FRAGILITY AND INDESTRUCTIBILITY II

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Abstract. In this paper we continue work from a previous paper on the fragility and indestructibility of the tree property. We present the following:

1. A preservation lemma implicit in Mitchell's PhD thesis, which generalizes all previous versions of Hamkins' Key lemma.
2. A new proof of theorems the 'superdestructibility' theorems of Hamkins and Shelah.
3. An answer to a question from our previous paper on the apparent consistency strength of the assertion "The tree property at $\aleph_2$ is indestructible under $\aleph_2$-directed closed forcing".
4. Two models for successive failures of weak square on long intervals of cardinals.

Techniques for preserving the tree property are central to a growing literature of consistency results obtaining the tree property as successive regular cardinals (see for example [11, 1, 3, 15, 20]). These techniques can be viewed abstractly as indestructibility results, which typically arise from either integration of preparation forcing or preservation lemmas. In our original paper [18] we proved results using both of these methods. In particular, using methods of Abraham and Cummings and Foreman [3, 1] we showed that modulo the existence of a supercompact cardinal it is consistent that the tree property holds at ω_2 and is indestructible under ω_2-directed closed forcing. Further, by proving a new preservation lemma we showed that the tree property at ω_2 in a model of Mitchell [11] is indestructible under the forcing to add an arbitrary number of Cohen reals. It follows that the tree property at ω_2 is consistent with $2^\omega > \omega_2$. The preservation lemma was

Lemma 0.1. *Let τ, η be cardinals with η regular and $2^\tau \geq \eta$. Let $\mathbb{P}$ be τ^+-cc and $\mathbb{R}$ be τ^+-closed. Let $\dot{T}$ be a $\mathbb{P}$-name for an η-tree. Then in $V[\mathbb{P}]$ forcing with $\mathbb{R}$ cannot add a branch through T.*

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This lemma was applied in work of Neeman [15] and inspired a lemma of Sinapova [16]. Indeed the idea that for some preservation properties the forcing only needs to be *formerly* closed seems to be quite powerful. This paper provides further applications of the above lemma, this time to obtain successive failures of weak square.

The subject of fragility is more subtle. In the original paper we proved that in Mitchell's model there is a cardinal preserving forcing which adds $\square_{\omega_1}$. In some sense this is the easiest kind of fragility result. The forcing is specifically designed to add a strong sufficient condition for the existence of an Aronszajn tree. More interesting fragility results consist in proving that the tree property fails in an extension where we hoped that it might hold. In this paper we give a definition of an Aronszajn tree which occurs as the tree of attempts to construct an object in some generic extension. This gives a new proof of theorems of Hamkins, and Hamkins and Shelah and an instance of fragility where we do not force a strong sufficient condition.

The results of the current paper are as follows:

- In Section 1 we prove a strong generalization of Hamkins' Key lemma and provide a few applications. We also remark on a few applications of a related lemma appearing in another of the author's papers.
- In Section 2 we provide new proofs of the main theorems of Hamkins [5], and Hamkins and Shelah [7]. As a special case of Theorem 2.8, we have that if κ is inaccessible, $\mathbb{P}$ has size less than κ and it is forced by $\mathbb{P}$ that $\dot{\mathbb{Q}}$ is κ-closed and adds a subset to κ, then in $V[\mathbb{P} * \dot{\mathbb{Q}}]$ there is a κ-Aronszajn tree. The more general statement gives the result of Hamkins and Shelah.
- In Section 3 we answer a question from our previous fragility and indestructibility paper on the apparent consistency strength of the statement "The tree property at ω_2 is indestructible under ω_2-directed closed forcing". In particular we show that any reasonable forcing to obtain the consistency of this statement requires a strongly compact cardinal.
- In Section 4 we prove some results which are generalizations of a result claimed without proof by Mitchell [11]. In particular we show that if there are infinitely many Mahlo cardinals, then there is a forcing extension in which $\square^*_{\aleph_n}$ fails for $1 \leq n < \omega$. In this model we achieve the most economical failure of GCH possible. In particular $2^{\aleph_n} = \aleph_{n+2}$ for all $n < \omega$. Going further we show that we can obtain the failure of $\square^*_\kappa$ for all κ in the interval $[\aleph_1, \aleph_{\omega^2}]$ from suitable large cardinals, but at the cost

that none of the $\aleph_{\omega\cdot m}$ are strong limit. Both of these results seem to require Lemma 0.1.

We make one notational remark before beginning the paper. We will write $V[\mathbb{P}]$ for a typical generic extension by a poset $\mathbb{P}$. Furthering this notation when we write $V[\mathbb{P}] \subseteq V[\mathbb{Q}]$ we mean that we can force over a given generic extension by $\mathbb{P}$ to obtain a generic extension by $\mathbb{Q}$.

1. Approximation lemmas

In this section we prove some preservation theorems. We start by defining the κ-approximation property.

Definition 1.1 (Hamkins)**.** *Let $V \subseteq W$ be models of set theory. The pair (V, W) has the* κ-approximation property *if and only if for every ordinal μ and every $b \subseteq \mu$ with $b \in W$, if for every $x \in \mathcal{P}_\kappa(\lambda)_V$, $b \cap x \in V$, then $b \in V$.*

We say that a poset $\mathbb{P}$ has the κ-approximation property if and only if for every $\mathbb{P}$-generic G, $(V, V[G])$ has the κ-approximation property.

In [5] Hamkins proved:

Lemma 1.2. *Let β be a regular cardinal suppose that $|\mathbb{P}| = \beta$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is β^+-closed. If $\mathrm{cf}(\lambda) > \beta$, then $\mathbb{P} * \dot{\mathbb{Q}}$ does not add subsets to λ all of whose initial segments are in the ground model.*

The lemma generalizes in a straightforward way to give the β^+-approximation property. The proof of Hamkins' lemma resembles the proof of a lemma in an early paper of Mitchell [11]. A close examination of Mitchell's paper reveals the following lemma.

Lemma 1.3. *Let κ be a regular cardinal. If $\mathbb{P} \times \mathbb{P}$ has the κ-cc and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is κ-closed, then $\mathbb{P} * \dot{\mathbb{Q}}$ has the κ-approximation property.*

This lemma is closely related to another more recent lemma of Mitchell [13]. One difference is that Mitchell's lemma is stated for posets which generalize a two step iteration. Further, Mitchell has only assumed that $\mathbb{Q}$ is forced to be strategically closed. Our lemma generalizes to give the analogous result using both the generalized two step iterations and strategic closure.

The main difference between our lemma and Mitchell's is that the condition $\mathbb{P} \times \mathbb{P}$ is κ-cc is replaced with the requirement that $\mathbb{P}$ has many strong master conditions for models of size less than κ. These two assumptions are somewhat different. In one direction, Mitchell's poset [12] for adding a club in ω_2 with finite conditions is strongly proper, but does not have the countable chain condition. In the other direction, the usual poset to add a dominating real has very strong countable chain condition, but does not add any Cohen reals and hence it is not strongly proper.

To prove the lemma we need some preliminaries. We say that a forcing $\mathbb{P}$ adds a *new* subset to a cardinal κ if there is a $\mathbb{P}$-name $\dot{b}$ for a subset of κ such that $\Vdash_{\mathbb{P}} \dot{b} \notin V$.

Proposition 1.4. *If $\mathbb{P}$ is nontrivial and κ-cc, then $\mathbb{P}$ adds a new subset of κ.*

Proof. Let λ be the least cardinal such that $\mathbb{P}$ adds a new subset to λ. We will show that $\lambda \leq \kappa$. Let $\dot{b}$ be a name for a subset of λ such that for all $\alpha < \lambda$, $\Vdash_{\mathbb{P}} \dot{b} \cap \alpha \in V$ and $\Vdash_{\mathbb{P}} \dot{b} \notin V$.

Let $T \in V$ be the tree of attempts to construct the characteristic function of the interpretation of $\dot{b}$. Note that T is normal and $\mathbb{P}$ adds a branch through T. By the κ-cc of $\mathbb{P}$, the levels of T have size less than κ.

If $\text{cf}(\lambda) > \kappa$, then it follows that there is a level $\gamma < \lambda$, such that for all $t \in T$ above level γ, any two extensions of t are comparable. So if we choose a condition which decides the value of the branch passed level γ, then that condition forces the branch to be in V, a contradiction. So we must have $\text{cf}(\lambda) \leq \kappa$.

We want to see that $\lambda \leq \kappa$. If we let S be the restriction of T to a cofinal set of levels, then S is a tree of height less than or equal to κ with levels of size less than κ. Moreover, $\dot{b}$ gives a name for a new cofinal branch through S. So there is a name for a new subset of S and S has size at most κ so we are done. $\square$

We also prove another lemma whose ideas come from Mitchell and which first appeared in its current form in another of the author's papers [19]. We repeat the proof because similar ideas will be used below. We start with a standard remark.

Remark 1.5. *Suppose that $\mathbb{P}$ is a poset and $\dot{b}$ is a $\mathbb{P}$-name for a subset of some ordinal μ. Assume that for all $z \in \mathcal{P}_\kappa(\mu)$, $\Vdash_{\mathbb{P}} \dot{b} \cap z \in V$, but $\Vdash_{\mathbb{P}} \dot{b} \notin V$. Then for all $p \in \mathbb{P}$ and all $y \in \mathcal{P}_\kappa(\mu)$, there are $p_1, p_2 \leq p$ and $z \supseteq y$ such that p_1, p_2 decide the value of $\dot{b} \cap z$ and they decide different values.*

Lemma 1.6. *Let κ be a regular cardinal. If $\mathbb{P} \times \mathbb{P}$ has the κ-cc, then $\mathbb{P}$ has the κ-approximation property.*

Proof. Suppose that the lemma is false. Then we have a poset $\mathbb{P}$ and a name $\dot{b}$, which fails to be approximated. We work by recursion to construct an antichain of size κ in $\mathbb{P} \times \mathbb{P}$. In particular, we construct

$\langle (p^0_\alpha, p^1_\alpha) \mid \alpha < \kappa\rangle$, a function $f : \kappa \to \mathcal{P}_\kappa(\mu)$ and sets x^0_α, x^1_α for $\alpha < \kappa$ such that

(1) $p^i_\alpha \Vdash \dot{b} \cap f(\alpha) = x^i_\alpha$ for $i \in 2$,
(2) $f(\alpha) \supseteq \bigcup f``\alpha$ and
(3) $x^0_\alpha \neq x^1_\alpha$, but $x^0_\alpha \cap \bigcup f``\alpha = x^1_\alpha \cap \bigcup f``\alpha$.

Assume that for some $\beta < \kappa$, we have constructed (p^0_α, p^1_α) for $\alpha < \beta$, $f \restriction \beta$ and x^0_α, x^1_α for $\alpha < \beta$. Let $y = \bigcup f``\beta$ which is in $\mathcal{P}_\kappa(\mu)$. Choose $p_\beta \in \mathbb{P}$ which decides the value of $\dot{b} \cap y$ to be x_β. Apply Remark 1.5 to p_β and y to obtain conditions p^0_β, p^1_β and $f(\beta) \in \mathcal{P}_\kappa(\mu)$ such that p^0_β, p^1_β decide different values for $\dot{b} \cap f(\beta)$. Record the values that each condition decides as x^0_β, x^1_β. This completes the construction.

We claim that $\{(p^0_\alpha, p^1_\alpha) \mid \alpha < \kappa\}$ is an antichain of size κ. Suppose that we had $\alpha < \beta$ such that $(p^0_\alpha, p^1_\alpha), (p^0_\beta, p^1_\beta)$ are compatible. Then $x^k_\beta \cap f(\alpha) = x^k_\alpha$ for $k = 0, 1$. Note that $x^0_\beta \cap f(\alpha) = x^1_\beta \cap f(\alpha) = x_\beta \cap f(\alpha)$ by the choice of x_β. This implies that $x^0_\alpha = x^1_\alpha$ a contradiction. □

We are now ready to prove Lemma 1.3.

Proof. Let μ be an ordinal and $\dot{b}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a subset of μ such that for all $y \in \mathcal{P}_\kappa(\mu)_V$, $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \dot{b} \cap y \in V$.

Claim 1.7. *There is a condition $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ such that for all x, y and for all $(p', \dot{q}') \leq (p, \dot{q})$ if $(p', \dot{q}') \Vdash \dot{b} \cap y = x$, then $(p, \dot{q}') \Vdash \dot{b} \cap y = x$.*

Proof of Claim. Suppose otherwise. We claim that the following statement holds:

$(\star)$ For all $(p, \dot{q})$ there are $x_0, x_1, y \in \mathcal{P}_\kappa(\mu)$, $p_0, p_1 \leq p$ and $\dot{q}'$ such that $x_0 \neq x_1$, $\Vdash_{\mathbb{P}} \dot{q}' \leq \dot{q}$ and for $i \in 2$, $(p_i, \dot{q}') \Vdash \dot{b} \cap y = x_i$.

Let $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ and apply the negation of the claim to obtain x_0, y and $(p_0, \dot{q}_0)$ such that $(p_0, \dot{q}_0) \Vdash \dot{b} \cap y = x_0$ and $(p, \dot{q}_0) \nVdash \dot{b} \cap y = x_0$. Extend $(p, \dot{q}_0)$ to $(p_1, \dot{q}_1)$ which forces $\dot{b} \cap y = x_1$ for some $x_1 \neq x_0$. We may assume that $\Vdash_{\mathbb{P}} \dot{q}_1 \leq \dot{q}_0 \leq \dot{q}$. So $(\star)$ follows.

Using $(\star)$ we construct a large antichain in $\mathbb{P} \times \mathbb{P}$ much like we did in the proof of Lemma 1.6. In this case we also construct a sequence of elements of $\dot{\mathbb{Q}}$ which are forced to be decreasing. More precisely we construct $p^0_\alpha, p^1_\alpha, p_\alpha, \dot{q}_\alpha, x^0_\alpha, x^1_\alpha$ for $\alpha < \kappa$ and a function $f : \kappa \to \mathcal{P}_\kappa(\mu)_V$. Suppose that we have constructed all of the above for all $\alpha < \beta$ and also $f \restriction \beta$ such that

(1) $f \restriction \beta$ is increasing and
(2) for all $\alpha < \alpha' < \kappa$, $\Vdash_{\mathbb{P}} \dot{q}_{\alpha'} \leq \dot{q}_\alpha$.

Choose a condition $\dot{q}^*$ such that $\Vdash_{\mathbb{P}} \dot{q}^* \leq \dot{q}_\alpha$ for all $\alpha < \beta$ and there is a condition p_β such that $(p_\beta, \dot{q}^*)$ decides $\dot{b} \cap \bigcup f``\beta$. Apply condition $(\star)$ to the condition $(p_\beta, \dot{q}^*)$ to obtain $p^0_\beta, p^1_\beta, \dot{q}_\beta, x^0_\beta, x^1_\beta$ and $f(\beta)$ such that

(1) $\Vdash_{\mathbb{P}} \dot{q}_\beta \leq \dot{q}^*$,
(2) for $i \in 2$, $(p^i_\beta, \dot{q}_\beta) \Vdash \dot{b} \cap f(\beta) = x^i_\beta$ and
(3) $x^0_\beta \neq x^1_\beta$.

It is easy to see that we have maintained the induction hypotheses. Using the closure of the term ordering on $\mathbb{Q}$ we can continue for κ-many stages. Working as in the proof of Lemma 1.6, it is not hard to see that $\{(p^0_\alpha, p^1_\alpha) \mid \alpha < \kappa\}$ is an antichain of size κ in $\mathbb{P} \times \mathbb{P}$, a contradiction. So we have proved the claim. $\square$

Let $(p, \dot{q})$ be as in Claim 1.7. Assume for a contradiction that $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \dot{b} \notin V$. An easy argument gives an analog of Remark 1.5 for our situation.

$$(\star\star) \qquad \begin{array}{c} \text{For all } (p', \dot{q}') \leq (p, \dot{q}) \text{ and for all } y \in \mathcal{P}_\kappa(\mu)_V, \text{ there are} \\ y' \in \mathcal{P}_\kappa(\mu) \text{ and } \dot{q}_0, \dot{q}_1 \text{ such that } y' \supseteq y, \Vdash_{\mathbb{P}} \dot{q}_0, \dot{q}_1 \leq \dot{q}' \text{ and} \\ (p', \dot{q}_i) \text{ for } i \in 2 \text{ decide different values for } \dot{b} \cap y'. \end{array}$$

By Proposition 1.4, $\mathbb{P}$ adds a new subset to κ. Further by Lemma 1.6, there is an $\eta < \kappa$ such that the intersection of our subset with η is new. Let $\dot{a}$ be a $\mathbb{P}$-name for a function from some $\eta < \kappa$ to 2 such that for all $\alpha < \eta$, $\Vdash_{\mathbb{P}} \dot{a} \restriction \alpha \in V$, but $\Vdash_{\mathbb{P}} \dot{a} \notin V$.

Using $(\star\star)$ build a binary tree of names for elements of $\mathbb{Q}$, $\langle \dot{q}_s \mid s \in 2^{<\eta} \rangle$ such that for all s, $\dot{q}_{s \frown 0}$ and $\dot{q}_{s \frown 1}$ are obtained by applying $(\star\star)$ to the condition $(p, \dot{q}_s)$ and some y_α where α is the length of s and the y_α are chosen inductively. We may assume that for all $\alpha < \eta$, for all s of length α, $(p, \dot{q}_s)$ decides $\dot{b} \cap y_\alpha$. Moreover we can assume that $y_\alpha \subseteq y_\beta$ for all $\alpha < \beta < \eta$.

Now $\langle \dot{q}_{\dot{a} \restriction \alpha} \mid \alpha < \eta \rangle$ is a $\mathbb{P}$-name for a decreasing sequence. Let $\dot{q}^*$ be a name forced to be a lower bound for the sequence. Extending $(p, \dot{q}^*)$ if necessary we may assume that it decides the value of $\dot{b} \cap \bigcup_{\alpha < \eta} y_\alpha$ to be x. We can now define the realization of $\dot{a}$ in V which will be a contradiction. Let $\alpha < \eta$ and define $a \restriction \alpha$ to be the unique s of length α such that $(p, \dot{q}_s) \Vdash \dot{b} \cap y_\alpha = x \cap y_\alpha$. $\square$

The lemma above greatly expands the class of posets which are known to have the approximation property. We mention a few applications. First, we note that Lemma 1.3 generalizes all previous versions of Hamkins' Key Lemma (see for example [6]) and so expands the class of forcings to which we can apply his theorems about forcing with the approximation and covering properties [6]. It also generalizes Proposition 1.1 of [2].

We also remark that Lemma 1.6 provides an easy proof that typical iterations have approximation properties. In particular, any iteration which has the κ-Knaster property will have the κ-approximation property. This easily gives Lemma 5.2 of Viale and Weiss [21]. Lemma 1.6 also improves Claim 3.4 of Neeman [14].

2. Results of Hamkins and Shelah

In this section we provide a new proof of the following theorems of Hamkins [5] and Hamkins and Shelah [7].

Theorem 2.1 (Hamkins)**.** *If κ is weakly compact, $|\mathbb{P}| < \kappa$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is κ-closed and adds a new subset to κ, then it is forced by $\mathbb{P} * \dot{\mathbb{Q}}$ that κ is not weakly compact.*

Theorem 2.2 (Hamkins-Shelah)**.** *If κ is strongly compact, $|\mathbb{P}| < \kappa$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is κ-closed and adds a new subset to some $\lambda \geq \kappa$, then it is forced by $\mathbb{P} * \dot{\mathbb{Q}}$ that κ is not λ-strongly compact.*

We start with a definition which comes from work of Jech [8] and was explored further by Weiss [22].

Definition 2.3. *Let $\kappa \leq \lambda$ be regular cardinals and η be an ordinal. A (κ, λ, η)-*tree *T is a collection of functions such that for all $f \in T$:*

1. $\mathrm{dom}(f) \in \mathcal{P}_\kappa(\lambda)$*,*
2. $\mathrm{rng}(f) \subseteq \eta$ *and*
3. *for all $y \subseteq \mathrm{dom}(f)$, $f \upharpoonright y \in T$,*

and for all $x \in \mathcal{P}_\kappa(\lambda)$, there is $f \in T$ with $\mathrm{dom}(f) = x$.

For $x \in \mathcal{P}_\kappa(\lambda)$ we set $T_x = \{f \in T \mid \mathrm{dom}(f) = x\}$.

Definition 2.4. *A (κ, λ, η)-tree is* thin *if for all $x \in \mathcal{P}_\kappa(\lambda)$, $|T_x| < \kappa$.*

Definition 2.5. *A function $b : \lambda \to \eta$ is a* cofinal branch through *T if for all $x \in \mathcal{P}_\kappa(\lambda)$, $b \upharpoonright x \in T$.*

Note that if κ is inaccessible and $\eta < \kappa$, then every (κ, λ, η)-tree is thin. The more usual notion of a (κ, λ)-tree is a $(\kappa, \lambda, 2)$-tree under our notation. Note that if $\eta < \kappa$ and we have a thin (κ, λ, η)-tree T, then we can construct a thin $(\kappa, \lambda, 2)$-tree T' such that T has a cofinal branch if and only if T' does. We define a principle which we call the (κ, λ)-tree property or $\mathrm{TP}(\kappa, \lambda)$.

Definition 2.6. *Let $\kappa \leq \lambda$ be regular cardinals.* $\mathrm{TP}(\kappa, \lambda)$ *holds if every thin $(\kappa, \lambda, 2)$-tree has a cofinal branch.*

By the discussion of the previous paragraph, $\mathrm{TP}(\kappa, \lambda)$ holds if and only if for all $\eta < \kappa$, every (κ, λ, η)-tree has a cofinal branch. It is worth noting that $\mathrm{TP}(\kappa, \kappa)$ is precisely the tree property at κ. To obtain the theorems of Hamkins and Shelah from our theorem we need the following theorem of Jech [8].

Theorem 2.7 (Jech)**.** *κ is strongly compact if and only if κ is inaccessible and for every $\lambda \geq \kappa$,* $\mathrm{TP}(\kappa, \lambda)$*.*

We present the following theorem.

Theorem 2.8. *Let κ be an inaccessible cardinal. If $|\mathbb{P}| < \kappa$, $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is κ-closed and $\lambda \geq \kappa$ is the least cardinal such that $\mathbb{Q}$ adds a new subset of λ over $V[\mathbb{P}]$, then* $\mathrm{TP}(\kappa, \lambda)$ *fails in $V[\mathbb{P} * \mathbb{Q}]$.*

Combining this result with Theorem 2.7 gives the results of Hamkins and Shelah.

Proof. Assume the hypotheses. Let $|\mathbb{P}| = \beta < \kappa$. We construct a branchless $(\kappa, \lambda, 2^{2^\beta})$-tree. This is enough by remarks that we made prior to the definition of $\mathrm{TP}(\kappa, \lambda)$. Let $\mathbb{A}$ be the $\mathbb{P}$-term forcing for elements of $\dot{\mathbb{Q}}$. By standard facts $\mathbb{A}$ is κ-closed. Moreover $\mathbb{A}$ adds a $\mathbb{P}$-name for a function from λ to 2 so that whenever g is $\mathbb{P}$-generic, then the interpretation of this name is the characteristic function of the subset of λ added by the induced $\mathbb{Q}$-generic object over $V[g]$ where $\mathbb{Q}$ is the interpretation of $\dot{\mathbb{Q}}$ by g. Such a name is coded by a function $b : \lambda \to 2^{2^\beta}$ where for each $\alpha < \lambda$, $b(\alpha)$ is a code for a function from a maximal antichain of $\mathbb{P}$ to 2.

We consider the $(\kappa, \lambda, 2^{2^\beta})$-tree T defined in V, which is the tree of less than κ sized approximations of b. That is for $f \in T$, $\mathrm{dom}(f) \in \mathcal{P}_\kappa(\lambda)$ and f returns the same kind of values as b. Note that T is thin since κ is inaccessible.

Let $g * G$ be $\mathbb{P} * \dot{\mathbb{Q}}$-generic and $\mathbb{Q}$ be the interpretation of $\dot{\mathbb{Q}}$ by g. In $V[g*G]$ we define a refinement of T. Let $c : \lambda \to 2$ be the characteristic function of the new subset of λ added by $\mathbb{Q}$ over $V[g]$. Let $f \in S$ if and only if $f \in T$ and for all $\alpha \in \mathrm{dom}(f)$, there is a $p \in g \cap \mathrm{dom}(f(\alpha))$ such that $f(\alpha)(p) = c(\alpha)$. Recall that $f(\alpha)$ returns a function from a maximal antichain in $\mathbb{P}$ to 2.

We claim that S is a thin $(\kappa, \lambda, 2^{2^\beta})$-tree with no cofinal branch. Note that $\mathcal{P}_\kappa(\lambda)_V$ is cofinal in $\mathcal{P}_\kappa(\lambda)_{V[g*G]}$, so we have defined a thin tree if we can show that for all x in $\mathcal{P}_\kappa(\lambda)_V$, there is $f \in S$ with $\mathrm{dom}(f) = x$. To see this let $x \in \mathcal{P}_\kappa(\lambda)_V$, by the closure of $\mathbb{Q}$, $c \restriction x \in V[g]$. So in V there is a $\mathbb{P}$-name for $c \restriction x$, but we have already said that every such name is coded by an element of T.

Lastly we claim that S has no cofinal branch. Suppose that S has a cofinal branch b in $V[g * G]$. Since $\mathbb{P} * \dot{\mathbb{Q}}$ has the κ-approximation property, it follows that $b \in V$. We arranged that b is a code for a $\mathbb{P}$-name and by the choice of S, the interpretation of b by g is c. This is impossible as we assumed that $c \notin V[g]$. □

Remark 2.9. *In the case when $\kappa = \lambda$, the tree T is essentially a κ-Aronszajn tree. We note that this tree is not* special *in the sense of Todorcevic* [17]*, since it obtains a branch in a cardinal preserving extension $V[\mathbb{P} \times \mathbb{A}]$.*

With the results of the section in mind we ask

Question 2.10. *If $\dot{\mathbb{Q}}$ is an $\mathrm{Add}(\omega, \kappa)$-name for a κ-closed forcing which adds a subset to κ, is there a κ-Aronszajn tree in $V[\mathrm{Add}(\omega, \kappa) * \dot{\mathbb{Q}}]$? More generally can we reprove Theorem 2.8 if we relax the condition $|\mathbb{P}| < \kappa$?*

This question is designed to test whether the preparation forcing $\mathbb{U}$ in [15] is necessary. A similar scenario arises in a more complex context in [19].

3. The consistency strength of indestructibility

In this section we apply ideas of Viale and Weiss [21] to show that the apparent consistency strength of the statement "The tree property at ω_2 is indestructible under ω_2-directed closed forcing" is high. In particular we prove the following theorem:

Theorem 3.1. *Suppose that κ is inaccessible and there is a poset $\mathbb{P}$ such that*

1. *$\mathbb{P} \times \mathbb{P}$ is κ-cc,*
2. *$\Vdash_{\mathbb{P}} \kappa = \omega_2$ has the tree property and*
3. *$\Vdash_{\mathbb{P}}$ the tree property at ω_2 is indestructible under ω_2-directed closed forcing,*

then κ is strongly compact.

The theorem makes use of $\mathrm{TP}(\kappa, \lambda)$ as defined in the previous section. We also need the following theorem of Viale and Weiss.

Theorem 3.2. *Suppose that $V \subseteq W$ and $\kappa \in V$ is inaccessible. If (V, W) has the κ-approximation and the κ-covering property, and $\mathrm{TP}(\kappa, \lambda)$ holds in W, then $\mathrm{TP}(\kappa, \lambda)$ holds in V.*

Combining Lemma 1.6, Theorem 2.7 and Theorem 3.2 it is enough to show that "The tree property at ω_2 is indestructible under ω_2-directed closed forcing" implies $\mathrm{TP}(\omega_2, \lambda)$ for all $\lambda \geq \omega_2$. To prove this we need the notion of *thin* approximation.

Definition 3.3. *A poset $\mathbb{P}$ has the* thin κ-approximation property *if for every ordinal μ and every $\mathbb{P}$-name $\dot{d}$ for a subset of μ, if for all $z \in \mathcal{P}_\kappa(\mu)$, $\Vdash_{\mathbb{P}} \dot{d} \cap z \in V$ and $|\{x \in V \mid$ there is $p \in \mathbb{P}$, $p \Vdash \dot{d} \cap z = x\}| < \kappa$, then $\Vdash_{\mathbb{P}} \dot{d} \in V$.*

We also need a lemma about closed forcing and thin approximation. The following is an appropriate generalization of a lemma of Silver to the setting of $\mathcal{P}_\kappa(\lambda)$.

Lemma 3.4. *Suppose that χ is a cardinal with $\chi < \kappa$ and $2^\chi \geq \kappa$. If $\mathbb{P}$ is χ^+-closed, then $\mathbb{P}$ has the thin κ-approximation property.*

For a proof we refer the reader to Proposition 2.1.12 of [22].

Lemma 3.5. *If κ is a successor cardinal with the tree property and the tree property at κ is indestructible under κ-directed closed forcing, then for every $\lambda \geq \kappa$, $\mathrm{TP}(\kappa, \lambda)$ holds.*

Proof. Assume the hypotheses and let $\lambda \geq \kappa$. In V let T be a thin $(\kappa, \lambda, 2)$-tree. Let G be $\mathrm{Coll}(\kappa, \lambda)$-generic over V. Since G is generic for κ-directed closed forcing, κ has the tree property in $V[G]$. Let $f \in V[G]$ be a bijection from κ to λ. We define a thin $(\kappa, \kappa, 2)$-tree from T using f. Let $h \in S$ if and only if there is $g \in T$ such that $\mathrm{dom}(h) = f^{-1}[\mathrm{dom}(g)]$ and for all $\alpha \in \mathrm{dom}(h)$, $h(\alpha) = g(f(\alpha))$.

Clearly S is a thin $(\kappa, \kappa, 2)$-tree. It follows that S has a branch d_S in $V[G]$. Let $d = d_S \circ f^{-1}$. Clearly d is a branch through T. It will be enough to show that $d \in V$. Let $\dot{d}$ be a $\mathrm{Coll}(\kappa, \lambda)$ name for d. We claim that $\dot{d}$ satisfies the hypotheses of the thin κ-approximation property. We may assume that it is forced that $\dot{d}$ is a branch through T. So it is forced that $\dot{d} \cap x \in T_x \subseteq V$. Further the set of possibilities for $\dot{d} \cap x$ has size at most $|T_x| < \kappa$, since T is thin. To finish the proof we note that $\mathrm{Coll}(\kappa, \lambda)$ has the thin κ-approximation property in V. This is clear from Lemma 3.4 if we let χ be the predecessor of κ. $\square$

This finishes the proof of Theorem 3.1.

4. Successive failures of weak square

In this section we prove some theorems about successive failures of weak square. In the first of our two theorems, we prove that one can obtain for all $n \geq 1$, the failure of $\square^*_{\aleph_n}$ with a cardinal arithmetic pattern similar to the one in the model of Cummings and Foreman for the tree property at $\aleph_n$ for $n \geq 2$. In the second of our theorems we prove that one can obtain further successive failures of weak square at the cost that many singular cardinals are not strong limit.

In some cases we will show that weak square fails by arguing that there are no special Aronszajn trees. This is equivalent by a theorem of Jensen [9]. Recall that a μ^+-tree T is special if there is a function f from T to μ such that $f(s) \neq f(t)$ whenever $s <_T t$.

For our first theorem of this section, we prove:

Theorem 4.1. *Assuming there are infinitely many Mahlo cardinals, there is a forcing extension in which $\square^*_{\aleph_n}$ fails for $1 \leq n < \omega$.*

The forcing for Theorem 4.1 is a full support product of posets like Mitchell's original forcing. We describe an abstract version of Mitchell's forcing $\mathbb{M}$ with three cardinal parameters.

Definition 4.2. *Let $\kappa < \lambda < \mu$ be regular cardinals. We define $\mathbb{P} = \mathbb{P}(\kappa, \mu)$ to be $\mathrm{Add}(\kappa, \mu)$ and further let $\mathbb{P} \restriction \alpha = \mathrm{Add}(\kappa, \alpha)$ for all $\alpha < \mu$. Next we define $\mathbb{M} = \mathbb{M}(\kappa, \lambda, \mu)$ to be the collection of pairs (p, f) such that $p \in \mathbb{P}$ and f is a function such that $|f| < \lambda$, $\mathrm{dom}(f)$ is a subset of the interval (λ, μ) and for all $\alpha \in \mathrm{dom}(f)$, $f(\alpha)$ is a $\mathbb{P} \restriction \alpha$-name for a condition in $\mathrm{Add}(\lambda, 1)_{V[\mathbb{P}\restriction\alpha]}$.*

We set $(p_1, f_1) \leq (p_2, f_2)$ if and only if $p_1 \leq p_2$, $\mathrm{dom}(f_1) \supseteq \mathrm{dom}(f_2)$ and for all $\alpha \in \mathrm{dom}(f_2)$ $p \restriction \alpha \Vdash f_1(\alpha) \leq f_2(\alpha)$.

Mitchell's original forcing for obtaining the tree property at ω_2 is $\mathbb{M}(\omega, \omega_1, \kappa)$ where κ is weakly compact. The following sequence of lemmas are straightforward modifications of lemmas of Abraham [1].

Lemma 4.3. *$\mathbb{M}$ is κ-closed and μ-Knaster.*

Lemma 4.4. *There are projections from $\mathbb{M}$ to $\mathbb{P}$ and $\mathbb{P}\restriction\alpha * \mathrm{Add}(\lambda, 1)_{V[\mathbb{P}\restriction\alpha]}$ for all $\alpha \in (\lambda, \mu)$.*

Lemma 4.5. *There is a λ-closed, μ-Knaster forcing $\mathbb{R}$ such that $\mathbb{M}$ is the projection of $\mathbb{P} \times \mathbb{R}$.*

For all α in the interval (λ, μ), let $\mathbb{M} \restriction \alpha$ be the natural restriction of $\mathbb{M}$ to functions whose domains are a subset of α.

Lemma 4.6. *For all $\alpha \in (\lambda, \mu)$, in $V[\mathbb{M} \restriction \alpha]$ the poset $\mathbb{M}/\mathbb{M} \restriction \alpha$ is the projection of $\mathbb{P}^* \times \mathbb{R}^*$ where $\mathbb{P}^* = \mathrm{Add}(\kappa, \mu \setminus \alpha)$ and $\mathbb{R}^*$ is λ-closed and μ-Knaster.*

Lemma 4.7. *In $V[\mathbb{M}]$ $2^\kappa = \mu$, λ is preserved and $\mu = \lambda^+$.*

We define $\mathbb{S}$ in $V[\mathbb{M}]$ to be $(\mathbb{P} \times \mathbb{R})/\mathbb{M}$. Implicit in the proof of the above lemma is the following.

Lemma 4.8. *In $V[\mathbb{M}]$ forcing with $\mathbb{S}$ preserves cardinals.*

We are now ready to define the main forcing used in the proof of Theorem 4.1. We work in a model of GCH with an increasing sequence of Mahlo cardinals $\langle \lambda_n \mid n < \omega \rangle$. For ease of notation we define $\lambda_{-2} = \omega$ and $\lambda_{-1} = \omega_1$. For all $n < \omega$ we define $\mathbb{M}_n$ to be $\mathbb{M}(\lambda_{n-2}, \lambda_{n-1}, \lambda_n)$. The main forcing is $\mathbb{Q}$ the full support product of the $\mathbb{M}_n$. We transfer all of the notation from the abstract version of $\mathbb{M}$ to each $\mathbb{M}_n$. For example we write $\mathbb{M}_n$ as the projection of $\mathbb{P}_n \times \mathbb{R}_n$.

Theorem 4.9. *In $V[\mathbb{Q}]$, $\square^*_{\aleph_k}$ fails for all $k \geq 1$.*

We start by proving a lemma about the cardinal structure of $V[\mathbb{Q}]$.

Lemma 4.10. *In $V[\mathbb{Q}]$, for all $n \geq -1$, λ_n is preserved and for all $n \geq -2$, $2^{\lambda_n} = \lambda_{n+2} = \aleph_{n+4}$.*

Proof. Let $n \geq -1$. $\mathbb{M}_{n+1}$ is the projection of a product of λ_n-cc forcing and λ_n-closed forcing. $\prod_{i \leq n} \mathbb{M}_i$ is λ_n-cc and $\prod_{i > n+1} \mathbb{M}_i$ is λ_n closed. It follows that λ_n is preserved in $V[\mathbb{Q}]$. Furthermore every $< \lambda_n$-sequence from $V[\mathbb{Q}]$ is in $V[\prod_{i \leq n} \mathbb{M}_i \times \mathbb{P}_{n+1}]$. Since this forcing is λ_n-cc of size λ_{n+1} which is inaccessible in V, we have $2^{\lambda_{n-1}} \leq \lambda_{n+1}$. Since $\mathbb{Q}$ projects to $\mathbb{P}_{n+1}$, we have $2^{\lambda_{n-1}} \geq \lambda_{n+1}$. It remains to show that for all $n \geq -2$, $\lambda_{n+2} = \aleph_{n+4}$. It will be enough to show that for all $n \geq -2$, λ_{n+1} is the successor of λ_n. This is immediate from Lemma 4.7 and the fact that forcing with $\mathbb{Q}$ preserves each λ_n. $\square$

We are now ready to prove Theorem 4.1.

Proof. Let $n < \omega$. It is enough to show that there are no special λ_n-trees in $V[\mathbb{Q}]$. In the proof of Lemma 4.10 we showed that every λ_n-sequence from $V[\mathbb{Q}]$ is a member of $V[\prod_{i \leq n+1} \mathbb{M}_i \times \mathbb{P}_{n+2}]$. So it is enough to show that there are no special λ_n-trees in this model. It is not hard to see that $\mathbb{S}_{n+1}$ preserves cardinals over $V[\prod_{i \leq n} \mathbb{M}_i \times \mathbb{P}_{n+2}]$. So it is enough to show that there are no special λ_n-trees in $V[\prod_{i \leq n} \mathbb{M}_i \times \mathbb{P}_{n+1} \times \mathbb{R}_{n+1} \times \mathbb{P}_{n+2}]$.

We reorganize the forcing to do the λ_n-closed forcing first. Note that λ_n is still Mahlo in $V[\mathbb{R}_{n+1} \times \mathbb{P}_{n+2}]$, since both forcings are λ_n-closed. We also note that all chain conditions and closure properties of forcings $\mathbb{M}_i$ for $i \leq n$ and $\mathbb{P}_{n+1}$ are preserved in this extension. Further even chain condition and closure properties of $\mathbb{P}_i$ and $\mathbb{R}_i$ for $i \leq n$ are preserved. For ease of notation we let $W = V[\mathbb{R}_{n+1} \times \mathbb{P}_{n+2}]$.

Suppose for a contradiction that $\dot{T}, \dot{h}$ are $\prod_{i \leq n} \mathbb{M}_i \times \mathbb{P}_{n+1}$-names for a λ_n-tree and specializing function. We may assume that it is forced that the underlying set of $\dot{T}$ is λ_n and $\dot{h}$ is a function from λ_n to λ_{n-1}. The forcing $\prod_{i \leq n} \mathbb{M}_i \times \mathbb{P}_{n+1}$ is λ_n-cc and hence there is an $X \subseteq \lambda_{n+1}$ with $|X| = \lambda_n$ such that $T, h \in W[\prod_{i \leq n} \mathbb{M}_i \times \mathbb{P}_{n+1} \restriction X]$. By rearranging the forcing we may assume that $X = \lambda_n$.

We need to locate each initial segment of T and h more precisely. Working in $W[\prod_{i \leq n} \mathbb{M}_i \times \mathbb{P}_{n+1} \restriction \lambda_n]$ we see that for each $\alpha < \lambda_n$ there is a $\nu_\alpha < \lambda_n$ such that $T \restriction \alpha, h \restriction \alpha \in W[\prod_{i \leq n} \mathbb{M}_i \restriction \nu_\alpha \times \mathbb{P}_{n+1} \restriction \nu_\alpha]$. So in the extension there is a club of closure points of the map $\alpha \mapsto \nu_\alpha$. Since the forcing to add T and h over W is λ_n-cc (hence preserves stationary subsets of λ_n), there is a V-inaccessible closure point γ.

Note that we have $T \restriction \gamma, h \restriction \gamma \in W[\prod_{i \leq n} \mathbb{M}_i \restriction \gamma \times \mathbb{P}_{n+1} \restriction \gamma]$ and this extension is by γ-cc forcing. It follows that $T \restriction \gamma$ is a special γ-tree as witnessed by $h \restriction \gamma$. It will be enough to show that the forcing to get from $W[\prod_{i \leq n} \mathbb{M}_i \restriction \gamma \times \mathbb{P}_{n+1} \restriction \gamma]$ up to $W[\prod_{i \leq n} \mathbb{M}_i \times \mathbb{P}_{n+1}]$ cannot add a branch through $T \restriction \gamma$. This is a contradiction since $T \restriction \gamma$ would be a branchless initial segment of T.

By Lemma 4.6 it is enough to show that forcing with $\mathbb{P}_{n+1} \restriction [\gamma, \lambda_n) \times \mathbb{R}^*_n \times \mathbb{P}^*_n$ could not have added a branch through $T \restriction \gamma$ over $W[\prod_{i \leq n} \mathbb{M}_i \restriction \gamma \times \mathbb{P}_{n+1} \restriction \gamma]$. It is clear that $\mathbb{P}_{n+1} \restriction [\gamma, \lambda_n)$ cannot add a branch, since it is γ-cc and $T \restriction \gamma$ is special. Next we need to show that $\mathbb{R}^*_n$ satisfies the hypotheses of Lemma 0.1 in the current model.

By Lemma 4.6, $\mathbb{R}^*_n$ is λ_{n-1}-closed and $2^{\lambda_{n-2}} = \gamma$ in $W[\mathbb{M}_n \restriction \gamma]$. Moreover $\prod_{i<n} \mathbb{M}_i \times \mathbb{P}_{n+1} \restriction \lambda_{n+1}$ is the product of λ_{n-1}-Knaster and λ_{n-1}-closed forcing in W. The hypotheses of Lemma 0.1 follow. So we have proved that $T \restriction \gamma$ is still branchless in $W[\prod_{i \leq n} \mathbb{M}_i \times \mathbb{P}_{n+1} \restriction \lambda_n \times \mathbb{R}^*_n]$. Moreover in this model γ is collapsed to have cofinality $\lambda_{n-1} = \aleph_{n+1}$. Straightforward applications of Easton's Lemma show that $(\mathbb{P}^*_n)^2$ is λ_{n-1}-cc in this model and hence by Lemma 1.6 applied to $T \restriction \gamma$ we have our desired contradiction. □

In our second theorem of the section, we show that we can obtain further successive failures of weak square. Recall that if $2^\mu = \mu^+$, then there is a μ^{++}-Aronszajn tree. So to have the failure of weak square at μ^+, we must have $2^\mu > \mu^+$. This issue is particularly important when μ is singular, since we have to deal with the singular cardinals problem. A problem of interest in this area is an open question of Woodin's from the 1980's whether is it consistent to have $\aleph_\omega$ strong limit, $2^{\aleph_\omega} > \aleph_{\omega+1}$

and the failure of $\square^*_{\aleph_\omega}$. A positive answer to this question is required for us to extend our first theorem from this section in the case where $\aleph_\omega$ is strong limit.

In the following theorem, we avoid the difficulties of Woodin's question by forcing $\aleph_\omega$ not to be a strong limit cardinal.

Theorem 4.11. *Suppose there is a supercompact cardinal with an increasing sequence of Mahlo cardinals of order type ω^2 above it. There is a generic extension in which $\square^*$ fails for all cardinals in the interval $[\aleph_1, \aleph_{\omega^2}]$.*

For this theorem, we will use Neeman's [15] formulation of Mitchell's posets. We first present an abstract version of a poset of Mitchell-like collapses.

Definition 4.12. *Let $\tau < \chi < \chi' \leq \eta$ be regular cardinals and let $\mathbb{P} = \mathrm{Add}(\tau, \eta)$. Let $\mathbb{C}(\mathbb{P}, \chi, \chi')$ be the collection of partial functions f of size $< \chi$ whose domain is a subset of (χ, χ') such that for all $\alpha \in \mathrm{dom}(f)$, $f(\alpha)$ is a $\mathbb{P} \restriction \alpha$-name for an element of $\mathrm{Add}(\chi, 1)_{V[\mathbb{P}\restriction\alpha]}$. We order the poset by $f_1 \leq f_2$ if and only if $\mathrm{dom}(f_1) \supseteq \mathrm{dom}(f_2)$ and for all $\alpha \in \mathrm{dom}(f_2)$, $\Vdash_{\mathbb{P}\restriction\alpha} f_1(\alpha) \leq f_2(\alpha)$.*

Note that such posets are easily 'enriched' in the sense of Neeman to give Mitchell like collapses. The *enrichment* of $\mathbb{C}(\mathbb{P}, \chi, \chi')$ by $\mathbb{P}$ is the poset defined in the generic extension by $\mathbb{P}$ with the same underlying set as $\mathbb{C}(\mathbb{P}, \chi, \chi')$, but with the order $f_1 \leq f_2$ if and only if $\mathrm{dom}(f_1) \supseteq \mathrm{dom}(f_2)$ and there is p in the generic for $\mathbb{P}$ such that for all $\alpha \in \mathrm{dom}(f_2)$, $p \restriction \alpha \Vdash f_1(\alpha) \leq f_2(\alpha)$. If P is $\mathbb{P}$-generic we will write $\mathbb{C}^{+P}$ for the enrichment of $\mathbb{C}$ by P. Note that the poset $\mathbb{C}(\mathbb{P}, \chi, \chi')$ is χ-closed, χ'-cc if χ' is inaccessible and collapses every regular cardinal in the interval (χ, χ') to have size χ. Hence it makes χ' into χ^+.

We are now ready to prove Theorem 4.11. Let $\langle \lambda_{m,n} \mid n < \omega, m < \omega \rangle$ be a sequence of cardinals such that $\lambda_{0,0} = \omega$ and for all $m < \omega$, $\lambda_{m+1,0} = \sup_{n<\omega} \lambda_{m,n}$ and $\lambda_{m,1} = \lambda^+_{m,0}$, and for all $n < \omega$ $\lambda_{m,n+2}$ is the least Mahlo cardinal greater than $\lambda_{m,n+1}$. The sole exception to this scheme is $\lambda_{0,2}$ which we assume is indestructibly supercompact. Our intention is to force $\lambda_{m,n} = \aleph_{\omega\cdot m+n}$. To do so we define a helpful function $k : \omega \times \omega \to \omega$ by

$$k(m,n) = \begin{cases} 0 & \text{if } (m,n) = (0,2) \\ m+1 & \text{if } n > 2 \\ m & \text{if } n = 2 \text{ and } m > 0 \end{cases}$$

which will form the association between posets that collapse cardinals and posets that add subsets.

Next we define the necessary posets:

- $\mathbb{A}_0 = \mathrm{Add}(\omega, \lambda_{0,2})$.
- For $m \geq 1$, $\mathbb{A}_m = \mathrm{Add}(\lambda_{m-1,1}, \lambda_{m,2})$.
- For $m < \omega$ and $2 \leq n < \omega$, $\mathbb{C}_{m,n} = \mathbb{C}(\mathbb{A}_{k(m,n)}, \lambda_{m,n-1}, \lambda_{m,n})$.

The model for the theorem is obtained by first forcing with the full support product $\prod_{m<\omega} \mathbb{A}_m$ to get generics A_m for $m < \omega$, then forcing with the full support product $\prod_{m<\omega, 2\leq n<\omega} \mathbb{C}_{m,n}^{+A_{k(m,n)}\restriction\lambda_{m,n}}$. We call this final model W and note that there is an outer model W' obtained by forcing with the full support product $\prod_{m<\omega} \mathbb{A}_m \times \prod_{m<\omega,2\leq n<\omega} \mathbb{C}_{m,n}$.

We have the following straightforward lemmas:

Lemma 4.13. *For all $m, n < \omega$, $\lambda_{m,n}$ is preserved in W'.*

Proof. It is not hard to see that for every $m < \omega$ and $n \geq 2$ the forcing to obtain W' can be written as a product of $\lambda_{m,n}$-cc forcing and $\lambda_{m,n}$-closed forcing and also as a product of $\lambda_{0,1} = \aleph_1$-cc and $\aleph_1$-closed forcing. It follows that $\lambda_{m,n}$ is preserved for all $m < \omega$ and $n \geq 2$. Together with the argument above, standard arguments allow us to show that in W' $\lambda_{m,0}$ and $\lambda_{m,1}$ are preserved for all $m < \omega$. $\square$

Lemma 4.14. *For every $m < \omega$, $2 \leq n < \omega$ and cardinal α in the interval $(\lambda_{m,n-1}, \lambda_{m,n})$, α is collapsed to have cardinality $\lambda_{m,n-1}$ in W.*

This follows easily from the definition of the collapse posets $\mathbb{C}_{m,n}$. As a corollary we have the expected cardinal structure.

Corollary 4.15. *In W (and also W') we have that for all $m, n < \omega$, $\lambda_{m,n} = \aleph_{\omega\cdot m+n}$.*

Straightforward modifications of the arguments from the proof of Theorem 4.1 give a proof of the following:

Lemma 4.16. *For all $m < \omega$ and $n \geq 1$, $\square^*_{\aleph_{\omega\cdot m+n}}$ fails in W.*

For the proof of Theorem 4.11 it remains to show:

Lemma 4.17. *For all $1 \leq l \leq \omega$, $\square^*_{\aleph_{\omega\cdot l}}$ fails in W.*

Proof. Assume for a contradiction that for some $l \geq 1$, $\square^*_{\omega\cdot l}$ holds in W. For all pairs (m, n) such that $k(m, n) = 1$ we force over W to refine the generic for $\mathbb{C}_{m,n}^{+A_{k(m,n)}}$ to a generic for $\mathbb{C}_{m,n}$ and obtain a model W^* with $W \subseteq W^* \subseteq W'$. It follows that W and W^* have the same cardinals and hence $\square^*_{\aleph_{\omega\cdot l}}$ holds in W^*.

By passing to W^* we have ensured that the forcing above $\lambda_{0,2}$, which is

$$\Big(\prod_{m\geq 2} \mathbb{A}_m * \prod_{\{(m,n)\mid k(m,n)>1\}} \mathbb{C}_{m,n}^{+A_{k(m,n)}}\Big) \times \prod_{\{(m,n)\mid k(m,n)=1\}} \mathbb{C}_{m,n},$$

is $\lambda_{0,2}$-directed closed in V. We collect this forcing and call it $\mathbb{X}$. It is now clear that W^* is the extension of V by the product $\mathbb{X}\times(\mathbb{A}_0*\mathbb{C}_{0,2}^{+\mathbb{A}_0})\times\mathbb{A}_1$. Recall that we chose $\lambda_{0,2}$ to be indestructibly supercompact in V and hence there is an embedding $j : V[\mathbb{X}] \to M$ witnessing that $\lambda_{0,2}$ is θ-supercompact for some $\theta > \sup_{m<\omega} \lambda_{m,0}$.

By standard arguments, the posets

- $\hat{\mathbb{A}}_0 = j(\mathbb{A}_0) \restriction [\lambda_{0,2}, j(\lambda_{0,2}))$,
- $\hat{\mathbb{A}}_1 = j(\mathbb{A}_1) \restriction (j(\lambda_{1,2}) - j``\lambda_{1,2})$ and
- $\hat{\mathbb{C}} = j(\mathbb{C}_{0,2})^{+A_0} \restriction [\lambda_{0,2}, j(\lambda_{0,2}))$

are enough to extend the embedding j to W^*. Moreover, $\aleph_{\omega\cdot l+1}$ has been collapsed to have cofinality ω_1 in $W^*[\hat{\mathbb{A}}_0 \times \hat{\mathbb{A}}_1 \times \hat{\mathbb{C}}]$.

Let $\langle \mathcal{C}_\alpha \mid \alpha < \aleph_{\omega\cdot l+1}\rangle$ be a weak square sequence in W^*. We can assume that for all α there is a club $C \in \mathcal{C}_\alpha$ with $\mathrm{ot}(C) = \mathrm{cf}(\alpha)$ and for all α and all $C \in \mathcal{C}_\alpha$, $\mathrm{ot}(C) < \aleph_{\omega\cdot l}$. Using our extended j, we let E be a club of order type ω_1 in $j(\vec{\mathcal{C}})_{\sup j``\aleph_{\omega\cdot l+1}}$. Working in $W^*[\hat{\mathbb{A}}_0 \times \hat{\mathbb{A}}_1 \times \hat{\mathbb{C}}]$, we can define $D^* = \{\alpha \mid j(\alpha) \in E\}$. Since $j``\aleph_{\omega\cdot l+1}$ is ω-closed, we have that D^* is club in $\aleph_{\omega\cdot l+1}$ of order type ω_1.

It is not hard to see that $\hat{\mathbb{A}}_0$ is isomorphic to the forcing to add $j(\lambda_{0,2})$ Cohen reals and hence is ccc in $W^*[\hat{\mathbb{A}}_1 \times \hat{\mathbb{C}}]$. Since $\hat{\mathbb{A}}_0$ is ccc, there is a club $D \in W^*[\hat{\mathbb{A}}_1 \times \hat{\mathbb{C}}]$ with $D \subseteq D^*$. Variants of the following two claims appear in the literature on successive cardinals with the tree property. For example the first is an application of Lemma 2.13 of [3] and the second is essentially Claim 4.15 from [15].

Claim 4.18. *$\hat{\mathbb{A}}_1$ is $\lambda_{0,2} = \aleph_2$-cc and $< \omega_1$-distributive in W^*.*

It follows that $\hat{\mathbb{A}}_1$ preserves cardinals over W^* and hence $\vec{\mathcal{C}}$ remains a weak square sequence in $W^*[\hat{\mathbb{A}}_1]$.

Claim 4.19. *$\hat{\mathbb{C}}$ is countably closed in W^* and hence in $W^*[\hat{\mathbb{A}}_1]$.*

Corollary 4.20. *W^* and $W^*[\hat{\mathbb{A}}_1 \times \hat{\mathbb{C}}]$ have the same ω-sequences.*

It follows that for all $\alpha < \aleph_{\omega\cdot l+1}$, $D \cap \alpha$ is in W^* and hence in the domain of j. So for all $\alpha \in \lim(D)$, we have $j(D \cap \alpha) = j``D \cap \alpha \subseteq E \cap j(\alpha) \in j(\vec{\mathcal{C}})_{j(\alpha)}$ and so by elementarity there is a $C \in \mathcal{C}_\alpha$ such that $D \cap \alpha \subseteq C$.

We arrive at a contradiction by showing that the countably closed forcing $\hat{\mathbb{C}}$ cannot add such a club D. This follows from a small modification of the first claim in Section 5 of Foreman and Magidor's [4], which is an adaptation of a special case of Theorem 2.1 in a paper of Magidor and Shelah [10]. For completeness we give a sketch of the argument.

Claim 4.21. *For all $q \in \hat{\mathbb{C}}$ and all $\alpha < \aleph_{\omega \cdot l+1}$, there is $\beta > \alpha$ such that the set $\{s \mid \exists q' \leq q,\ q' \vDash s = D \cap \beta\}$ has size $\aleph_{\omega \cdot l}$.*

Otherwise we have a condition q and a bound $\delta < \aleph_{\omega \cdot l}$ such that for all $\beta < \aleph_{\omega \cdot l+1}$, the size of the set of possible values for $D \cap \beta$ decided by conditions below q is at most δ. An argument similar to the one in Lemma 1.4 shows that q forces D to be in $W^*[\hat{\mathbb{A}}_1]$, which is impossible since D has order type ω_1.

Now that we have many levels on which the set of possible values for $D \cap \beta$ is large, we can build a tree of possible values for initial segments of D indexed by elements of $\aleph_{\omega \cdot l}^{<\omega}$. Using the above claim we can ensure that different branches through the tree force different values for initial segments of D. So we obtain an ordinal β such that the set of possible values for $D \cap \beta$ is $\aleph_{\omega \cdot l+1}$. We will show that this is impossible. Any value for $D \cap \beta$ must be contained in some club $C \in \mathcal{C}_\beta$, but each $C \in \mathcal{C}_\alpha$ was assumed to have order type less than $\aleph_{\omega \cdot l}$. So the number of possible values for $D \cap \beta$ is at most $\aleph_{\omega \cdot l}$, since $\mu^\omega < \aleph_{\omega \cdot l}$ for all $\mu < \aleph_{\omega \cdot l}$ in $W^*[\hat{\mathbb{A}}_1]$. This finishes the proof of Lemma 4.17 and with it the proof of Theorem 4.11. ☐

We end this section with a few remarks on the proof of Theorem 4.11. There is significant freedom in the choice of the $\mathbb{A}$ posets and hence we can create other GCH patterns in the final model. We can obtain more successive failures of weak square provided that we avoid singular strong limit cardinals with the exception of the end of the interval. So in particular we can obtain the failure of weak square at every cardinal less than or equal to $\beth_\omega$. We chose this particular presentation, because it is close to the GCH pattern that might occur if we successfully answer the following question.

Question 4.22. *Is it consistent that $\square^*$ fails for all cardinals in the interval $[\aleph_1, \aleph_{\omega^2}]$, $\aleph_{\omega^2}$ is strong limit and $2^{\aleph_{\omega^2}} > \aleph_{\omega^2+1}$?*

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