GITIK'S GAP 2 SHORT EXTENDER FORCING WITH COLLAPSES

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In this somewhat expository paper we write up some of the details required to interleave collapses with Gitik's gap 2 short extender forcing [1]. In a previous paper [4] we gave a distilled view of the chain condition argument for the gap 2 forcing. A definition of the appropriate forcing was given in [3], but most of the details are omitted. In this paper we prove the following theorem.

Theorem 0.1 (Gitik). *Assuming there is a sequence $\langle \kappa_n \mid n < \omega \rangle$ with $\kappa = \sup_{n<\omega} \kappa_n$ and each κ_n is κ_n^{+n+2}-strong, then there is a forcing extension in which $\kappa = \aleph_\omega$ is strong limit and $\aleph_\omega^\omega = \aleph_{\omega+2}$.*

We assume familiarity with the presentation of the gap 2 forcing as presented in either our previous paper [4] or in Gitik's [2]. We present the poset P^* with the Prikry ordering first and highlight the way that the gap 2 short extender forcing $(P, \leq)$ sits inside it. The restricted ordering on P^* which satisfies the κ^{++}-cc is obtained by restricting the Prikry ordering to use $(P, \rightarrow)$ instead of $(P, \leq)$. The main obstacle in this case is to reconstruct the proof of the Prikry lemma with the added collapses.

1. The main forcing

Recall that for $n < \omega$ we have $(\kappa_n, \kappa_n^{+n+2})$-extenders E_n witnessing that κ_n is κ_n^{+n+2}-strong. Let P be the gap 2 short extender forcing. We define a new poset P^* with some extra features. In order to collapse between the κ_n's we record the diagonal Prikry sequence corresponding to the normal measure in each extender. We use this sequence to break up the collapse. For notational convenience we let $\kappa_{-1} = \omega_1$.

Definition 1.1. *Let $p = \langle d \rangle \frown \langle p_n \mid n < \omega \rangle$ be in P^* if there is $l = \mathrm{lh}(p)$ such that for $n < l$ we have $p_n = (\rho_n, f_n, g_n, h_n)$ and for $n \geq l$ we have $p_n = (a_n, A_n, f_n, g_n, H_n)$ with the following properties*

1. *the sequence $\langle f_n \mid n < l \rangle \frown \langle (a_n, A_n, f_n) \mid n \geq l \rangle \in P$,*
2. *for $n < l$, $\rho_n \in (\kappa_{n-1}, \kappa_n)$,*
3. *for $n < l$, $(g_n, h_n) \in \mathrm{Coll}(\kappa_{n-1}^{+n+8}, <\rho_n) \times \mathrm{Coll}(\rho_n^{+n+4}, <\kappa_n)$,*
4. *for $n \geq l$, $g_n \in \mathrm{Coll}(\kappa_{n-1}^{+(n-1)+8}, <\kappa_n)$ and H_n is a function with domain $\pi_{\mathrm{mc}(a_n)0}$“A_n such that for all ρ in the domain, $H_n(\rho) \in \mathrm{Coll}(\rho^{+n+4}, <\kappa_n)$.*

To indicate that a particular component belongs to a particular poset we add a superscript p, so for instance we might have $p_n = (\rho_n^p, f_n^p, g_n^p, h_n^p)$. We also define the natural map $\sigma : P^* \to P$, which for a condition above returns the element of P mentioned in the first item. Let $p, q \in P^*$. We define $p \leq q$ if

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(1) *$\sigma(p) \leq \sigma(q)$ in $(P, \leq)$ in particular $lh(p) \geq \mathrm{lh}(q)$ and there is a sequence $\langle \nu_n \mid n \in [\mathrm{lh}(q), \mathrm{lh}(p)) \rangle$ of elements from relevant measure one sets witnessing this extension,*
(2) *for all $n < \mathrm{lh}(q)$, $h^p_n \leq h^q_n$ and $\rho^p_n = \rho^q_n$,*
(3) *for all n with $\mathrm{lh}(q) \leq n < \mathrm{lh}(p)$, $\pi_{\mathrm{mc}(a^q_n)0}(\nu_n) = \rho^p_n$ and $h^p_n \leq H^q_n(\rho^p_n)$,*
(4) *for all $n \leq \mathrm{lh}(p)$ and all $\rho \in \mathrm{dom}(H^p_n)$, $H^p_n(\rho) \leq H^q_n(\rho)$, and*
(5) *for all $n < \omega$, $g^p_n \leq g^q_n$.*

As usual we have $p \leq^* q$ if $p \leq q$ and $\mathrm{lh}(p) = \mathrm{lh}(q)$.

To ensure coherence between the selected ρ's, we work on the dense set of conditions such that for all n and for all $\eta \leq_{E_n} \delta \in \mathrm{rng}(a_n)$ and all $\nu \in A_n$, $\pi_{\delta 0}(\nu) = \pi_{\eta 0}(\pi_{\delta\eta}(\nu))$.

The definition makes it clear that σ is a projection map from $(P^*, \leq)$ to $(P, \leq)$. It follows that $\kappa^\omega \geq \kappa^{++}$ in the extension provided we can show that κ^{++} is preserved.

Definition 1.2. *We define another ordering $\to$ on P^* by replacing $\sigma(p) \leq \sigma(q)$ with $\sigma(p) \leftarrow \sigma(q)$ in clause (1) of the definition of the ordering.*

Again it is clear that σ defines a projection from $(P^*, \to)$ to $(P, \to)$.

Lemma 1.3. *For all $p \in P^*$ and all dense open D in $(P^*, \leq)$ there are $p^* \leq^* p$ and $n^* < \omega$ such that for all $\vec{\nu} \in \prod_{\mathrm{lh}(p^*) \leq i < \mathrm{lh}(p^*)+n^*} A^{p^*}_i$, $p^* \frown \vec{\nu} \in D$.*

Proof. Let $p \in P^*$ and $D \subseteq P^*$ be dense open. The argument proceeds in four rounds. In the first round we get a 'universal upper part'.

Claim 1.4. *There is a condition $q \leq^* p$ such that for all $r \leq q$, if $r \in D$, then $\langle r_n \mid n < \mathrm{lh}(r) \rangle \frown \langle q_n \mid n \geq \mathrm{lh}(r) \rangle \in D$.*

We work to diagonalize over possible extensions of p. For each $k < \omega$, we enumerate $\vec{g} = \langle g_n \mid n < \mathrm{lh}(p) + k \rangle$, $\vec{h} = \langle h_n \mid n < \mathrm{lh}(p) + k \rangle$ and $\vec{\nu} = \langle \nu_i \mid i < k \rangle$ such that for some $q \leq p \frown \vec{\nu}$, $g_n = g^q_n$ and $h_n = h^q_n$ for $n < \mathrm{lh}(q)$. We note that there are just $\kappa_{\mathrm{lh}(p)+k-1}$ such sequences and the forcing restricted to coordinates greater than or equal to $\mathrm{lh}(p) + k$ is $\kappa^+_{\mathrm{lh}(p)+k-1}$-closed.

We work by induction on $k < \omega$ to create a decreasing sequence of direct extensions of p. At stage k in the induction we diagonalize over the above enumeration. At a given stage in our enumeration, we have a condition $q \leq^* p$ and we work with some $\vec{g}$, $\vec{h}$ and $\vec{\nu}$, we ask if there is a direct extension r of the condition determined by q and $\vec{g}$, $\vec{h}$ and $\vec{\nu}$ which is in D. If one exists then we must "capture" it using a direct extension q. We work as follows:

- For $n < \mathrm{lh}(p)$, we decrease f^q_n to f^r_n.
- For $n \in [\mathrm{lh}(p), \mathrm{lh}(p) + k)$, we decrease f^q_n to $f^r_n \upharpoonright (\mathrm{dom}(f^r_n) \setminus \mathrm{dom}(a^q_n))$.
- For $n \geq \mathrm{lh}(p) + k$, we decrease q_n to r_n.

Using the closure of the relevant posets there are no issues with the limit stages of the construction. The desired q is a direct extension of all "q"'s appearing in the construction. Using the fact that every extension in P^* can be viewed as first adding a sequence $\vec{\nu}$ in a minimal way and then taking a direct extension, it is not hard to see that q has the desired universal property.

In the second round of the construction, we capture extensions of the top most collapse in the lower part of the condition. We use a similar enumeration as in round one of our construction except that we omit $h_{\mathrm{lh}(p)+k-1}$.

We define a sequence of sets Y_m for $m < \omega$ and a sequence $\langle q^m \mid m < \omega \rangle$ of direct extensions of q where we have only decreased the values of the constraining functions H_n and the f_n. We let Y_0 be the set $\{r \restriction n \mid r \restriction n \frown q \restriction [n, \omega) \in D\}$ and $q_0 = q$

Assuming that we have defined q_m and Y_m for some $m < \omega$, we seek to define Y_{m+1} and q_{m+1}. We work with q_{m+1} first and construct it by diagonalizing over the enumerations mentioned above by induction on k. We work with some $k < \omega$ and at each stage in the diagonalization we work with some $\vec{g}, \vec{h}$ and $\vec{\nu}$ (note that the $\vec{h}$ is shorter this time). By induction, we have a direct extension $\hat{q}$ of q_m and we ask if there are an extension $\vec{f}$ of $\langle f^{\hat{q}}_n \mid n < \mathrm{lh}(q) + k \rangle$ and an extension h of $H^{q^m}_{\mathrm{lh}(q_m)+k-1}(\pi_{\mathrm{mc}(a^{q_m}_n)0}(\nu_{k-1}))$ such that $\hat{q} \restriction (\mathrm{lh}(q_m) + k)$ strengthened by $\vec{g}$, $\vec{f}$ and h is in Y_m. If there is then we update the $f^{\hat{q}}_n$'s using $\vec{f}$ and the relevant constraining function in $\hat{q}$ to return h on the relevant coordinate.

Again there is no issue at limit stages since we may assume that there are just $\rho^{+\,\mathrm{lh}(q_m)+k+2}$ many ν's which project to a given ρ and the contraining functions at ρ take values in $\rho^{+\,\mathrm{lh}(q_m)+k+4}$-closd forcing. Further, the forcing in the f-parts is κ^+-closed. We let q_{m+1} be a lower bound for the construction.

We define

$$\begin{aligned} Y_{m+1} = & \{r \restriction n \mid r \in P^*, \mathrm{lh}(r) = n \text{ and } (\exists g \leq g^{q_{m+1}}_n)(\exists A \in E_{n\,\mathrm{mc}(a^q_n)}) \\ & (\forall \nu \in A) \text{ if } \rho = \pi_{\mathrm{mc}(a^{q_{m+1}}_n)0}(\nu) \text{ then} \\ & r \restriction n \frown (\rho, (a^{q^{m+1}}_n, A^{q^{m+1}}_n, f^{q_{m+1}}_n) \frown \nu, g, H^{q_{m+1}}_n(\rho)) \in Y_m\} \end{aligned}$$

We note that Y_{m+1} is closed under strengthening the f, g and h parts of $r \restriction n$. At this point it is not clear that we can meet the challenge of getting lower parts in to the Y_m. We only know that if we manage to succeed, then we did not need to decrease the topmost collapsing condition nor the f-part below the length of the condition.

We choose a condition q_ω below each q_m. In the third round of the construction we wish to capture the g parts which witness membership in each Y_m. It is not hard to show that we can find $q_{\omega+1} \leq q_\omega$ such that for all m and for all $r \leq q_{\omega+1}$ of length n if $r \restriction n \in Y_m$ and this is witnessed by some $g \leq g^{q_{\omega+1}}_n$, then it is also witnessed by $g^{q_{\omega+1}}_n$. This is accomplished by another diagonalization very similar to the first round of the construction.

In the fourth and final round of the construction we define measure one sets which capture membership (and non-membership) of each $r \restriction n$ in each Y_m. To do so we need a claim.

Claim 1.5. *Let $n, m < \omega$ and $r \in P^*$ be a condition of length n. If $r \restriction n \notin Y_{m+1}$, then $\{\nu < \kappa_n \mid$ for all $g \leq g^{q_{m+1}}_n$ setting $\rho = \pi_{\mathrm{mc}(a^q_n)0}(\nu)$ if $\sup(\mathrm{rng}(g)) < \rho$, then $r \restriction n \frown (\rho, (a^{q^{m+1}}_n, A^{q^{m+1}}_n, f^{q_{m+1}}_n) \frown \nu, g, H^{q_{m+1}}_n(\rho)) \notin Y_m\} \in E_{n\,\mathrm{mc}(a^q_n)}$.*

Assuming $r \restriction n \notin Y_{m+1}$, for each g we have a measure one set A_g in $E_{n\,\mathrm{mc}(a^q_n)}$ such that for all $\nu \in A_g$, the natural way of extending $r \restriction n$ by ν is not a member of Y_m. It follows that the set $A = \{\nu \mid$ for all g, if $\sup \mathrm{rng}(g) < \pi_{\mathrm{mc}(a^q_n)0}(\nu)$, then $\nu \in A_g\} \in E_{n\,\mathrm{mc}(a^q_n)}$. The set A is the one required for the claim.

Let S_n be the set of $\langle(\rho_k, g_k, h_k) \mid k < n$ which satisfy (1) and (2) in the definition of P^*. To make a given $s \in S_n$ an initial segment of a condition in P^* we only need to add a sequence of f parts. For each $s \in S_n$ and $m < \omega$, we let $A_{s,m}$ witness that

$s \oplus \langle f_k^{q_{\omega+1}} \mid k < n \rangle \in Y_m$ if possible and otherwise we let it be the measure one set witnessing the previous claim. We let A_n^* be the intersection of the $A_{s,m}$ for $s \in S_n$ and $m < \omega$ and we strengthen $q_{\omega+1}$ to $q_{\omega+2}$ by restricting it's measure one sets to $\langle A_n^* \mid n < \omega \rangle$.

We are now ready to complete the argument. Let $r \leq q_{\omega+2}$ with $r \in D$. Let $n^* = \mathrm{lh}(r) - \mathrm{lh}(q_{\omega+2})$ and let $p^* = r \restriction \mathrm{lh}(q_{\omega+2}) \frown q_{\omega+2} \restriction [\mathrm{lh}(q_{\omega+2}, \omega)$. A straightforward inductive argument shows that $p^* \restriction \mathrm{lh}(q_{\omega+2}) = r \restriction \mathrm{lh}(q_{\omega+1}) \in Y_{n^*}$. Another inductive argument using round three of the construction shows that every n^* step extension of p^* is in D. $\square$

Let G be $(P^*, \leq)$-generic and derive D and sequences $\vec{\rho}$ and $\vec{C}$ where D is $\mathrm{Coll}(\omega_1, < \rho_0)$-generic, $\vec{\rho}$ is the union of the sequences $\vec{\rho}^p$ for $p \in G$ and $\vec{C}$ is a sequence of generics such that $\vec{C} \restriction n$ is generic for $\prod_{i<n} \mathrm{Coll}(\rho_i^{+i+4}, < \kappa_i) \times \mathrm{Coll}(\kappa_i^{+i+8}, < \rho_{i+1})$.

Corollary 1.6. *If X is a bounded subset of κ in $V[G]$, then $X \in V[D \times \vec{C} \restriction n]$ for some $n < \omega$.*

Corollary 1.7. *In $V[G]$, $\kappa = \aleph_\omega$ and $\kappa^+ = \aleph_{\omega+1}$.*

It is not hard to see that we have added κ^{++} ω-sequences to κ. So we are finished if we can show that κ^{++} is preserved in the extension by $(P^*, \rightarrow)$ which is induced by G. To do this we prove the following lemma.

Lemma 1.8. *$(P^*, \rightarrow)$ has the κ^{++}-cc.*

The proof is a fairly straight forward combination of the fact that $(P, \rightarrow)$ has the κ^{++}-cc and so do the collapsing functions when viewed as a large product. The following claim is standard. Here we use i_n for the ultrapower map derived from E_{n0}.

Claim 1.9. *$\prod_{n<\omega} \mathrm{Coll}(\kappa_n^{+n+4}, < i_n(\kappa_n))_{\mathrm{Ult}(V, E_{n0})} \times \mathrm{Coll}(\kappa_n^{+n+8}, < \kappa_{n+1})$ has the κ^{++}-cc*

Proof of Lemma 1.8. Let $\langle p_\alpha \mid \alpha < \kappa^{++} \rangle$ be a sequence of elements of P^*. We may assume that for all $\alpha, \beta < \kappa^{++}$,

(1) $\sigma(p_\alpha)$ and $\sigma(p_\beta)$ are compatible in the $\rightarrow$ ordering on P,
(2) p_α and p_β have the same length and
(3) for all $n < \mathrm{lh}(p_\alpha)$, $g_n^{p_\alpha}$ and $g_n^{p_\beta}$, and $h_n^{p_\alpha}$ and $h_n^{p_\beta}$ are compatible.

The proof is complete by noticing that the collapsing conditions above the common length of the conditions comes from a poset which is subsumed by the product from the previous claim. $\square$

References

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