \cong \mathrm{Ult}(N_n, i_n(U)),$$

and the direct limit embedding $i_\omega : V \to N_\omega$. It is well known that the sequence of critical points $\langle \kappa_n \mid n < \omega\rangle$, $\kappa_{n+1} = i_{n,n+1}(\kappa_n)$ is Prikry generic over N_ω and that $N_\omega[\langle \kappa_n \mid n < \omega\rangle] = \bigcap_n N_n$ ([1, 3]).

Let $\mathbb{Q}$ be the Prikry name for the forcing for adding a disjoint club from $(S^{\kappa^+}_\kappa)^V = \kappa^+ \cap \mathrm{Cof}^V(\kappa)$ and $\mathbb{Q}_\omega = i_\omega(\mathbb{Q})^{\langle \kappa_n \mid n<\omega\rangle} \in N_\omega[\langle \kappa_n \mid n < \omega\rangle]$ is isomorphic to the forcing for adding a κ^+-Cohen set over V, and likewise, to adding a κ_n^+-Cohen set over N_n, for each $n < \omega$.

Moreover, taking a $\mathbb{Q}_\omega$-generic filter H over V, we have that both H and $i_{0,1}$“H generate mutually generic filters over N_1 for $i_1(\mathbb{Q}_\omega) = \mathbb{Q}_\omega$. More generally, for each n, the sequence $H^n_0, \dots, H^n_n$, where each H^n_k is generated by $i_{k,n}$“$H \subset i_n(\mathbb{Q}_\omega) = \mathbb{Q}_\omega$ for $1 \le k \le n$, are mutually generic filters for $\mathbb{Q}_\omega$ over N_n. With this choice of “shifts” of H, we obtain that for each $n < k$, the standard iterated ultrapower map $i_{n,k} : N_n \to N_k$ extends to $i^*_{n,k} : N_n[\langle H^n_0, \dots, H^n_n\rangle] \to N_k[\langle H^k_0, \dots, H^k_n\rangle] \subseteq N_k[\langle H^k_0, \dots, H^k_k\rangle]$. For each n, the sequence $\langle H^n_0, \dots, H^n_n\rangle$ is denoted by $\mathcal{H}_n$. The final extension $N_\omega[\mathcal{H}]$ of N_ω is given by the sequence $\mathcal{H} = \langle H^\omega_n \mid n < \omega\rangle$, were H^ω_n is the filter generated by $i_{n,\omega}$“H, which achieves the critical equality

$$N_\omega[\mathcal{H}] = \bigcap_n N_n[\mathcal{H}_n].$$

From this equality it follows at once that:

(1) $N_\omega[\mathcal{H}]$ is closed under its κ-sequences;
(2) $\kappa_\omega = i_\omega(\kappa)$ is singular in $N_\omega[\mathcal{H}]$, as $\langle \kappa_n \mid n < \omega\rangle$ belongs to each $N_n[\mathcal{H}_n]$; and
(3) $H \in N_\omega[\mathcal{H}]$, as $H = H^n_n$ for all $n < \omega$.

Therefore every stationary subset S of κ^+_ω in $N_\omega[\mathcal{H}]$ can be assumed to concentrate at some cofinality $\rho < \kappa_\omega$. Say for simplicity that $\rho < \kappa_0$, one shows that S reflects in $N_\omega[\mathcal{H}]$ by examining its pull backs $S_n = i^{-1}_{n,\omega}(S) \subseteq \kappa^+_n$. If we have that $\kappa_n = i_n(\kappa)$ is Π^1_1-subcompact in the Cohen extension $N_n[\mathcal{H}_n]$ of N_n, then we have that if S_n is stationary then it must reflect at some $\delta < \kappa^+_n$ of cofinality $\delta < \kappa_n$. This can then translated by $i_{n,\omega}$ to S reflecting at $i_{n,\omega}(\delta)$ in $N_\omega[\mathcal{H}]$. To rule out the other option, of having all

$S_n \subseteq \kappa_n^+$ being nonstationary in $N_n[\mathcal{H}_n]$, one takes witnessing disjoint clubs $C_n \subseteq \kappa_n^+$ and uses the fact that for each $n < \omega$, $i_{n,\omega} : N_n \to N_\omega$ extends to $i^*_{n,\omega} : N_n[\mathcal{H}_n] : N_\omega[i_{n,\omega}``\mathcal{H}_n] \subset N_\omega[\mathcal{H}]$. This allows us to show that the club $D_n = i^*_{n,\omega}(C_n) \subseteq \kappa_\omega^+$ belongs to $N_\omega[\mathcal{H}]$ for each n. Since $N_\omega[\mathcal{H}]$ is closed under its κ sequences, it computes $\langle D_n \mid n < \omega\rangle$ correctly and thus also $D = \bigcap_n D_n$, that would have to be disjoint from S, a contradiction.

We turn now to the new construction and prove Theorem 1.4. Let V' be a model which contains a κ^+-Π^1_1-subcompact cardinal κ, which also carries a (κ, κ^{++})-extender.

Recall that κ is κ^+-Π^1_1-subcompact if for every set $A \subseteq H(\kappa^+)$ and every Π^1_1-statement Φ such that $\langle H(\kappa^+), \in, A\rangle \models \Phi$, there are $\rho < \kappa$, $B \subseteq H(\rho^+)$, and an elementary embedding $j : \langle H(\rho^+), \in, B\rangle \to \langle H(\kappa^+), \in, A\rangle$ with $cp(j) = \rho$, and $\langle H(\rho^+), \in, B\rangle \models \Phi$.

Let V be obtained from V' by an Easton-support iteration of products $\mathrm{Add}(\alpha^+, \alpha^{++})$ for inaccessible $\alpha \leq \kappa$.

By Lemma 42 of [11], κ remains κ^+-Π^1_1-subcompact in V and even in a further extension by $\mathrm{Add}(\kappa^+, \kappa^{++})$. Moreover, by standard argument it is routine to verify that κ still carries a (κ, κ^{++})-extender in V. We note that as a consequence of κ^+-Π^1_1-subcompactness in V, simultaneous reflection holds for collections of fewer than κ many stationary subsets of $S^{\kappa^+}_{<\kappa}$. Further this property is indestructible under $\mathrm{Add}(\kappa^+, \kappa^{++})$.

Working in V, let E be a (κ, κ^{++})-extender. Let

$$\langle j_{m,n} : M_m \to M_n \mid m \leq n \leq \omega\rangle$$

be the iteration by E and

$$\langle i_{m,n} : N_m \to N_m \mid m \leq n \leq \omega\rangle$$

be the iteration by the normal measure E_κ where $V = M_0 = N_0$. We write j_n for $j_{0,n}$ and i_n for $i_{0,n}$.

We describe a generic extension of M_ω in which $j_\omega(\kappa^+)$ satisfies the conclusion of the theorem. It follows by elementarity that there is such an extension of V. We are able to isolate the forcing used, but this is not required in the proof.

We start by constructing a generic object for $j_\omega(\mathbb{P}_E)$ over M_ω, where $\mathbb{P}_E$ is the extender based forcing of Merimovich. Although for the most part, we will refer to Merimovich's arguments in [13], our presentation follows a more up-do-date presentation of the forcing, given by Merimovich in [14, Section 2].

We recall that conditions $p \in \mathbb{P}_E$ are pairs of the form $p = \langle f, T\rangle$, where $f : d \to [\kappa]^{<\omega}$ is a partial function from κ^{++} to $[\kappa]^{<\omega}$ with domain $d \in [\kappa^{++}]^{\leq\kappa}$ with $\kappa \in d$, and T is tree of height ω, whose splitting sets are all measure one with respect to a a measure $E(d)$ on V_κ, derived from the extender E. The generator of $E(d)$ is the function $\mathrm{mc}(d) = \{\langle j(\alpha), \alpha\rangle \mid \alpha \in d\}$. Therefore, a typical node in the tree is an increasing sequence of functions $\langle \nu_0, \ldots, \nu_{k-1}\rangle$ where each ν_i is a partial, order preserving function $\nu_i : \mathrm{dom}(\nu_i) \to \kappa$, with

$\kappa \in \mathrm{dom}(\nu_i)$ and $|\nu_i| = \nu_i(\kappa)$. The sequence $\langle \nu_0, \ldots, \nu_{k-1} \rangle$ is increasing in the sense that $\nu_i(\kappa) < \nu_{i+1}(\kappa)$ for all i. When extending conditions $p = \langle f, T \rangle \in \mathbb{P}_E$ we are allowed to (i) extend f in as Cohen conditions (namely, add points $\gamma < \kappa^{++}$ to $\mathrm{dom}(f)$ and arbitrarily choose $f(\gamma) \in [\kappa]^{<\omega}$), and modify the tree properly; (ii) shrink the tree T; and (iii) choose a point $\langle \nu \rangle \in \mathrm{succ}_{\emptyset}(T)$ to extend $p = \langle f, T \rangle$ to $p_{\langle \nu \rangle} = \langle f_{\langle \nu \rangle}, T_{\langle \nu \rangle} \rangle$, where $f_{\langle \nu \rangle}$ is defined by $\mathrm{dom}(f_{\langle \nu \rangle}) = \mathrm{dom}(f)$ and

$$f_{\langle \nu \rangle}(\alpha) = \begin{cases} f(\alpha) \cup \{\nu(\alpha)\} & \text{if } \alpha \in \mathrm{dom}(\nu) \text{ and } \nu(\alpha) > \max(f(\alpha)) \\ f(\alpha) & \text{otherwise} \end{cases}$$

Any extension of p is obtained by finite combination of (i)-(iii), and the direct extensions of p are those which are obtained by (i),(ii). The poset $\mathbb{P}^*_E$ is the suborder of $\mathbb{P}_E$ whose extension consists only of the Cohen type extension (i). Clearly, $\mathbb{P}^*_E$ is isomorphic to $\mathrm{Add}(\kappa^+, \kappa^{++})$.

We turn to describe the construction of a $j_\omega(\mathbb{P}_E)$ generic following [13]. Let G_0 be $\mathbb{P}^*_E$ generic. We work by induction to define G_n for $n < \omega$. Suppose that we have defined $G_n \subseteq j_n(\mathbb{P}^*_E)$ for some $n < \omega$. First, let G'_{n+1} be the closure of the set $j_{n,n+1}``G_n$. Then, we take G_{n+1} to be obtained from G'_{n+1} by adding the ordinal α to G'_{n+1} at coordinate $j_{n,n+1}(\alpha)$, for all $\alpha < j_n(\kappa^{++})$.

Claim 6.1. *G_{n+1} is M_{n+1}-generic for $j_{n+1}(\mathbb{P}^*_E)$.*

Proof. This is a straightforward application of Woodin's surgery argument [6], so we only sketch the proof. Let D be a dense open subset of $j_{n+1}(\mathbb{P}^*_E)$. Let E be the set of all f in D such that all $j_n(\kappa)$ sized modifications of f are in D. E is dense using the closure of $j_{n+1}(\mathbb{P}^*_E)$. Now the fact that G'_{n+1} meets E implies that G_{n+1} meets D, since each condition in G_{n+1} is a $j_n(\kappa)$ sized modification of one in G'_{n+1}. □

We make a few remarks.

Remark 6.2. For each n, G_n can be obtained directly from the upwards closure of $j_n``G_0$ by combining the alterations used in construction of G_i for $i < n$.

Since the proof of the previous claim can be repeated in any suitably closed forcing extension, we have the following.

Remark 6.3. If H is generic for $j_n(\kappa^+)$-closed forcing and mutually generic to the upwards closure of $j_n``G_0$, then H and G_n are mutually generic.

Let G_ω be the $j_\omega(\mathbb{P}_E)$-generic obtained from G_0 as in [13].

Claim 6.4. *$M_\omega[G_\omega]$ is closed under κ-sequences.*

Proof. It is enough to show that $M_\omega[G_\omega]$ is closed under κ-sequences of ordinals. Let $\langle \gamma_\delta \mid \delta < \kappa \rangle$ be a sequence of ordinals. We can assume that each

γ_δ is of the form $j_\omega(g_\alpha)(\alpha_\delta, j(\alpha_\delta), \dots j_{n_\delta - 1}(\alpha_\delta))$ for some $g_\delta : [\kappa]^{n_\delta} \to \kappa^{++}$ and $\alpha_\delta < \kappa^{++}$. We refer the reader to [13] Corollary 2.6 for a proof. Since $\langle j_\omega(g_\delta) \mid \delta < \kappa\rangle = j_\omega(\langle g_\delta \mid \delta < \kappa\rangle) \restriction \kappa \in M_\omega$, it is enough to show that $\langle(\alpha_\delta, \dots j_{n_\delta-1}(\alpha_\delta)) \mid \delta < \kappa\rangle \in M_\omega[G_\omega]$. To see this note that $\{\alpha_\delta \mid \delta < \kappa\} = \mathrm{dom}(f)$ for some $f \in G_0$ and that the sequence $f(\alpha_\delta) \frown (\alpha_\delta, \dots j_{n_\delta-1}(\alpha_\delta))$ is an initial segment of the ω-sequence with index $j_\omega(\alpha_\delta)$ in G_ω. This is enough to compute $\langle(\alpha_\delta, \dots j_{n_\delta-1}(\alpha_\delta)) \mid \delta < \kappa\rangle$, since j_ω"$f \in M_\omega$. $\square$

Lemma 6.5. $\bigcap_{n<\omega} M_n[G_n] = M_\omega[G_\omega]$.

Proof. We defined G_n so that generating the extender based generic in $M_n[G_n]$ with G_n as the starting Cohen generic gives exactly G_ω. It follows that $M_\omega[G_\omega] \subseteq \bigcap_{n<\omega} M_n[G_n]$. For the other direction suppose that $x \in \bigcap_{n<\omega} M_n[G_n]$ is a set of ordinals. Let $x_n \in M_n[G_n]$ be the set $\{\alpha \mid j_{n,\omega}(\alpha) \in x\}$.

Now we have a $j_n(\mathbb{P}^*_E)$-name $\dot{x}_n$ for x_n. We can view $\dot{x}_n$ as a $j_n(\mathbb{P}_E)$-name $\dot{x}^*_n$ by adding trees conditions in $j_n(\mathbb{P}^*_E)$. It follows that $\alpha \in x_n$ if and only if $j_{n,\omega}(\alpha)$ is in $j_{n,\omega}(\dot{x}^*_n)$ as evaluated by G_ω.

Since $M_\omega[G_\omega]$ is closed under ω-sequences, the sequence of evaluations of $j_{n,\omega}(\dot{x}^*_n)$ is in $M_\omega[G_\omega]$. Hence x is definable by $\zeta \in x$ if and only if for all large n, ζ is in $j_{n,\omega}(\dot{x}^*_n)$ as evaluated by G_ω. $\square$

We are now ready to describe the generic extension of M_ω. We recall some of the basic ideas from [11]. Let $\dot{\mathbb{Q}}$ be the canonical name in the Prikry forcing defined from E_κ for the poset to shoot a club disjoint from the set of $\alpha < \kappa^+$ such that $\mathrm{cf}^V(\alpha) = \kappa$. Let $\mathbb{Q}_\omega$ be the forcing $i_\omega(\dot{\mathbb{Q}})$ as interpreted by the critical sequence $\langle i_n(\kappa) \mid n < \omega\rangle$. By the argument following Claim 39 of [11], $\mathbb{Q}_\omega$ is equivalent to the forcing $\mathrm{Add}(\kappa^+, 1)$ in V. In fact, for every $n < \omega$, in N_n it is equivalent to $\mathrm{Add}(i_n(\kappa^+), 1)$.

Let $k_n : N_n \to M_n$ be the natural embedding given by

$$k_n(i_n(f)(\kappa, \dots i_{n-1}(\kappa))) = j_n(f)(\kappa, \dots j_{n-1}(\kappa)).$$

We discussed the notion of a *width* of an elementary embedding in the previous section. We recall a fundamental result concerning lifting embeddings and their widths (see [2]).

Lemma 6.6. *Suppose that $k : N \to M$ has width κ, $\mathbb{Q} \in N$ is a κ^+-distributive forcing, and $H \subseteq \mathbb{Q}$ is generic over N, then k"$H \subset k(\mathbb{Q})$ generates a generic filter $\langle k$"$H\rangle$ for M.*

Claim 6.7. *For all $1 \leq n < \omega$, k_n has width $\leq (i_{n-1}(\kappa)^{++})^{N_n}$.*

Proof. Let $x = j_n(g)(\alpha, j(\alpha), \dots j_{n-1}(\alpha))$ be an element of M_n where $g : [\kappa]^n \to V$ and $\alpha < \kappa^{++}$. Consider $h = i_n(g) \restriction (i_{n-1}(\kappa)^{++})^{N_n}$. Then $k(h)(\alpha, j(\alpha), \dots j_{n-1}(\alpha))$ makes sense and is equal to x, as required. $\square$

We are now ready to define the analogs of $\mathcal{H}$ and $\mathcal{H}_n$ for our situation. Recall that for subset X of some poset, we write $< X >$ for the upwards closure in that poset. Which poset is meant will be clear from the context.

Let H be generic for $\mathbb{Q}_\omega$ over $V[G_0]$ and recall that $\mathbb{Q}_\omega = i_n(\mathbb{Q}_\omega)$ is also a member of N_n for every $n \geq 1$, and is $i_n(\kappa^+)$-closed. We note that clearly, $i_n(\kappa^+) > i_{n-1}(\kappa^{++}) = \mathrm{width}(k_n)$.

Let

$$\mathcal{H}_n = \langle <j_{m,n} \circ k_m``H> \mid m \leq n\rangle$$

and

$$\mathcal{H} = \langle <j_{m,\omega} \circ k_m``H> \mid m < \omega\rangle.$$

Note that $\mathcal{H}_n$ is a subset of $\prod_{m \leq n} j_{m,n}(k_m(\mathbb{Q}_\omega))$. We do not yet know that it is generic.

We prove a sequence of claims about $\mathcal{H}_n$ and $\mathcal{H}$. The first is straightforward.

Claim 6.8. *For all $n < \omega$,*

$$\mathcal{H}_{n+1} = <j_{n,n+1}``\mathcal{H}_n> \frown <k_{n+1}``H>.$$

Let $\bar{\mathcal{H}}_n = \langle <i_{m,n}``H> \mid m \leq n\rangle$.

Claim 6.9. *$\mathcal{H}_n = <k_n``\bar{\mathcal{H}}_n>$.*

This is immediate from the following. For $m \leq n$, we have

$$<k_n``<i_{m,n}``H>> = <j_{m,n}``<k_m``H>>$$

since $k_n \circ i_{m,n} = j_{m,n} \circ k_m$. In particular, since $\bar{\mathcal{H}}_n$ is generic for $i_n(\kappa^+)$-closed forcing over N_n and k_n has width less than $i_n(\kappa)$, $\mathcal{H}_n$ is generic over M_n.

Claim 6.10. *$\mathcal{H}_n$ is mutually generic with G_n over M_n.*

Proof. Since G_0 and H are mutually generic over V, we can repeat the argument from Claim 18 of [11] to see that $\bar{\mathcal{H}}_n$ is generic over the model $N_n[<i_n``G_0>]$. By the previous claim, $<j_n``G_0>$ and $\mathcal{H}_n$ are mutually generic over M_n. By Remark 6.3, $\mathcal{H}_n$ is mutually generic with G_n. □

Lemma 6.11. *$M_\omega[G_\omega][\mathcal{H}]$ is closed under κ-sequences.*

Proof. It is enough to show that every κ-sequence of ordinals from the larger model $M_\omega[G_\omega][\mathcal{H}]$ is in $M_\omega[G_\omega]$. Let $\vec{\alpha}$ be such a sequence. By construction, $\vec{\alpha} \in V[G_0][H]$ and hence in $V[G_0]$. However, $M_\omega[G_\omega]$ is closed under κ-sequences in $V[G_0]$, so $\vec{\alpha} \in M_\omega[G_\omega]$, as required. □

Lemma 6.12. *$\bigcap_{n<\omega} M_n[G_n][\mathcal{H}_n] = M_\omega[G_\omega][\mathcal{H}]$.*

Proof. The proof is similar to the proof of Lemma 6.5. First, note that the embedding $j_{n,\omega} : M_n \to M_\omega$ lifts to an elementary embedding

$$j_{n,\omega} \colon M_n[\mathcal{H}_n] \to M_\omega[<j_{n,\omega}``\mathcal{H}_n>],$$

and $M_\omega[<j_{n,\omega}``\mathcal{H}_n>] \subseteq M_\omega[\mathcal{H}]$. Further, the construction of G_ω is the same whether we start in M_n or $M_n[\mathcal{H}_n]$.

It is immediate that $\bigcap_{n<\omega} M_n[G_n][\mathcal{H}_n] \supseteq M_\omega[G_\omega][\mathcal{H}]$. For the other inclusion, we work as before and suppose that $x \in \bigcap_{n<\omega} M_n[G_n][\mathcal{H}_n]$ is a set of ordinals. For each n, we can define $x_n = \{\gamma \mid j_{n,\omega}(\gamma) \in x\}$. Continuing as before, but working in $M_n[\mathcal{H}_n]$, we have that $x_n = \dot{x}_n^{G_n}$ for some name $\dot{x}_n$.

Now as before we can transform $\dot{x}_n$ to a $j_n(\mathbb{P})$-name $\dot{x}_n^*$ with the property that for every ordinal α, $j_{n,\omega}(\alpha) \in j_{n,\omega}(\dot{x}_n)^{<j_{n,\omega}``G_n>}$ if and only if it is in $j_{n,\omega}(\dot{x}_n^*)^{G_\omega}$. The only difference here is that we use the extension of $j_{n,\omega}$ to $M_n[\mathcal{H}_n]$.

By the previous lemma, $M_\omega[G_\omega][\mathcal{H}]$ has the sequence $\langle j_{n,\omega}(\dot{x}_n^*) \mid n < \omega\rangle$ and hence the sequence of interpretations. So we can define x as the set of ζ such that $\zeta \in j_{n,\omega}(\dot{x}_n^*)^{G_\omega}$ for all sufficiently large $n < \omega$. $\square$

Let $k_\omega : N_\omega \to M_\omega$ be the natural elementary embedding. Note that k_ω naturally extends to an embedding between the two Prikry generic extensions,

$$k_\omega^* : N_\omega[\langle i_m(\kappa) \mid m < \omega\rangle] \longrightarrow M_\omega[\langle j_m(\kappa) \mid m < \omega\rangle].$$

Similarly, For each $n < \omega$ we define $W_{n,\omega}$ to be the limit ultrapower obtained by starting in M_n and iterating the measure $j_n(E_\kappa)$, the normal measure of $j_n(E)$. As with N_ω. Denote the critical points of the iteration by $j_n(E_\kappa)$ and its images by $\langle \kappa_m^n\rangle_{m<\omega}$, and let $i_\omega^n : V \to W_{n,\omega}$ denote the composition of the finite ultrapower map $j : V \to M_n$ with the latter infinite iteration map by the normal measures. Therefore elements $y_n \in W_{n,\omega}$ are of the form $i_\omega^n(f)(\alpha_0, \ldots, \alpha_{n-1}, \kappa_0^n, \ldots, \kappa_{m-1}^n)$ for some $n, m < \omega$, $f : \kappa^{n+m} \to V$ in V, and $\alpha_\ell \in j_\ell(\kappa^{++})$ for each $\ell < n$.

The structures $W_{n,\omega}$ are naturally connected via maps $k_{n,k} : W_{n,\omega} \to W_{k,\omega}$ for $n \le k < \omega$, given by

$$\begin{aligned} &k_{n,k}(i_\omega^n(f)(\alpha_0, \ldots, \alpha_{n-1}, \kappa_0^n, \ldots, \kappa_{m-1}^n)) \\ &= \ i_\omega^k(f)(\alpha_0, \ldots, \alpha_{n-1}, \kappa_0^k, \ldots, \kappa_{m-1}^k) \end{aligned}$$

It is straightforward to verify that the limit of the directed system

$$\{W_{n,\omega}, k_{n,k} \mid n \le k < \omega\}$$

is M_ω, and the direct limit maps $k_{n,\omega} : W_{n,\omega} \to M_\omega$, which are defined by

$$\begin{aligned} &k_{n,k}\left(i_\omega^n(f)(\alpha_0, \ldots, \alpha_{n-1}, \kappa_0^n, \ldots, \kappa_{m-1}^n)\right) \\ &= j_\omega(f)(\alpha_0, \ldots, \alpha_{n-1}, j_n(\kappa), j_{n+1}(\kappa), \ldots, j_{n+m-1}(\kappa)) \end{aligned}$$

naturally extend to the generic extensions by the suitable Prikry sequences

$$k_{n,\omega}^* : W_{n,\omega}[\langle j_m(\kappa)\rangle_{m<n}{}^\frown\langle i_m(j_n(\kappa))\rangle_{m<\omega}] \longrightarrow M_\omega[\langle j_m(\kappa)\rangle_{m<\omega}].$$

We denote $W_{n,\omega}[\langle j_m(\kappa)\rangle_{m<n}{}^\frown\langle i_m(j_n(\kappa))\rangle_{m<\omega}]$ by $W_{n,\omega}^*$, for each $n < \omega$.

Finally, we note that following implies that $M_\omega[\langle j_m(\kappa) \mid m < \omega\rangle]$ is the direct limit of the system of Prikry generic extensions $\{W_{n,\omega}^*, k_{n,k}^* \mid n \le k < \omega\}$

Claim 6.13. *$<k_\omega^*``H> \in M_\omega[G_\omega][\mathcal{H}]$ is generic for $j_\omega(\dot{\mathbb{Q}})^{\langle j_n(\kappa) \mid n<\omega\rangle}$ over $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle]$.*

Proof. Let $D \in M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle]$ be a dense open subset of the forcing $j_\omega(\dot{\mathbb{Q}})^{\langle j_n(\kappa)|n<\omega\rangle}$. Then there are $n < \omega$ and $\bar{D} \in W^*_{n,\omega}$ such that $k^*_{n,\omega}(\bar{D}) = D$. It follows from our arguments above that $< k_n``H >$ is generic for $k_n(\mathbb{Q}_\omega)$ over $M_n$. Now $k_n(\mathbb{Q}_\omega) \in W^*_{n,\omega}$ and $\bar{D}$ is a dense subset of it. So $< k_n``H > \cap \bar{D}$ is nonempty. Let $q \in H$ be such that $k_n(q) \in \bar{D}$. It follows that $k^*_\omega(q) = k^*_{n,\omega}(k^*_n(q)) \in D$. So we have shown that $< k^*_\omega``H >$ is generic over $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle]$. It remains to show that it is a member of $M_\omega[G_\omega][\mathcal{H}]$. Since $< k^*_\omega``H > = < k^*_{n,\omega}`` < k^*_n``H >>$, we have that $< k^*_\omega``H > \in M_n[\mathcal{H}_n]$ for all $n < \omega$. By Claim 6.12, it follows that $< k^*_\omega``H > \in M_\omega[G_\omega][\mathcal{H}]$. □

To complete the proof of Theorem 1.4 we need a finer control of the relationship between $\mathbb{P}$ and $\mathbb{P}^*$ names. To this end we make some definitions.

Definition 6.14. Let $f \in \mathbb{P}^*$, α be an ordinal and $\dot{C}$ be a name for a club subset of κ^+. We say that f *stably forces* $\check{\alpha} \in \dot{C}$ ($f \Vdash^s \check{\alpha} \in \dot{C}$) if every alteration of f on fewer than κ many coordinates forces $\check{\alpha} \in \dot{C}$.

We have the following straightforward claims.

Claim 6.15. *If $g \geq f$ in $\mathbb{P}^*$ and $f \Vdash^s \check{\alpha} \in \dot{C}$, then $g \Vdash^s \check{\alpha} \in \dot{C}$.*

Claim 6.16. *If $f \Vdash^s \check{\alpha} \in \dot{C}$, then for every finite sequence $\vec{\nu}$ from some tree associated to f, $f \frown \vec{\nu} \Vdash^s \check{\alpha} \in \dot{C}$. If in addition $f = f' \frown \vec{\nu}$ for some finite sequence $\vec{\nu}$, then $f' \Vdash^s \check{\alpha} \in \dot{C}$.*

We define $\dot{C}^s = \{(\check{\alpha}, f) \mid f \Vdash^s \check{\alpha} \in \dot{C}\}$. Clearly it is forced that that $\dot{C}^s \subseteq \dot{C}$. It is also straightforward to see that $\dot{C}^s$ is forced to be closed.

Claim 6.17. *$\dot{C}^s$ is forced to be unbounded in κ^+.*

Proof. Fix some $f \in \mathbb{P}^*$ and $\alpha_0 < \kappa^+$. Take some sufficiently large θ and some $N \prec H_\theta$ of size κ such that $\mathbb{P}^*, f, \dot{C}, \alpha_0 \in N$ and ${}^{<\kappa}N \subseteq N$. Since $\mathbb{P}^*$ is κ^+-closed we can find a $(\mathbb{P}^*, N)$-generic condition $f^* \geq f$ with $\mathrm{dom}(f^*) = N \cap \kappa^{++}$. Let $\alpha = \sup(N \cap \kappa^{++})$.

Let f' be any condition obtained by altering f^* on a set of size less than κ. Since ${}^{<\kappa}N \subseteq N$, the alteration is a member of N. Hence a standard argument shows that f' is also $(\mathbb{P}^*, N)$-generic and so $f' \Vdash \check{\alpha} \in \dot{C}$. □

The name $\dot{C}^s$ behaves well when translated to a $\mathbb{P}$-name. Let

$$\dot{E} = \{(\check{\alpha}, p) \mid f^p \Vdash^s \check{\alpha} \in \dot{C}\}.$$

Claim 6.18. *$\dot{E}$ is forced by $\mathbb{P}$ to be a club subset of κ^+.*

Proof. By the arguments of Claim 6.17, $\dot{E}$ is unbounded. Let us show first that $\dot{E}$ is forces to be closed.

Let $p = \langle f, T\rangle$ be a condition that forces that δ is an accumulation point of $\dot{E}$. Therefore, for every $\alpha < \delta$ there are $\vec{\nu} = \langle \nu_0, \ldots, \nu_{n-1}\rangle \in T$, $p' = \langle f', T'\rangle \leq^* p^\frown\vec{\nu}$, and $\gamma < \delta$ such that $f' \Vdash^s \gamma \in \dot{C}$. Clearly, f' is a Cohen extension of $f^\frown\vec{\nu}$. Denote by f^* the Cohen extension of f for which $f' = f^{*\frown}\vec{\nu}$. Since $f' \Vdash^s \gamma \in \dot{C}$, $f^* \Vdash^s \gamma \in \dot{C}$ as well. It follows that the condition

$p^* = \langle f^*, T^* \rangle$, $T^* = \pi^{-1}_{\mathrm{dom}(f^*),\mathrm{dom}(f)}(T)$ is a direct extension of p and forces "$\check{\gamma} \in \dot{E}$".

Using the fact the Cohen extension order on the function f is κ^+-closed and $\mathrm{cf}(\delta) \le \kappa$, we can repeat this process and construct a sequence of Cohen extensions $\langle f_i \mid i \le \mathrm{cf}(\delta) \rangle$ of f, and an increasing sequence $\langle \gamma_i \mid i < \mathrm{cf}(\delta) \rangle$ cofinal in δ, such that $f_i \Vdash^s \check{\gamma}_i \in \dot{C}$. Let $\bar{f} = f_{\mathrm{cf}(\delta)}$. Since $\dot{C}$ is a Cohen name of a club, $\bar{f} \Vdash^s \check{\delta} \in \dot{C}$. Setting $\bar{T} = \pi^{-1}_{\mathrm{dom}(\bar{f}),\mathrm{dom}(f)}(T)$ and $\bar{p} = \langle \bar{f}, \bar{T} \rangle$, we have that $\bar{p}$ is a direct extension of p, and forces "$\check{\delta} \in \dot{E}$. □

Remark 6.19. The proofs of the previous two claims work for any name for a club $\dot{C}$ in any κ^+-closed generic extension.

We are now ready to finish the proof of Theorem 1.4.

Claim 6.20. *In $M_\omega[G_\omega][\mathcal{H}]$, every finite collection of stationary subsets of $j_\omega(\kappa^+)$ reflects at a common point.*

Proof. Let S_i for $i < k$ be a sequence of stationary sets in $M_\omega[G_\omega][\mathcal{H}]$. By Claim 6.13, $<k_\omega``H>$ is a club in $j_\omega(\kappa^+)$ disjoint from $S^{j_\omega(\kappa^+)}_{j_\omega(\kappa)}$ as computed in $M_\omega$ and $<k_\omega``H> \in M_\omega[G_\omega][\mathcal{H}]$. So we can assume that each S_i concentrates on a fixed cofinality below $j_\omega(\kappa)$. For simplicity we assume that the cofinalities of the S_i are bounded below κ.

Let T^i_n be the set of α such that $j_{n,\omega}(\alpha) \in S_i$. For a fixed $n < \omega$, T^i_n for $i < k$ is definable in $M_n[G_n][\mathcal{H}_n]$. By the indestructibility of stationary reflection at $j_n(\kappa^+)$ in M_n, if each T^i_n for $i < k$ is stationary, then they reflect at a common point. Suppose that δ is this common reflection point. Then there are sets $A_i \in M_n$ for $i < k$ with $\mathrm{ot}(A) = \mathrm{cf}(\delta)$ such that A_i is stationary in δ and $A_i \subseteq T^i_n$. It follows that for each $i < k$, $j_{n,\omega}(A_i) = j_{n,\omega}``A_i \subseteq S_i$ and hence the collection of S_i reflect at $j(\delta)$.

We assume for the sake of a contradiction that for each $n < \omega$ at least one of the T^i_n is nonstationary. Let $\dot{C}_n$ be a name whose interpretation by G_n and $\mathcal{H}_n$, C_n, is a club disjoint from some $T^{i_n}_n$. By the arguments above and Remark 6.19, we can work in $M_n[\mathcal{H}_n]$ and translate $\dot{C}_n$ to a $j_n(\mathbb{P})$-name $\dot{E}_n$ for a club subset of $j_n(\kappa^+)$. By the construction of $\dot{E}_n$, we have that for $\alpha < j_n(\kappa^+)$, $j_{n,\omega}(\alpha)$ is in $j_{n,\omega}(\dot{E}_n)^{G_\omega}$ if and only if it is in $j_{n,\omega}(\dot{C}_n)^{<j_{n,\omega}``G_n>}$.

By the closure of $M_\omega[G_\omega]$ under ω-sequences, $\langle j_{n,\omega}(\dot{E}_n) \mid n < \omega \rangle$ is in $M_\omega[G_\omega]$. Hence we can interpret it using G_ω and $j_{n,\omega}``\mathcal{H}_n$. Let D_n denote the resulting club. Let $i^* < k$ be such that $i^* = i_n$ for infinitely many n. We claim that $\bigcap_{n<\omega} D_n$ is disjoint from S_{i^*}. Otherwise, we have $j_{n,\omega}(\alpha) \in D_n \cap S_{i^*}$ for some n such that $i^* = i_n$. It follows that $\alpha \in C_n \cap T^{i_n}_n$, a contradiction. □

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