

Last Passage Percolation and the Slow Bond Problem

by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Statistics

in the

Graduate Division

of the

University of California, Berkeley

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Spring 2019

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## Abstract

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Last passage percolation models are fundamental examples in statistical mechanics where the energy of a path is maximized over all directed paths with given endpoints in a random environment, and the maximizing paths are called *geodesics*. Here we consider the Poissonian last passage percolation (LPP) and the exponential directed last passage percolation (DLPP), the latter having a standard coupling with another classical interacting particle system, the totally asymmetric simple exclusion process or TASEP. These belong to the so-called KPZ universality class, for which exact algebraic formulae have led to precise results for fluctuations and scaling limits. However, such formulae are not very robust and studying the geometry of the geodesics can often provide new insights into these models. Here we consider three problems in each of these three models; exponential DLPP, TASEP and Poissonian LPP, and see how geometric and probabilistic techniques solve such problems.

In the first problem, we study finer properties of the coalescence structure of finite and semi-infinite geodesics for exactly solvable models of last passage percolation. We consider directed last passage percolation on  $\mathbb{Z}^2$  with i.i.d. exponential weights on the vertices. Fix two points  $v_1 = (0, 0)$  and  $v_2 = (0, \lfloor k^{2/3} \rfloor)$  for some  $k > 0$ , and consider the maximal paths  $\Gamma_1$  and  $\Gamma_2$  starting at  $v_1$  and  $v_2$  respectively to the point  $(n, n)$  for  $n \gg k$ . Our object of study is the point of coalescence, i.e., the point  $v \in \Gamma_1 \cap \Gamma_2$  with smallest  $|v|_1$ . We establish that the distance to coalescence  $|v|_1$  scales as  $k$ , by showing the upper tail bound  $\mathbb{P}(|v|_1 > Rk) \leq R^{-c}$  for some  $c > 0$ . We also consider the problem of coalescence for semi-infinite geodesics. For the almost surely unique semi-infinite geodesics in the direction  $(1, 1)$  starting from  $v_3 = (-\lfloor k^{2/3} \rfloor, \lfloor k^{2/3} \rfloor)$  and  $v_4 = (\lfloor k^{2/3} \rfloor, -\lfloor k^{2/3} \rfloor)$ , we establish the optimal tail estimate  $\mathbb{P}(|v|_1 > Rk) \asymp R^{-2/3}$ , for the point of coalescence  $v$ . This answers a question left open by Pimentel [70] who proved the corresponding lower bound.

Next, we study the “slow bond” model, where the totally asymmetric simple exclusion process (TASEP) on  $\mathbb{Z}$  is modified by adding a slow bond at the origin. The slow bond increases the particle density immediately to its left and decreases the particle density immediately to its right. Whether or not this effect is detectable in the macroscopic current started from the step initial condition has attracted much interest over the years and this

question was settled recently in [16] where it was shown that the current is reduced even for arbitrarily small strength of the defect. Following non-rigorous physics arguments in [49, 48] and some unpublished works by Bramson, a conjectural description of properties of invariant measures of TASEP with a slow bond at the origin was provided by Liggett in [62]. We establish Liggett's conjectures and in particular show that, starting from step initial condition, TASEP with a slow bond at the origin, as a Markov process, converges in law to an invariant measure that is asymptotically close to product measures with different densities far away from the origin towards left and right. Our proof exploits the correspondence between TASEP and the last passage percolation on  $\mathbb{Z}^2$  with exponential weights and uses the understanding of geometry of maximal paths in those models.

Finally, we study the modulus of continuity of polymer fluctuations and weight profiles in Poissonian LPP. The geodesics and their energy in Poissonian LPP can be scaled so that transformed geodesics cross unit distance and have fluctuations and scaled energy of unit order, and we refer to scaled geodesics as *polymers* and their scaled energies as *weights*. Polymers may be viewed as random functions of the vertical coordinate and, when they are, we show that they have modulus of continuity whose order is at most  $t^{2/3}(\log t^{-1})^{1/3}$ . The power of one-third in the logarithm may be expected to be sharp and in a related problem we show that it is: among polymers in the unit box whose endpoints have vertical separation  $t$  (and a horizontal separation of the same order), the maximum transversal fluctuation has order  $t^{2/3}(\log t^{-1})^{1/3}$ . Regarding the orthogonal direction, in which growth occurs, we show that, when one endpoint of the polymer is fixed at  $(0, 0)$  and the other is varied vertically over  $(0, z)$ ,  $z \in [1, 2]$ , the resulting random weight profile has sharp modulus of continuity of order  $t^{1/3}(\log t^{-1})^{2/3}$ . In this way, we identify exponent pairs of  $(2/3, 1/3)$  and  $(1/3, 2/3)$  in power law and polylogarithmic correction, respectively for polymer fluctuation, and polymer weight under vertical endpoint perturbation. The two exponent pairs describe [42, 43, 44] the fluctuation of the boundary separating two phases in subcritical planar random cluster models.

To my parents, Dada and all my teachers

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## Acknowledgments

First and foremost, I would like express my heartfelt thanks to my advisor Dr. Alan Hammond who has been a constant support for me throughout my graduate years. He readily agreed to be my advisor when I first approached him in my second year, for which I am eternally grateful. He has been a wonderful advisor, with his immense help and suggestions on academic and administrative things, and has always been very accessible, extraordinarily patient and enthusiastic to help. I have learnt a lot from the many academic discussions we had, and from our collaborations. I am grateful to him for kindly allowing me to include our joint work in this dissertation.

I am extremely grateful to Dr. Allan Sly for all the academic and other discussions we had, for allowing me to visit him at Princeton, and for granting me the permission to include our joint works in this dissertation. I am also thankful to him for convincing me to come to Berkeley in the first place.

I would also like to express my heartfelt gratitude to Dr. Riddhipratim Basu, for being a wonderful collaborator, and a super helpful senior and permitting me to use our joint work here.

I learnt a lot from my collaborators, both at UC Berkeley and outside, including Dr. Parthanil Roy, Dr. Manjunath Krishnapur, Dr. Shirshendu Ganguly, Dr. Lionel Levine, Dr. Yuval Peres and Dr. Omer Angel. I thank Yuval Peres and Microsoft Research Redmond for their hospitality during my visits at Microsoft.

I would like to thank Dr. Nike Sun, Dr. David Aldous and Dr. Fraydoun Rezakhanlou for kindly agreeing to be on my dissertation committee. I am grateful to Nike Sun for many academic and nonacademic interactions we had when she was at Berkeley. I would also like to thank David Aldous for always being very helpful, and supporting me as a GSR over a summer.

Berkeley is probably the best place to do one's Ph.D.. I learnt a lot from the many interactions with and the classes I took from Dr. Jim Pitman, Dr. Steve Evans, Dr. Luca Trevisan and Dr. Marc Rieffel. I thank Dr. Pranab Bardhan from the Economics Department at Berkeley for the many informal interactions we had and for guiding me to all the cultural events at Berkeley, particularly those organized by ISAS. Many thanks are due to La Shana Porlaris, Mary Melinn and the other department staff members. They have been exceptionally helpful with all the administrative procedures and many day-to-day requirements.

My first two years at UC Berkeley were generously supported by the Loève Fellowship, for which I am deeply grateful.

I sincerely cherish the warm friendship of Zsolt Bartha, Nick Bhattacharya, Jacob Calvert, Milind Hegde, Brett Kolesnik, Koulik Khamaru, Sören Künzle, Satyaki Mukherjee, Soumendu Sundar Mukherjee, Sujayam Saha, Jason Wu and Yumeng Zhang at Berkeley. Special thanks are due to Sujayam Saha and Soumendu Sundar Mukherjee for allowing me to stay at their place for a week when I first came to Berkeley and their immense help in getting an apartment and settling into an alien environment. I would also like to thank Sumanta Basu,

Nirupam Chakrabarty, Paromita Dubey, Abhishek Roy, Nilanjana Laha, Pragya Sur, Subhabrata Sen, Moumanti Podder, Promit Ghosal, Swarnadip Ghosh, Suyash Gupta, Sohom Bhattacharya and Ujan Gangopdhyay. I am particularly grateful to Pragya Sur and Subhabrata Sen for being such good friends and seniors and for being hospitable during my many visits to Stanford.

Last but not the least, I thank my parents, Sujit Sarkar and Gita Sarkar and my elder brother Souvik Sarkar for always being there for me. I am eternally indebted to my extended family, my cousins and friends back in India, particularly, Rituparna Paul, Ankita Srivastava, Rahul Mandal, Dhriti Ghose and Archisman Neogi and all my teachers since childhood, who have moulded me in many ways into the person I am today.

# Chapter 1

## Introduction

The last passage percolation (LPP) is a classical statistical mechanics model where a directed path in independent local randomness maximizes a random energy determined by the environment and the maximising paths are called geodesics. It has wide applications, such as in polymer growth models, queueing systems, random matrix theory and random permutations. In 1986, Kardar, Parisi, and Zhang [53] predicted universal scaling behaviour for many random growth processes in two dimensions (sometimes interpreted as 1+1 dimension), including first and last passage percolation as well as corner growth processes, though rigorous validation has been subsequently provided for only a handful of them. In such models, fluctuation in the direction of growth is governed by an exponent of one-third, with this fluctuation enduring on a scale governed by an exponent of two-thirds in the orthogonal, or transversal, direction. Baik, Deift and Johansson [6] established the  $n^{1/3}$ -order fluctuation of the maximum number of Poisson points on an increasing path from  $(0, 0)$  to  $(n, n)$ , deriving the GUE Tracy-Widom distributional limit of the scaled energy. Later Johansson [51] proved the transversal fluctuation exponent of two-thirds in this model; he also proved the  $n^{1/3}$  fluctuation and Tracy-Widom scaling limit in directed last passage percolation on  $\mathbb{Z}^2$  with i.i.d. passage times distributed according to either Geometric or Exponential distribution [50]. These are the exactly solvable models, for which many exact distributional formulae are available, typically using some deep machinery from algebraic combinatorics or random matrix theory, and certain duality properties from queueing theory in some cases. Over the last twenty years there has been tremendous progress in achieving a detailed understanding in these and a handful of other exactly solvable models, and a rich limiting theory has emerged; see [21] for an excellent survey of this line of works. However, such exact formulae are not very robust and studying the geometry of the geodesics can often provide new insights into these models. The two classical exactly solvable models in the KPZ universality class are the Poissonian LPP and the totally asymmetric simple exclusion process (TASEP), which can be coupled with the exponential last passage percolation. In this dissertation, we will focus on three problems in each of these three models: exponential LPP, Poissonian LPP and TASEP, and see how geometric and probabilistic techniques help us in understanding and solving these problems. First, we briefly introduce the models.

## 1.1 Model Definitions

### 1.1.1 Poissonian last passage percolation

Let  $\Pi$  be a Poisson point process of unit intensity on  $\mathbb{R}^2$ . Let  $X_n$  denote the maximum number of points in  $\Pi$  along any increasing path from  $(0, 0)$  to  $(n, n)$ , which we call the *energy* of the maximal path; we call all such maximal paths *geodesics*. Conditional on the number of points  $N$  in the square  $[0, n]^2$ ,  $X_n$  is distributed as the length of the longest increasing subsequence of a random permutation of the first  $N$  natural numbers, which associates it to the classical *Ulam's problem* or the *Ulam-Hammersley's problem*. If  $\ell_n$  denotes the length of the longest increasing subsequence of a random permutation of the first  $n$  natural numbers, then Hammersley in 1972 [37] first showed that the limit of  $\mathbb{E}\ell_n/\sqrt{n}$  as  $n \rightarrow \infty$  exists by using Kingsman's subadditive ergodic theorem. Using a correspondence with Young-Tableaux, Vershik and Kerov [77] and Logan and Shepp [63] established that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}\ell_n}{\sqrt{n}} = 2.$$

(Aldous and Diaconis gave an alternative proof using interacting particle systems in [2]). The next breakthrough came with the work of Baik, Deift and Johansson [6] who found the scaling limit of  $X_n$  in terms of the GUE Tracy-Widom distribution, denoted by  $F_2$ .<sup>1</sup> See [71] for a good exposition on the Poissonian LPP and the Ulam's problem.

### 1.1.2 Exponential directed last passage percolation (DLPP)

The exponential directed last passage percolation model on  $\mathbb{Z}^2$  is defined as follows. For each vertex  $v \in \mathbb{Z}^2$  associate i.i.d. weight  $\xi_v$  distributed as  $\text{Exp}(1)$ . Define  $u \preceq v$  if  $u$  is co-ordinate wise smaller than  $v$  in  $\mathbb{Z}^2$ . For any oriented path  $\gamma$  from  $u$  to  $v$  let the passage time of  $\gamma$  be defined by

$$\ell(\gamma) := \sum_{v' \in \gamma \setminus \{v\}} \xi_{v'}.$$

---

<sup>1</sup>The Tracy-Widom distribution is defined in terms of the Airy function, a special function in mathematical analysis, given by

$$\text{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right) dt, \quad (x \in \mathbb{R}).$$

The Airy kernel  $\mathbf{A} : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$  is defined by

$$\mathbf{A}(x, y) = \begin{cases} \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x-y} & x \neq y, \\ \text{Ai}'(x)^2 - x\text{Ai}(x)^2 & x = y. \end{cases} \quad (1.1)$$

Then the cumulative distribution function  $F_2 : \mathbb{R} \mapsto \mathbb{R}$  is defined as

$$F_2(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_t^\infty \dots \int_t^\infty \det_{i,j=1}^n (\mathbf{A}(x_i, x_j)) dx_1 \dots dx_n.$$

For  $u \preceq v$  define the last passage time from  $u$  to  $v$ , denoted  $T_{u,v}$ , by  $T_{u,v} := \max_{\gamma} \ell(\gamma)$ , where the maximum is taken over all up/right oriented paths from  $u$  to  $v$ . Let  $\Gamma_{u,v}$  denote the (almost surely unique) path between  $u$  and  $v$  that attains the last passage time  $T_{u,v}$ . We call the path  $\Gamma_{u,v}$  the geodesic between  $u$  and  $v$ . Johansson [50] established the Tracy-Widom scaling limit for the last passage times in this model.

### 1.1.3 Totally asymmetric simple exclusion process (TASEP)

Totally Asymmetric Simple Exclusion Process (TASEP) is an interacting particle system modelling one-way traffic movement. On the line, the dynamics is as follows: each particle jumps to the right after waiting for a time that is an independent exponential random variable with rate one provided the site to its right is empty. This process has been studied in detail for more than forty years in both the statistical physics and the probability literature, and a rich understanding of its behaviour has emerged. Stationary measures for TASEP were identified by Liggett [59] as early as in 1976 when he showed that product Bernoulli measures are all non-trivial extremal stationary measures for the TASEP dynamics. This and a sequence of works [58, 60, 61, 20] have characterised the stationary measures and proved ergodic theorems for symmetric and asymmetric exclusion processes for various different settings. Utilizing this progress, Rost [72] in 1981 evaluated the asymptotic current and hydrodynamic density profile when the process starts from the step initial condition, i.e., with one particle each at every nonpositive site of  $\mathbb{Z}$  and no particles at positive sites. In particular, he showed that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}T_n}{n} = 4, \quad (1.2)$$

where  $T_n$  is the passage time, that is, the amount of time taken by the particle at position  $-n$  to jump to site 1 starting from the step initial condition. The reciprocal of the expression in (1.2) is called the *current* in the TASEP. Subsequently, TASEP was identified [50] to be one of the canonical *exactly solvable* models in the so-called KPZ universality class, and thus very fine information about the process was obtained using exact determinantal formulae that included the Tracy-Widom scaling limits for the current fluctuations.

Under the standard coupling between the exponential DLPP and the TASEP, the time for the particle in the TASEP from position  $-n$  to move to 1 starting from the step initial condition is distributed as the last passage time between  $(0,0)$  and  $(n,n)$  in exponential DLPP.

### 1.1.4 The slow bond problem

Whether a localized microscopic defect, especially if it is small, will affect the macroscopic behaviour of a system is a fundamental question in statistical mechanics. For the TASEP on  $\mathbb{Z}$ , this problem became famous in the physics and mathematics community as the “slow-bond” problem; when TASEP on  $\mathbb{Z}$  starting from the step initial condition is imputed with a slow bond at the origin, that is the wait time of jump for a particle at the origin is increased

to an exponential with a mean time  $1/(1 - \varepsilon)$ , is this effect detectable in the macroscopic current? Originally posed by Janowsky and Lebowitz in 1992 in [49], this question was settled in [16] by showing that the current is reduced even for arbitrarily small strength of the defect  $\varepsilon$ . That is, if  $T_n^\varepsilon$  denotes the passage time in the slow bond model starting from the step initial condition, then [16] shows that for all  $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}T_n^\varepsilon}{n} > 4. \quad (1.3)$$

For a complete background on this problem, see [16]. We briefly review the background here. The decrease in jump rate at the origin will increase the particle density to the immediate left of such a “slow bond” and decrease the density to its immediate right. The difficulty in analysing this model arises because the model no longer remains exactly solvable. If  $\varepsilon$  is close to one, it is easy to see that the above limit in (1.3) is strictly greater than 4. Whether the limit changes for all  $\varepsilon > 0$ , or if there is a critical  $\varepsilon_c > 0$  such that the limit remains unchanged for all  $\varepsilon < \varepsilon_c$  was the subject of intense dispute. The problem can be posed in various forms: as a stochastic driven transport through narrow channels with obstructions [36]; as a growth model with defect line [67]; or as a polymer pinning problem of a one-dimensional interface [47, 4]. In Poissonian LPP, the diagonal is reinforced by an independent one dimensional Poisson process of intensity  $\lambda$ . If  $X_n^\lambda$  is the maximum number of points on an increasing path from  $(0, 0)$  to  $(n, n)$ , the authors in [16] also showed that, for all  $\lambda > 0$ ,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}X_n^\lambda}{n} > 2.$$

For slow bond TASEP, using a non-rigorous mean field argument, Janowsky and Lebowitz in [48] suggested that the current behaves like  $(1 - \varepsilon)/(2 - \varepsilon)^2$  if the jump rate at the origin is  $1 - \varepsilon$ , thus they predicted that  $\varepsilon_c = 0$ . Theoretical renormalization group arguments and heuristic “influence percolation” arguments also supported this conjecture in [47] and [17] respectively. In a more recent work [24], using a non-rigorous theoretical argument and analysis of the first sixteen terms of formal power series expansion of the current, authors suggested that for small values of  $\varepsilon > 0$  the current should behave as  $1/4 - \gamma \exp(-a/\varepsilon)$  with  $a \approx 2$ .

By approximating the model with an exclusion process whose rates vary more regularly in space, an upper bound for  $\varepsilon_c$  was derived in [26]. Also [62] provided an alternative bound. The general hydrodynamic limit results were proved in [75] for all values  $0 < \varepsilon < 1$  of the slow-down. However the hydrodynamic limit cannot answer the question about the critical value of the slow-down rate.

Simultaneously, a contending class of theoretical arguments, mostly from the theoretical physics literature, supported also by numerical data, hinted that  $\varepsilon_c > 0$ . See [52], [36] and [74].

In a closely related model by Baik and Rains [5], the authors considered the so-called “symmetrized” version of the maximal increasing subsequence with a defect line, for which they showed that  $\lambda_c = 1$ .

## 1.2 Organization of the dissertation

I end this chapter by briefly describing the contents of all the chapters that follow.

### 1.2.1 Chapter 2: Coalescence of geodesics.

Coalescence structure of the geodesics in LPP is a central question in this model. Simply put, what is the distance at which two geodesics, whose starting points are close, meet? Using the KPZ scaling exponent  $2/3$  for transversal fluctuation of finite geodesics, one predicts that the distance to coalescence will scale as  $k$  if the distance between their starting points was  $k^{2/3}$ . We prove this in this chapter.

To be precise, consider exponential DLPP on  $\mathbb{Z}^2$ . Fix two points  $v_1 = (-\lfloor k^{2/3} \rfloor, \lfloor k^{2/3} \rfloor)$  and  $v_2 = (\lfloor k^{2/3} \rfloor, -\lfloor k^{2/3} \rfloor)$  for some  $k > 0$ , and consider the geodesics  $\Gamma_1$  and  $\Gamma_2$  starting at  $v_1$  and  $v_2$  respectively to the point  $(n, n)$  for  $n \gg k$ . Then the point of coalescence, that is, the point  $v \in \Gamma_1 \cap \Gamma_2$  with smallest  $\|v\|_1$  satisfies

$$\mathbb{P}(\|v\|_1 > Rk) \leq R^{-c},$$

for some  $c > 0$ .

We also consider the problem of coalescence for semi-infinite geodesics. For the almost surely unique semi-infinite geodesics in the direction  $(1, 1)$  starting from  $v_1$  and  $v_2$ , we establish the optimal tail estimate

$$\mathbb{P}(\|v\|_1 > Rk) \asymp R^{-2/3},$$

for the point of coalescence  $v$ . This resolves a problem left open by Pimentel [70] who proved the corresponding lower bound.

In addition to these theorems, we also prove a local fluctuation estimate for the geodesic, which is of independent interest (for example, it is used in [15] and [45]). Roughly, it says that the geodesic from  $(0, 0)$  to  $(n, n)$  has a fluctuation of the order of  $\ell^{2/3}$  at  $(\ell, \ell)$  for any fixed  $\ell \in (0, n)$ . (The global  $n^{2/3}$  fluctuation of the geodesic was shown in [51] and [16].)

We solve these using a multiscale analysis of the geometry of the field around the geodesics, taking advantage of the moderate deviation estimates of the geodesic lengths from the integrable probability literature [7],[16]. The results here are also crucially applied in the subsequent chapter. The contents of this chapter appeared in [14] in a joint work with Riddhipratim Basu and Allan Sly.

### 1.2.2 Chapter 3: Invariant measures for TASEP with a slow bond.

Following non-rigorous physics arguments [49], [48] and some unpublished works by Bramson, a conjectural description of the properties of the invariant measures of TASEP with a slow bond at the origin was provided by Liggett in [62]. In this chapter, we establish

Liggett’s conjectures and in particular show that TASEP with a slow bond at the origin, started from step initial condition, converges in law to an invariant measure that is “*asymptotically equivalent*” to product measures with different densities far away from the origin towards left and right. Our proof exploits the correspondence between TASEP and DLPP on  $\mathbb{Z}^2$  mentioned before, together with the coalescence of geodesics and a novel coupling argument.

We also prove a central limit theorem for the last passage time  $T_n^\varepsilon$  in the slow bond model. The Gaussian scaling limit in this case is in contrast with the Tracy-Widom scaling limit of the passage time  $T_n$  in the regular TASEP. This chapter is based on a joint work [15] with Riddhipratim Basu and Allan Sly.

### 1.2.3 Chapter 4: Modulus of continuity for fluctuations and weight profiles.

The geometric and continuity properties of the maximizing paths in last passage percolation, the geodesics, have been of significant interest. Precise modulus of continuity results are known for Brownian motion paths, for example, but can such precise continuity results be given for the geodesics in LPP? Clearly, the problem is much harder for LPP due to the lack of independent increments. In this chapter, we show that it is indeed possible to give precise modulus of continuity for LPP geodesics and their weight profiles.

The geodesics and their energy in last passage percolation can be scaled so that transformed geodesics cross unit distance and have fluctuations and scaled energy of unit order, and we refer to scaled geodesics as *polymers* and their scaled energies as *weights*. Polymers may be viewed as random functions of the vertical coordinate and, when they are, we show that they have modulus of continuity whose order is at most  $t^{2/3}(\log t^{-1})^{1/3}$ . The power of one-third in the logarithm may be expected to be sharp and in a related problem we show that it is: among polymers in the unit box whose endpoints have vertical separation  $t$  (and a horizontal separation of the same order), the maximum transversal fluctuation has order  $t^{2/3}(\log t^{-1})^{1/3}$ . Regarding the orthogonal direction in which growth occurs, we show that, when one endpoint of the polymer is fixed at  $(0, 0)$  and the other is varied vertically over  $(0, z)$ ,  $z \in [1, 2]$ , the resulting random weight profile has sharp modulus of continuity of order  $t^{1/3}(\log t^{-1})^{2/3}$ . In this way, we identify exponent pairs of  $(2/3, 1/3)$  and  $(1/3, 2/3)$  in power law and polylogarithmic correction, respectively for polymer fluctuation and polymer weight under vertical endpoint perturbation. The two exponent pairs describe [42, 43, 44] the fluctuation of the boundary separating two phases in subcritical planar random cluster models. This chapter is based on a joint work [45] with Alan Hammond.

# Chapter 2

## Coalescence of geodesics

### 2.1 Introduction

Coalescence of geodesics has been an interesting tool to study the geometry of first and last passage percolation models. In first passage percolation, the study was initiated by Newman and co-authors as summarised in his ICM paper [68] which proved certain coalescence results under curvature assumptions on the limit shape. Much progress has been made in recent years in understanding the geodesics starting with the breakthrough idea of Hoffman [46] of studying infinite geodesics using Busemann functions. These techniques have turned out to be extremely useful, providing a great deal of geometric information on the structure of geodesics in first passage percolation [28, 1, 27].

In recent years there has been a great deal of interest in studying the coalescence of polymers (maximal paths which we shall also refer to as geodesics) in last passage percolation models as well [32, 25, 70]. Much can be established in certain exactly solvable settings including the existence and uniqueness of semi-infinite geodesics starting at a given point along a given direction, and coalescence of geodesics along deterministic directions. Some of these results have recently been proved beyond exactly solvable models as well [35, 34]. Here, we shall restrict ourselves to the exactly solvable setting of Exponential directed last passage percolation on  $\mathbb{Z}^2$ , and establish the precise order of the distance to coalescence for two semi-infinite geodesics along the same direction started at distinct points (see Theorem 2) with the optimal tail estimate answering an open question from [70] who proved the corresponding lower bound. We, however, are also interested in the finite variants of the question, where we consider distance to coalescence of geodesics from two distinct points to a far away point. This variant is more important for some applications. We prove a similar scaling in this finite setting also (see Theorem 1), albeit with a worse tail estimate. Our arguments combine moderate deviations from the exactly solvable literature with tools from percolation to understand geometry of a geodesic together with the environment around it. By way of the proof of this main result we also obtain a local transversal fluctuation result for the geodesics in last passage percolation (see Theorem 3) that is of independent interest.

We now move towards precise model definition and the statement of the main results.

### 2.1.1 Model definition and main results

We recall the exponential DLPP here. Consider the following last passage percolation model on  $\mathbb{Z}^2$ . For each vertex  $v \in \mathbb{Z}^2$  associate i.i.d. weight  $\xi_v$  distributed as  $\text{Exp}(1)$ . Define  $u \preceq v$  if  $u$  is co-ordinate wise smaller than  $v$  in  $\mathbb{Z}^2$ . For any oriented path  $\gamma$  from  $u$  to  $v$  let the passage time of  $\gamma$  be defined by

$$\ell(\gamma) := \sum_{v' \in \gamma \setminus \{v\}} \xi_{v'}.$$

For  $u \preceq v$  define the last passage time from  $u$  to  $v$ , denoted  $T_{u,v}$  by  $T_{u,v} := \max_{\gamma} \ell(\gamma)$  where the maximum is taken over all up/right oriented paths from  $u$  to  $v$ . Let  $\Gamma_{u,v}$  denote the (almost surely unique) path between  $u$  and  $v$  that attains the last passage time  $T_{u,v}$ . We shall call the path  $\Gamma_{u,v}$  the geodesic between  $u$  and  $v$ .<sup>1</sup>

#### Coalescence of finite geodesics

We now proceed towards statements of our main results. We first state our result in the finite setting. Let  $\mathbf{n}$  denote the point  $(n, n)$ . Let  $k > 0$  be arbitrary and let  $v_1 = (0, 0)$  and  $v_2 = (0, k^{2/3})$  (assume without loss of generality that  $k^{2/3}$  is an integer, the same result holds with  $\lfloor k^{2/3} \rfloor$  otherwise). Let  $v_* = (v_{*,1}, v_{*,2})$  be a leftmost common point between  $\Gamma_{v_1, \mathbf{n}}$  and  $\Gamma_{v_2, \mathbf{n}}$  (observe that  $v_{*,1}$  is well-defined even if there is no unique leftmost common vertex). Our main result shows that  $v_{*,1}$  is of order  $k$  when  $n \gg k$ .

**Theorem 1.** *There exist positive constants  $R_0, C, c > 0$  such that for all  $R > R_0$  and for all  $k > 0$  the coalescence location satisfies*

$$\limsup_{n \rightarrow \infty} \mathbb{P}(v_{*,1} > Rk) \leq CR^{-c}.$$

It is natural to predict this scaling from the KPZ transversal fluctuation exponent of  $2/3$  which says that the geodesic between two points at distance  $n$  fluctuates at scale  $n^{2/3}$  away from the straight line joining the two points. However, to prove Theorem 1, we need finer local control on the transversal fluctuation of the geodesic (see Theorem 3) below.

This coalescence result is robust in the following ways. We do not need to consider geodesics to have the same endpoint, merely that their distance is at the correct scale of transversal fluctuation. Thus Theorem 1 will be valid for the leftmost common point of  $\Gamma_{v_1, \mathbf{n}}$  and  $\Gamma_{v_2, (n, n+n^{2/3})}$  as well. Furthermore, the choice of direction is arbitrary. The same result

<sup>1</sup>Observe that this is a little different from the usual definition of last passage percolation as we exclude the final vertex while adding weights. This is done for convenience as our definition allows  $\ell(\gamma) = \ell(\gamma_1) + \ell(\gamma_2)$  where  $\gamma$  is the concatenation of  $\gamma_1$  and  $\gamma_2$ . As the difference between the two definitions is minor while considering last passage times between far away points, and the geodesics are same, all our results will be valid for both our and the usual definition of LPP.

holds for geodesics to  $(n, hn)$  for any fixed  $h \in (0, \infty)$ . See Corollary 2.3.2 for a precise statement. Finally, it would also be clear from the proof that we need not have taken  $v_1$  and  $v_2$  on a vertical line; the same proof would have worked on two points at distance  $k^{2/3}$  on the line  $x + y = 0$ , say. As a matter of fact, the same proof will show that a similar tail estimate works for  $v_{*,2}$  and hence for  $|v_*|_1$  as well, as claimed before.

We also mention that we work with the Exponential last passage percolation merely for concreteness. Our proof depends only on the Tracy-Widom limit and one point upper and lower tail moderate deviation estimates for the last passage times (see Theorem 2.1.1 and Theorem 2.1.2) and should work equally well for other exactly solvable models where such estimates are available. Indeed such estimates are available for Poissonian directed last passage percolation in continuum [64, 65] and last passage percolation on  $\mathbb{Z}^2$  with geometric passage times [9, 23], and variants of our results should apply to those models as well.

### Semi-infinite geodesics

As already mentioned for the case of semi-infinite geodesics, one can obtain a more precise asymptotic result. Before a statement of the result in a proper context, we need to introduce a few definitions and develop some background. The study of semi-infinite geodesics in last passage percolation with Exponential passage times was initiated in [32, 25] where the following general picture was established. Starting from any  $x \in \mathbb{Z}^2$  there exists an almost surely unique semi-infinite path  $\Gamma_x = \{x = x_0, x_1, x_2, \dots\}$  such that for each  $i < j$  the section of  $\Gamma_x$  between  $x_i$  and  $x_j$  is the geodesic between  $x_i = (x_{i,1}, x_{i,2})$  and  $x_j$ , and such that  $\lim_{n \rightarrow \infty} \frac{x_{n,1}}{x_{n,2}} = 1$ . Such a path is called the semi-infinite geodesic starting at  $x$  in direction  $(1, 1)$ . Moreover, any sequence of finite geodesics from  $x$  to points  $y_n$  in the asymptotic direction  $(1, 1)$  converges to  $\Gamma_x$  almost surely. Finally, this collection of semi-infinite geodesics  $\{\Gamma_x\}_{x \in \mathbb{Z}^2}$  almost surely coalesce, i.e., for any  $x, x' \in \mathbb{Z}^2$ , the number of vertices in  $\Gamma_x \Delta \Gamma_{x'}$  is finite ( $\Delta$  denotes the symmetric difference between the two sets of vertices). The same result holds for any positive quadrant direction bounded away from the coordinate axial directions.

This set of results closely parallels the results of Newman and co-authors in early 90s as summarized in [68] in the context of first passage percolation under certain assumptions on curvature of the limit shape. In recent years the coalescence structure of semi-infinite geodesics has become a central object of study and a lot of progress has been made using Busemann functions [46, 1, 27, 28]. However, with the help of integrable structure much finer results can be established in the LPP setting with exponential weights.

Distance to coalescence for semi-infinite geodesics along the same direction is a natural object of study in the integrable setting and was considered in [70], and a similar scaling was predicted. Using Burke's duality, and Busemann functions, [70] established, among other things, a lower bound to this effect, and the upper bound remained open. For technical convenience, let us change the setting of Theorem 1 slightly.

Fix  $k \in \mathbb{N}$ . Consider the straight line  $\mathbb{L} = \{(x, y) \in \mathbb{Z}_2 : x + y = 0\}$ . on  $\mathbb{L}$  (assume, as before, without loss of generality that  $k^{2/3}$  is an integer). For  $v$  on the line  $\mathbb{L}$ , let  $\Gamma_v$  denote the

almost surely unique semi-infinite geodesic starting at  $v$  in the direction  $(1, 1)$ . Recall that the collection  $\{\Gamma_v\}_{v \in \mathbb{L}}$  is coalescing, i.e., for any  $v, v' \in \mathbb{L}$ , almost surely  $\Gamma_v$  and  $\Gamma_{v'}$  coalesce. Let  $c(v, v') = (x(v, v'), y(v, v'))$  denote the point at which  $\Gamma_v$  and  $\Gamma_{v'}$  coalesce. Let  $\text{dist}(v, v') = x(v, v') + y(v, v')$  denote the distance to coalescence. Now consider  $v_3 = (-k^{2/3}, k^{2/3})$  and  $v_4 = (k^{2/3}, -k^{2/3})$  (assume without loss of generality that  $k^{2/3} \in \mathbb{N}$ ). Translated to this setting, [70] proved that  $\lim_{R \rightarrow 0} \mathbb{P}(\text{dist}(v_3, v_4) > Rk) \rightarrow 1$  as  $R \rightarrow 0$  uniformly in  $k$ , further it was conjectured that, this is the correct scaling, i.e.,  $\lim_{R \rightarrow 0} \mathbb{P}(\text{dist}(v_3, v_4) > Rk) \rightarrow 0$  as  $R \rightarrow \infty$ . Moreover, using some calculations in the limiting Airy process, [70] conjectured that  $\mathbb{P}(\text{dist}(v_3, v_4) > Rk) \asymp R^{-2/3}$  as  $R \rightarrow \infty$  uniformly in  $k$ . Our second main result settles this conjecture.

**Theorem 2.** *In the above set-up, there exists  $C_1, C_2, R_0 > 0$  such that for all  $k > 0$  and  $R > R_0$ , we have*

$$C_1 R^{-2/3} \leq \mathbb{P}(\text{dist}(v_3, v_4) > Rk) \leq C_2 R^{-2/3}.$$

As a matter of fact a lower bound to this effect was already proved in [70] (see [70, Section 3] for a discussion of the difficulty of the approach therein to get a matching upper bound), hence the main work goes in proving the upper bound. However we shall also provide a short proof for the lower bound for completeness.

Observe also that we took the points  $v_3$  and  $v_4$  on the anti-diagonal line  $x + y = 0$  rather than on a vertical line as in the set-up of Theorem 1. It is merely for technical convenience, the same result will still hold for points on the vertical line, as well as for semi-infinite geodesics in other directions (except the axial directions) with minor changes to the proof.

We finish this subsection with a further discussion about the importance of studying the coalescence of finite geodesics in models of KPZ universality class. The coalescence structure of geodesics in exactly solvable polymer models is important in understanding scaling limits of the random geometric structures. See [70] for connections to this question to the Airy Sheet. As mentioned above, in Brownian last passage percolation, where one has a strong resampling property called Brownian Gibbs property [22], detailed structure of coalescent polymer trees has been explored and used to make progress towards the important question of Brownian regularity of the Airy processes [39, 40, 38, 41]. Using techniques of [70], local Brownian regularity has also been explored in [69].

Finally we advocate a further reason for studying the geometry of geodesics in the context of exactly solvable polymer models. A detailed understanding of the geometry of geodesics, beyond what can be obtained from the integrable techniques have recently proved useful in study of certain models that arise from adding local defects to the integrable models. Even though the local defects destroy the integrable structure, the more geometric understanding of the geodesics are still useful. One example of this principle was obtained in [16], where TASEP with a slow bond at the origin was studied using the correspondence to Exponential last passage percolation, and geometric understanding of the geodesic was used to show that a slow bond at the origin of arbitrarily small strength changes the current, thus settling the “slow bond problem”. As a matter of fact, we use the results to study the invariant

measures of TASEP with a slow bond [15] (see next chapter), and establish a conjecture of Liggett from [62]. We remark that for these applications, it is crucial to have the coalescence statement in the finite setting. Hence even without the optimal tail decay in Theorem 1, the result turns out to be important and useful.

### 2.1.2 Inputs from integrable probability; Tracy-Widom limit, and $n^{2/3}$ fluctuations

In this subsection, we recall the basic inputs from the integrable probability literature that we shall be using throughout. As mentioned before Exponential DLPP is one of the handful of models for which the KPZ scaling result and much more has been rigorously established. The Tracy-Widom scaling limit for exponential DLPP is due to Johansson [50].

**Theorem 2.1.1** ([50]). *Let  $h > 0$  be fixed. Let  $v = (0, 0)$  and  $v_n = (n, \lfloor hn \rfloor)$ . Then*

$$\frac{T_{v,v_n} - (1 + \sqrt{h})^2 n}{h^{-1/6}(1 + \sqrt{h})^{4/3} n^{1/3}} \xrightarrow{d} F_{TW}. \quad (2.1)$$

where the convergence is in distribution and  $F_{TW}$  denotes the GUE Tracy-Widom distribution.

GUE Tracy-Widom distribution is a very important distribution in random matrix theory that arises as the scaling limit of largest eigenvalue of GUE matrices; see e.g. [6] for a precise definition of this distribution. For our purposes moderate deviation inequalities for the centred and scaled variable as in the above theorem will be important. Such inequalities can be deduced from the results in [7], as explained in [16]. We quote the following result from there.

**Theorem 2.1.2** ([16], Theorem 13.2). *Let  $\psi > 1$  be fixed. Let  $v, v_n$  be as in Theorem 2.1.1. Then there exist constants  $N_0 = N_0(\psi)$ ,  $t_0 = t_0(\psi)$  and  $c = c(\psi)$  such that we have for all  $n > N_0, t > t_0$  and all  $h \in (\frac{1}{\psi}, \psi)$*

$$\mathbb{P}[|T_{v,v_n} - n(1 + \sqrt{h})^2| \geq tn^{1/3}] \leq e^{-ct}.$$

Theorem 2.1.2 provides much information about the geometry and regularity of geodesics in the DLPP model; which was exploited crucially in [16], and will be extensively used by us again. Most fundamental among those is the  $n^{2/3}$  transversal fluctuation of the geodesic between points at distance  $n$ . Let  $\gamma$  denote the (almost surely unique) geodesic between two fat away points  $u$  and  $v$ . For simplicity let us assume  $u = (0, 0)$  and  $v = (n, n)$ . The transversal fluctuation of  $\gamma$ , denoted  $\text{TF}_n$  is defined by  $\sup_{(x,y) \in \gamma} |x - y|$  and denotes the maximum vertical distance between a point on  $\gamma$  to the straight line joining the two points. It follows from Theorem 2.1.2 that  $\text{TF}_n$  is an order  $n^{2/3}$  random object. The scaling exponent

$2/3$  was identified in [51]; it follows from the arguments of [16], cf. Theorem 11.1 there (see also Theorem 2.5 in [10] for an argument using Burke's duality) that

$$\mathbb{P}(\text{TF}_n > kn^{2/3}) \leq e^{-ck^2}$$

uniformly in large  $n$  for some  $c > 0$ .

Observe that Theorem 11.1 in [16] provides a global upper bound on transversal fluctuation. However, from points  $(x, y) \in \gamma$  with  $x = \ell \ll n$ , one expects a much smaller transversal fluctuation of order  $\ell^{2/3}$ . Such a local fluctuation estimate would be useful to us and is also of independent interest. We now move towards a precise statement to this effect.

Let  $\Gamma$  be the geodesic from  $(0, k')$  to  $(n, n + k)$ . For  $\ell \in \mathbb{Z}$ , let  $\Gamma(\ell) \in \mathbb{Z}$  be the maximum number such that  $(\ell, \Gamma(\ell)) \in \Gamma$  and  $\Gamma^{-1}(\ell) \in \mathbb{Z}$  be the maximum number such that  $(\Gamma^{-1}(\ell), \ell) \in \Gamma$ . The following theorem is our final main result in this chapter.

**Theorem 3.** *Fix  $L > 0$ . Then there exist positive constants  $n_0, \ell_0, s_0, c$  depending only on  $L$ , such that for all  $n \geq n_0, s \geq s_0 \vee 2L, \ell \geq \ell_0$ , and  $|k'| \leq L\ell^{2/3}, |k| \leq Ln^{2/3}$  we have*

$$\mathbb{P}[|\Gamma(\ell) - \ell| \geq s\ell^{2/3}] \leq e^{-cs^2};$$

$$\mathbb{P}[|\Gamma^{-1}(\ell) - \ell| \geq s\ell^{2/3}] \leq e^{-cs^2}.$$

**Remark 2.1.3.** *It will be clear from the proof that the exponent  $s^2$  here is determined by the exponents in the moderate deviation tail estimates. In the Poissonian LPP, where optimal moderate deviation bounds are known [64, 65], one can improve the bound further to  $e^{-cs^3}$ , which is optimal. Using the tails estimates for largest eigenvalue of Laguerre Unitary Ensemble available in [55], it is possible to get the optimal exponents in for Exponential LPP as well, however it is not needed for our purposes.*

This result is of independent interest as it provides information on local transversal fluctuation of the geodesics, and has already been useful in several different contexts. For example, this has been used to study the locally Brownian nature of the pre-limiting Airy process profile for Exponential LPP on short scales and to study the time correlation of the same [11], and also the modulus of continuity for polymer fluctuations and weight profiles in Poissonian LPP [45]. For our purposes here we shall need a more refined version of this estimate; see Theorem 2.2.1 below.

A variant of this result also holds for semi-infinite geodesics; see Proposition 2.6.2. We shall use this result to prove Theorem 2. Using Proposition 2.6.2, one can also provide an alternative proof of the lower bound in [70] (see Remark 2.6.5).

### 2.1.3 Outline of the proof of Theorem 1

We describe now the basic outline of our proof of Theorem 1. For some large fixed number  $M$ , we try to achieve coalescence at length scales  $M^i k$  for different values of  $i$ . We show that at each scale coalescence happens with probability bounded below independent of  $i$ ,

and these events are approximately independent, i.e., failure to coalesce at one scale does not make coalescence at the next scale much less likely. Trying at a large number of length scales one obtains Theorem 1. More precisely we establish the following.

Let  $L$  be a large fixed number. For  $r \in \mathbb{N}$  consider the points  $a_r = (r, r + Lr^{2/3})$  and  $b_r = (r, r - Lr^{2/3})$  (assume without loss of generality that  $r^{2/3}$  is an integer). For some large  $M \in \mathbb{N}$  let  $\Gamma_1$  denote the geodesic from  $a_r$  to  $a_{Mr}$ , and  $\Gamma_2$  denote the geodesic from  $b_r$  to  $b_{Mr}$ . Let  $\text{Coal}_{r,M}$  denote the event that  $\Gamma_1$  and  $\Gamma_2$  share a common vertex. We have the following theorem.

**Theorem 2.1.4.** *Fix  $L > 0$ . Then for all sufficiently large  $M$  and all  $r > 0$  we have*

$$\mathbb{P}(\text{Coal}_{r,M}) \geq \frac{1}{2}.$$

Theorem 2.1.4 says that if the geodesics  $\Gamma_{v_1, \mathbf{n}}$  and  $\Gamma_{v_2, \mathbf{n}}$  from Theorem 1 does not have an atypically high transversal fluctuation at distances  $r$  and  $Mr$ , then with probability at least  $\frac{1}{2}$ , they coalesce in  $[r, Mr]$ . Observe that Theorem 3 says that atypical transversal fluctuation at a given point is exponentially unlikely, later we establish a refinement of this showing that such events are also roughly independent if  $M$  and  $r$  are sufficiently large (see Theorem 2.2.1), thereby establishing Theorem 1.

Most of the work in this chapter goes into the proof of Theorem 2.1.4. This is done via a bootstrapping argument. We first show that the probability is bounded below by an arbitrary small constant independent of  $r$  and  $M$  (see Proposition 2.3.1). This follows from showing at some horizontal length scale distance  $D$  (where  $r \ll D \ll Mr$ ) with a probability bounded away from zero there exists a barrier of width  $O(D)$  and height  $O(D^{2/3})$  just above the geodesic  $\Gamma_2$ , such that any path passing through the barrier is penalised a lot. Using Theorem 3 and the FKG inequality we show that in presence of such a barrier and in environment that is typical otherwise,  $\Gamma_2$  will merge with  $\Gamma_1$  before crossing the barrier region. The construction of the barrier here is similar to one present in [16], and we shall quote many of the probabilistic estimates in that paper throughout our proof.

We note here that events forcing coalescence of finite and infinite geodesics have been studied in a number of works in different settings. Some of these are done using ergodicity in non-integrable settings and are inherently not quantitative [68, 56, 1], while the preprint [30] considers a rectangle of size  $n^{1+o(1)} \times n^{2/3}$  and show that best paths constrained to stay within this rectangle coalesce with rather weak probability lower bound of  $n^{-o(1)}$ . Our approach of further developing the combination of geometric techniques and integrable inputs, introduced in [16], together with the control on local fluctuations of geodesics (Theorem 3), in contrast, leads to a proof that coalescence happens with uniformly positive probability at the correct length scale.

## 2.1.4 Notations

For easy reference purpose, let us collect here a number of notations, some of which have already been introduced, that we shall use throughout the remainder of this chapter. Define

the partial order  $\preceq$  on  $\mathbb{Z}^2$  by  $u = (x, y) \preceq u' = (x', y')$  if  $x \leq x'$ , and  $y \leq y'$ . For  $a, b \in \mathbb{Z}^2$  with  $a \preceq b$ , let  $\Gamma_{a,b}$  denote the geodesic from  $a$  to  $b$  in the Exponential LPP, and  $T_{a,b}$  denotes the weight of the geodesic  $\Gamma_{a,b}$ .

For an increasing path  $\gamma$  and  $\ell \in \mathbb{Z}$ ,  $\gamma(\ell) \in \mathbb{Z}$  will denote the maximum number such that  $(\ell, \gamma(\ell)) \in \gamma$  and  $\gamma^{-1}(\ell) \in \mathbb{Z}$  be the maximum number such that  $(\gamma^{-1}(\ell), \ell) \in \gamma$ . Let  $\ell(\gamma)$  denote the weight of the increasing path  $\gamma$ . Also for  $a < b < c < d \in \mathbb{Z}$ , and  $\gamma$  an increasing path from  $(a, a')$  to  $(d, d')$ , we define

$$\gamma[b, c] = \{\gamma(x) : b \leq x \leq c\}$$

as the part of  $\gamma$  between the vertical lines  $x = b$  and  $x = c$ .

For  $u = (x, y) \preceq u' = (x', y')$  in  $\mathbb{Z}^2$ , let  $d(u, u') = (x' - x) + (y' - y)$  denote the  $\ell_1$  distance between  $u$  and  $u'$ . Define

$$\begin{aligned} \tilde{T}_{u,u'} &= T_{u,u'} - \mathbb{E}(T_{u,u'}), \\ \hat{T}_{u,u'} &= T_{u,u'} - 2d(u, u'). \end{aligned}$$

It is easy to see that  $\hat{T}_{u,u'} \leq \tilde{T}_{u,u'}$ . Roughly speaking, if the slope of the line joining  $u$  and  $u'$  is close to 1, then  $\tilde{T}_{u,u'}$  can be well approximated by  $\hat{T}_{u,u'}$  (see Section 9 of [16]).

Also for any set  $R \subseteq \mathbb{R}^2$ , let  $T_{u,v}^R$  denote the weight of the maximal path from  $u$  to  $v$  that avoids the region  $R$ . Let  ${}^R T_{u,v}$  denote the weight of the maximal path from  $u$  to  $v$  that intersects the region  $R$ . Also define  $\tilde{T}_{u,v}^R = T_{u,v}^R - \mathbb{E}T_{u,v}^R$  and  $\hat{T}_{u,v}^R = T_{u,v}^R - 2d(u, v)$ . Similarly define  ${}^R \tilde{T}_{u,v}$  and  ${}^R \hat{T}_{u,v}$ .

We shall use the notation  $\llbracket \cdot, \cdot \rrbracket$  to denote discrete intervals, i.e.,  $\llbracket a, b \rrbracket$  will denote  $[a, b] \cap \mathbb{Z}$ . We shall often assume without loss of generality that fractional powers of integers i.e.,  $k^{2/3}$  or rational multiples of integers as integers themselves. This is done merely to avoid the notational overhead of integer parts, and it is easy to check that such assumptions do not affect the proofs in any substantial way. In the various theorems and lemmas, the values of the constants  $C, C', c, c'$  appearing in the bounds change from one line to the next, and will be chosen small or large locally.

### 2.1.5 Organisation of the chapter

The rest of the chapter is organised as follows. In Section 2.2 we prove Theorem 3 and also prove a refinement Theorem 2.2.1. In Section 2.3, we start prove Theorem 2.1.4 by reducing it to the key Proposition 2.3.1, and use it to establish Theorem 1. The next two sections are devoted to the proof of Proposition 2.3.1. In Section 2.4, we define a geometric structure and a number of key events used in the rest of the proof. Section 2.5 constructs a collection of coalescing paths on a combination of the key events and estimates the corresponding probabilities, and concludes the proof of Proposition 2.3.1. Section 2.6 contains the proof of Theorem 2. This final section is independent of the rest of the chapter except that we use a variant of Theorem 3.

## 2.2 Local transversal fluctuation of geodesics

Our objective in this section is to prove Theorem 3 and prove a refinement of the same. The proof of Theorem 3 is reminiscent of an argument in [68] in the context of first passage percolation, however is much stronger than the result there as for first passage percolation one has much weaker information about fluctuation of passage times and transversal fluctuations.

*Proof of Theorem 3.* We show only that  $\mathbb{P}[(\Gamma(\ell) - \ell) \geq sr^{2/3}] \leq e^{-cs^2}$ . The other parts are similar. Also for convenience, we assume that  $k' = 0$ , so that  $\Gamma$  is the geodesic from  $(0, 0)$  to  $(n, n + k)$ . For general  $k' \leq L\ell^{2/3}$ , the argument is similar.

Choose  $\alpha = 2^{\frac{1}{6}}$ . For  $j \geq 0$ , let  $B_j$  denote the event that  $\Gamma(2^j\ell) - 2^j\ell \geq s((2\alpha)^j\ell)^{2/3}$  and  $\Gamma(2^{j+1}\ell) - 2^{j+1}\ell \leq s((2\alpha)^{j+1}\ell)^{2/3}$ . Note that as  $|\Gamma(n) - n| = |k| \leq Ln^{2/3} < sn^{2/3}$  for all  $s \geq 2L$ , hence,

$$\{\Gamma(\ell) - \ell \geq sl^{2/3}\} \subseteq \bigcup_{j \geq 0} B_j.$$

See Figure 2.1.

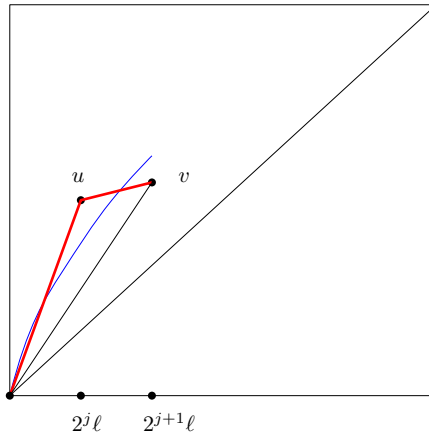


Figure 2.1: Proof of Theorem 3: the blue curve in the figure is the graph of the function  $y = x + sx^{2/3}$ . On the event  $\Gamma(\ell) - \ell \geq sl^{2/3}$ , there must exist some  $j \geq 0$  such that  $\Gamma$  crosses the blue curve on the interval  $[2^j\ell, 2^{j+1}\ell]$ . The event  $B_j$  is a slightly more involved variant of the event described above. On the event  $B_j$  one must have points  $u$  and  $v$  as above such that the path from  $(0, 0)$  to  $u$  and then to  $v$  is atypically long, hence this event is unlikely.

Hence it suffices to show that

$$\mathbb{P}(B_j) \leq e^{-cs^2\alpha^{2j/3}}.$$

Let  $B_{j,t,t'}$  denote the event that

$$\Gamma(2^j\ell) \in U_t := [2^j\ell + (s+t)((2\alpha)^j\ell)^{2/3}, 2^j\ell + (s+t+1)((2\alpha)^j\ell)^{2/3}]$$

and

$$\Gamma(2^{j+1}\ell) \in V_{t'} := [2^{j+1}\ell + (s - t')((2\alpha)^{j+1}\ell)^{2/3}, 2^{j+1}\ell + (s - t' + 1)((2\alpha)^{j+1}\ell)^{2/3}].$$

for  $t, t' = 0, 1, 2, \dots$ . Clearly,

$$B_{j,t,t'} \subseteq \left\{ \sup_{u \in U_t, v \in V_{t'}} (T_{0,(2^j\ell,u)} + T_{(2^j\ell,u),(2^{j+1}\ell,v)} - T_{0,(2^{j+1}\ell,v)}) \geq 0 \right\}.$$

If  $\mathcal{S}$  is the line segment joining  $\mathbf{0}$  to some vertex  $v \in V_{t'}$ , then it is easy to see that

$$\mathcal{S}(2^j\ell) - 2^j\ell \leq \frac{\alpha^{2/3}}{2^{1/3}}(s - t' + 1)((2\alpha)^j\ell)^{2/3}.$$

Thus computing expectations, it follows from Lemma 9.4 of [16]<sup>2</sup> and the fact that  $2^{1/2} > \alpha > 1$ , that there exists some constant  $c_1$  not depending on  $\ell, s, t, t', j$ , such that for all  $u \in U_t, v \in V_{t'}$ , and all  $s$  sufficiently large,

$$\mathbb{E}(T_{0,(2^j\ell,u)}) + \mathbb{E}(T_{(2^j\ell,u),(2^{j+1}\ell,v)}) \leq \mathbb{E}(T_{0,(2^{j+1}\ell,v)}) - c_1((s + t + t')\alpha^{\frac{2j}{3}})^2(2^j\ell)^{1/3}.$$

Using the moderate deviation estimates for supremum and infimum of the lengths of a collection of paths given in Proposition 10.1 and Proposition 10.5 of [16] (and breaking  $U_t$  and  $V_{t'}$  into consecutive intervals of length  $(2^j\ell)^{2/3}$  and taking  $\alpha^{4j/3}$ -many union bounds), and using similar arguments as in the proof of Lemma 11.3 of [16], this implies,

$$\mathbb{P}(B_{j,t,t'}) \leq e^{-c(s+t+t')^2\alpha^{\frac{2j}{3}}}.$$

Summing over  $t, t', j$  gives the result. □

### 2.2.1 An improved regularity estimate

Observe that Theorem 3 says that at any given length scale  $\ell$ , the geodesic is unlikely to have a transversal fluctuation that is much larger than  $\ell^{2/3}$ . Our next result will show a decorrelation between these unlikely events at well separated length scales.

Let  $\Gamma$  denote the geodesic from  $\mathbf{0}$  to  $\mathbf{n}$ . For  $k, M \in \mathbb{Z}$  fixed, and any vertex  $v \in \mathbb{Z}^2$ , let  $A_i^v$  denote the event that  $|\Gamma_{v,\mathbf{n}}(M^i k) - M^i k| \geq s(M^i k)^{2/3}$ . Also let  $A_i := A_i^{\mathbf{0}}$ . We have the following theorem.

---

<sup>2</sup>Using similar observations as made in Section 9 of [16] for Poissonian LPP, from Theorem 2.1.1 and Theorem 2.1.2, it follows that, in Exponential LPP model, for fixed  $\psi > 0$ , there exists  $r_0 = r_0(\psi)$  such that for points  $u = (x, y)$  and  $u' = (x', y')$  in  $\mathbb{Z}^2$  such that  $x' - x = r \geq r_0$ , and  $\frac{y' - y}{x' - x} \in (\frac{1}{\psi}, \psi)$ , one has,  $\mathbb{E}(T_{u,u'}) = (\sqrt{r} + \sqrt{y' - y})^2 + O(r^{1/3}) = d(u, u') + 2\sqrt{r(y' - y)} + O(r^{1/3})$ . Hence, Corollary 9.3, Lemma 9.4 and Lemma 9.5 from [16] continue to hold for Exponential LPP as well with the same proof.

**Theorem 2.2.1.** *There exist positive constants  $s_0, M_0, c, c'$  such that for all  $s > s_0, M > M_0$  and  $\ell, k \in \mathbb{Z}$  we have*

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left( \sum_{i=1}^{\ell} \mathbf{1}_{A_i} \geq 2e^{-cs} \ell \right) \leq e^{-c'\ell}.$$

Theorem 2.2.1 will follow from the next proposition.

**Proposition 2.2.2.** *Let  $F \subseteq [\ell]$ . Then there exist positive constants  $c, s_0, M_0$  such that for all  $s \geq s_0, M \geq M_0$ ,*

$$\mathbb{P}(|\Gamma(M^i k) - M^i k| \geq s(M^i k)^{2/3} \text{ for all } i \in F) \leq e^{-cs|F|}.$$

We postpone the proof of Proposition 2.2.2 for now, and first show how Theorem 2.2.1 follows from Proposition 2.2.2.

*Proof of Theorem 2.2.1.* Let  $B_i$ 's for  $i \in [\ell]$  be i.i.d. Bernoulli random variables with success probability  $e^{-cs}$ . Then Theorem 2.2.2 implies that  $(\mathbf{1}_{A_1}, \mathbf{1}_{A_2}, \dots, \mathbf{1}_{A_\ell}) \preceq_{ST} (B_1, B_2, \dots, B_\ell)$ . Hence,  $\sum_{i=1}^{\ell} \mathbf{1}_{A_i} \preceq_{ST} \sum_{i=1}^{\ell} B_i$ . Theorem 2.2.1 now follows from Hoeffding's inequality applied to  $\sum_{i=1}^{\ell} B_i$ .  $\square$

For the proof of Proposition 2.2.2 we will need the following lemma that is basic and was stated in [16]. As we would have several occasions to resort to this lemma, we restate it here without proof.

**Lemma 2.2.3** ([16], Lemma 11.2, Polymer Ordering). *Consider points  $a = (a_1, a_2), a' = (a_1, a_3), b = (b_1, b_2), b' = (b_1, b_3)$  such that  $a_1 < b_1$  and  $a_2 \leq a_3 \leq b_2 \leq b_3$ . Then we have  $\Gamma_{a,b}(x) \leq \Gamma_{a',b'}(x)$  for all  $x \in \llbracket a_1, b_1 \rrbracket$ .*

We shall now prove Proposition 2.2.2. Before starting with the technicalities of the proof let us explain the basic idea which is however simple. Consider the case  $k = 1$  and  $F = \{1, 2\}$ . Using constructions as in the proof of Theorem 3, for  $M$  sufficiently large we can approximate the event  $A_1 := |\Gamma(M) - M| \geq sM^{2/3}$ , up to a very small error in probability by an event  $B_1$  that depends only on the random field on  $\llbracket 0, D \rrbracket \times \mathbb{Z}$ , for some  $M \ll D \ll M^2$ . The main point is that even on the unlikely event  $A_1$ , it is very likely that  $|\Gamma(D) - D| \leq s'D^{2/3}$  for some  $s'$  that is not too large. This implies the event  $A_2 := |\Gamma(M^2) - M^2| \geq sM^{4/3}$  can then be well approximated by another unlikely event  $B_2$  that is measurable with respect to the random field on  $\llbracket D + 1, n \rrbracket \times \mathbb{Z}$ , and hence independent of  $B_1$ . The following proof makes this idea precise.

*Proof of Proposition 2.2.2.* Without loss of generality we assume  $k = 1$ . Also let  $F = [\ell]$ . For any fixed subset  $F \subseteq [\ell]$ , the proof follows similarly. Fix  $\alpha < 2^{1/2}$ . Choose  $C'$  large enough so that  $\alpha^{2C'/3} \geq 2$ . Let  $M = 2^C$  where  $C$  is large enough such that  $\alpha^{C'} < 2^{C-C'}$ . Let  $A_i^v$  denote the event that  $\Gamma_{v,\mathbf{n}}(M^i k) - M^i k \geq s(M^i k)^{2/3}$ . Also let  $A_i' := A_i^{\mathbf{0}}$ .

Fix  $\ell, s$ . For any  $r \in [\ell]$  and any  $0 < x \leq M^{r+1}$  and any  $s_1 \geq s$  such that  $s_1((2\alpha)^{C'}x)^{2/3} < s(M^{r+2})^{2/3}$ , let  $E_{r+1,x,s_1}^v = \{\Gamma(x) - x \geq s_1x^{2/3}\}$ . Define  $E_{r+1,x,s_1}^v$  similarly with the geodesic  $\Gamma$  replaced by  $\Gamma_{v,n}$ . Also let  $0 \leq z < x$  be such that  $x - z \geq \beta x$  for some fixed positive constant  $\beta$ , and let  $s_2$  be such that  $s_2z^{2/3} < \frac{s_1x^{2/3}}{2}$ . We claim that for any such  $x$  fixed, and any such  $z < x$  fixed and any such  $s_1, s_2$ , and for  $v = (z, z + s_2(z)^{2/3})$

$$\mathbb{P} \left( E_{r+1,x,s_1}^v \cap \bigcap_{i \in [r+2,\ell]} A_i^{v'} \right) \leq 2^{\ell-r} e^{-cs_1 - cs(\ell-r-1)}, \quad (2.2)$$

where  $c$  is some absolute constant.

We prove the statement (2.2) by induction on  $\ell - r$ . Clearly this holds when  $r = \ell - 1$  by simply applying Theorem 3. Assume (2.2) holds for  $r = k + 1$ , we prove this for  $r = k$ . Fix  $x \leq M^{k+1}$ ,  $z < x$  and  $s_1 \geq s$  such that  $x - z \geq \beta x$  and  $s_1((2\alpha)^{C'}x)^{2/3} < s(M^{k+2})^{2/3}$ . Parallel to the events in Theorem 3, for  $v = (z, z + s_2z^{2/3})$ , define  $B_j^v$  as the event that  $\Gamma_{v,n}(2^jx) - 2^jx \geq s_1((2\alpha)^{jx})^{2/3}$  and  $\Gamma_{v,n}(2^{j+1}x) - 2^{j+1}x \leq s_1((2\alpha)^{j+1}x)^{2/3}$  for all  $j = 0, 1, 2, \dots, C' - 1$ . Then

$$\begin{aligned} B &:= E_{k+1,x,s_1}^v \cap \bigcap_{i \in [k+2,\ell]} A_i^{v'} \\ &\subseteq \bigcup_{j=0}^{C'-1} \left( B_j^v \cap \bigcap_{i \in [k+2,\ell]} A_i^{v'} \right) \cup \left( \left\{ \Gamma_{v,n}(2^{C'}x) - 2^{C'}x \geq s_1((2\alpha)^{C'}x)^{2/3} \right\} \cap \bigcap_{i \in [k+3,\ell]} A_i^{v'} \right). \end{aligned}$$

Now let  $U_j = \{(2^jx, y) : y \geq 2^jx + s_1((2\alpha)^{jx})^{2/3}\}$  and  $V_j = \{(2^{j+1}x, y) : y \leq 2^{j+1}x + s_1((2\alpha)^{j+1}x)^{2/3}\}$  for  $j \leq C' - 1$ . Then, for each  $j \leq C' - 1$ ,

$$\begin{aligned} &\left( B_j^v \cap \bigcap_{i \in [k+2,\ell]} A_i^{v'} \right) \\ &\subseteq \left\{ \sup_{u \in U_j, w \in V_j} (T_{v,u} + T_{u,w} - T_{v,w}) \geq 0 \right\} \cap \left\{ \bigcup_{w \in V_j} \left\{ \bigcap_{i \in [k+2,\ell]} A_i^{w'} \right\} \right\} \end{aligned}$$

The two events in the intersection are independent. It follows from the proof of Theorem 3 that

$$\mathbb{P} \left\{ \sup_{u \in U_j, w \in V_j} (T_{v,u} + T_{u,w} - T_{v,w}) \geq 0 \right\} \leq e^{-cs_1\alpha^{2j/3}}.$$

Also for all  $j \leq C' - 1$ , using the induction hypothesis (2.2) with  $r = k + 1$ ,  $s'_1 = s$ ,  $x' = M^{r+2}$ , and  $v' = (2^{j+1}x, 2^{j+1}x + s_1((2\alpha)^{j+1}x)^{2/3})$ , the topmost vertex of  $V_j$ , and using polymer ordering Lemma 4.3.2,

$$\mathbb{P} \left\{ \bigcup_{w \in V_j} \left\{ \bigcap_{i \in [k+2,\ell]} A_i^{w'} \right\} \right\} \leq \mathbb{P} \left( \bigcap_{i \in [k+2,\ell]} A_i^{v'} \right) \leq 2^{\ell-k-1} e^{-cs(\ell-k-1)}.$$

Also the fact that  $\alpha^{C'} < 2^{C-C'}$  and the condition on  $s_1$  imply that  $s_1''((2\alpha)^{C'}x'')^{2/3} < s(M^{j+3})^{2/3}$ , where  $x'' = 2^{C'}x$  and  $s_1'' = s_1\alpha^{2C'/3}$  and  $\{\Gamma(2^{C'}x) - 2^{C'}x \geq s_1((2\alpha)^{C'}x)^{2/3}\} = E_{k+2,x'',s_1''}$ . Hence, applying statement (2.2) of the induction hypothesis again,

$$\begin{aligned} & \mathbb{P} \left( \left\{ \Gamma_{v,n}(2^{C'}x) - 2^{C'}x \geq s_1((2\alpha)^{C'}x)^{2/3} \right\} \bigcap_{i \in [k+3,\ell]} A_i^v \right) \\ &= \mathbb{P} \left( E_{k+2,x'',s_1''} \bigcap_{i \in [k+3,\ell]} A_i^v \right) \\ &\leq 2^{l-k-1} e^{-cs_1'' - cs(\ell-k-2)} = 2^{l-k-1} e^{-cs_1\alpha^{2C'/3} - cs(\ell-k-2)}. \end{aligned}$$

Hence, bringing all this together,

$$\mathbb{P}(B) \leq \sum_{j=1}^{C'-1} 2^{\ell-k-1} e^{-cs(\ell-k-1)} e^{-cs_1\alpha^{2j/3}} + 2^{l-k-1} e^{-cs_1\alpha^{2C'/3} - cs(\ell-k-2)}.$$

Since  $\alpha^{2C'/3} \geq 2$ , this proves that

$$\mathbb{P}(B) \leq 2^{\ell-k} e^{-cs_1 - cs(\ell-k-1)}.$$

This proves statement (2.2) of the induction hypothesis for  $j = k$  and completes the induction and proves the claim.

Hence, with  $r = 0$ ,  $x = M$  and  $s_1 = s$ ,  $z = 0$ , we get  $E_{j+1,x,s_1} = A_1'$ , and from the above claim,

$$\mathbb{P} \left( \bigcap_{i=1}^{\ell} A_i' \right) \leq 2^{\ell} e^{-c\ell s}.$$

hence, by taking a union bound over all  $2^{\ell}$  terms,

$$\mathbb{P}(|\Gamma(M^i) - M^i| \geq s(M^i)^{2/3} \text{ for all } i \in [\ell]) \leq 2^{2\ell} e^{-c\ell s}.$$

For all  $s \geq s_0$ , such that  $\frac{c}{2}s > \log 4$ , one has the result.  $\square$

Note that the arguments used in proving Theorem 3 and Theorem 2.2.1 would still go through if we considered the transversal fluctuation of the geodesic between two points such that the line segment joining them has slope  $m$  bounded away from 0 and  $\infty$ . We state this without proof in the following corollary.

**Corollary 2.2.4.** *Let  $\psi > 1$  and  $m \in [\frac{1}{\psi}, \psi]$  be fixed. Let  $\Gamma$  be the geodesic from  $(0, 0)$  to  $(n, mn)$ . Let  $\mathcal{S}$  be the line segment joining  $(0, 0)$  to  $(n, mn)$ . For  $\ell \in \mathbb{Z}$ , let  $\mathcal{S}(\ell)$  be such that  $(\ell, \mathcal{S}(\ell)) \in \mathcal{S}$ .*

(a) Then there exist positive constants  $n_0, \ell_0, s_0, c$  depending only on  $\psi$ , such that for all  $n \geq n_0, s \geq s_0, \ell \geq \ell_0$ ,

$$P[|\Gamma(\ell) - \mathcal{S}(\ell)| \geq s\ell^{2/3}] \leq e^{-cs}.$$

(b) For  $k, M \in \mathbb{Z}$  fixed, and any vertex  $v \in \mathbb{Z}^2$ , let  $A'_i$  denote the event that  $|\Gamma(M^i k) - \mathcal{S}(M^i k)| \geq s(M^i k)^{2/3}$ . Then there exist positive constants  $s_0, M_0, c, c'$  depending only on  $\psi$ , such that for all  $s > s_0, M > M_0$  and  $\ell, k \in \mathbb{Z}$  we have

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left( \sum_{i=1}^{\ell} \mathbf{1}_{A'_i} \geq 2e^{-cs}\ell \right) \leq e^{-c'\ell}.$$

## 2.3 Coalescence of finite geodesics

In this section we prove Theorem 1 following the strategy outlined in Section 2.1.3. We prove Theorem 2.1.4 modulo the key Proposition 2.3.1 stated below, and use it to complete the proof of Theorem 1.

Recall the set-up of Theorem 2.1.4. Let  $L$  be a large fixed number. For  $r \in \mathbb{N}$  consider the points  $a_r = (r, r + Lr^{2/3})$  and  $b_r = (r, r - Lr^{2/3})$  (assume without loss of generality that  $r^{2/3}$  is an integer). For some large  $M \in \mathbb{N}$  let  $\Gamma_1$  denote the geodesic from  $a_r$  to  $a_{Mr}$ , and  $\Gamma_2$  denote the geodesic from  $b_r$  to  $b_{Mr}$ . Let  $\text{Coal}_{r,M}$  denote the event that  $\Gamma_1$  and  $\Gamma_2$  share a common vertex. We want to show that the probability of this event is bounded below by  $1/2$ , for some suitably large  $M$ , uniformly in all large  $r$ . We first show that the probability is bounded below by an arbitrary small constant independent of  $r$  and  $M$ .

**Proposition 2.3.1.** *Fix  $z, r \in \mathbb{N}$  and  $0 \leq u_0 \leq \log \log z$  and  $0 \leq v_0 \leq \log \log z^2$ . Set  $a_1 = (zr, zr + u_0 z^{2/3} r^{2/3}), b_1 = (zr, zr - u_0 z^{2/3} r^{2/3}), a_2 = (z^2 r, z^2 r + v_0 (z^2)^{2/3} r^{2/3}), b_2 = (z^2 r, z^2 r - v_0 (z^2)^{2/3} r^{2/3})$ . Let  $\Gamma_0$  be the geodesic from  $a_1$  to  $a_2$  and  $\Gamma'_0$  be the geodesic from  $b_1$  to  $b_2$ . Let  $F$  be the event that  $\Gamma_0$  and  $\Gamma'_0$  meet one another. Then there exists an absolute positive constant  $\alpha$  not depending on  $z, r$ , such that*

$$\mathbb{P}(F) \geq \alpha > 0.$$

Proof of Proposition 2.3.1 is rather elaborate and the next two sections are devoted to it. Roughly the idea is as follows. We show that at some horizontal length scale  $D$  (where  $r \ll D \ll Mr$ ) with a probability bounded away from zero there exists a barrier of width  $O(D)$  and height  $O(D^{2/3})$  just above the geodesic  $\Gamma_2$ , such that any path passing through the barrier is penalised a lot. Using Theorem 3 and the FKG inequality we show that in presence of such a barrier and in environment that is typical otherwise,  $\Gamma_2$  will merge with  $\Gamma_1$  before crossing the barrier region. The construction of the barrier here is similar to one present in [16], and we shall quote many of the probabilistic estimates in that paper throughout our proof.

We first show how we can conclude Theorem 2.1.4 from Proposition 2.3.1.

*Proof of Theorem 2.1.4.* Fix  $r$ . By translation invariance, it is enough to look at the geodesic from  $e_0 = (0, Lr^{2/3})$  to  $e_1 = ((M-1)r, (M-1)r + L(Mr)^{2/3})$  and the geodesic from  $g_0 = (0, -Lr^{2/3})$  to  $g_1 = ((M-1)r, (M-1)r - L(Mr)^{2/3})$ . Let  $\Gamma = \Gamma_{e_0, e_1}$  and  $\Gamma' = \Gamma_{g_0, g_1}$ . Set

$$p_i = 2^{2^i} r, \text{ for } i = \frac{N}{2}, \frac{N}{2} + 1, \dots, N,$$

such that  $2^{2^N} = M - 1$ , i.e.,  $N = c \log \log(M - 1)$  for some absolute constant  $c$ . Also set  $a_i = (p_i, p_i + p_i^{2/3} i)$  and  $b_i = (p_i, p_i - p_i^{2/3} i)$ , for  $i = \frac{N}{2}, \frac{N}{2} + 1, \dots, N$ . Applying Lemma 2.2.4, it follows that  $\Gamma$  and  $\Gamma'$  pass between  $a_i$  and  $b_i$  for all  $i$  with probability at least

$$1 - 2 \sum_{i=\frac{N}{2}}^N e^{-c \frac{i}{2}} \geq 1 - C' e^{-c' N}.$$

From Proposition 2.3.1, with  $z = 2^{2^i}$ , it is easy to see that  $\Gamma_{a_i, a_{i+1}}$  and  $\Gamma_{b_i, b_{i+1}}$  meet with a positive probability  $\alpha$  not depending on  $i, r$ . Hence, due to polymer ordering (Lemma 4.3.2) and independence in each block,  $\Gamma$  and  $\Gamma'$  meet between 0 and  $(M-1)r$  with probability at least

$$1 - (1 - \alpha)^{N/2} - C' e^{-c' N} \geq 1 - \frac{C'}{(\log M)^{c'}},$$

for some positive constants  $C', c'$  that depend only on  $L$  but not on  $r$  or  $M$ . This proves the theorem by choosing  $M$  sufficiently large.  $\square$

We now complete the proof of Theorem 1. The proof is similar to the proof of Theorem 2.1.4, except that we use Theorem 2.2.1 instead of Theorem 3 and a union bound.

*Proof of Theorem 1.* Let  $M > M_0, s_0$  as in Theorem 2.2.1 and let  $e^{-cs_0} = \varepsilon < \frac{1}{8}$ . Let  $C_i$  denote the event that  $|\Gamma_{v_1, \mathbf{n}}(M^i k) - M^i k| \leq s_0 (M^i k)^{2/3}$  and  $|\Gamma_{v_2, \mathbf{n}}(M^i k) - M^i k| \leq s_0 (M^i k)^{2/3}$ . Let  $a_i = (M^i k, M^i k + s_0 (M^i k)^{2/3})$  and  $b_i = (M^i k, M^i k - s_0 (M^i k)^{2/3})$ . Let  $D_i$  denote the event that the geodesic from  $a_i$  to  $a_{i+1}$  meets the geodesic from  $b_i$  to  $b_{i+1}$ . By Theorem 2.1.4 and choosing  $r = M^i k$ , it follows that for  $M$  sufficiently large we have  $\mathbb{P}(D_i) \geq 1/2$  for all  $i$ . Since  $D_i$  are independent it follows that for all  $\ell$

$$\mathbb{P} \left( \sum_{i=1}^{\ell} \mathbf{1}_{D_i} \leq \frac{\ell}{4} \right) \leq e^{-c' \ell}$$

for some  $c' > 0$ . Using this together with Theorem 2.2.1 we get

$$\mathbb{P} \left( \sum_{i=1}^{\log R (\log M)^{-1}} \mathbf{1}_{C_i \cap D_i} = 0 \right) \leq R^{-c},$$

for some absolute positive constant  $c$ . This proves Theorem 1.  $\square$

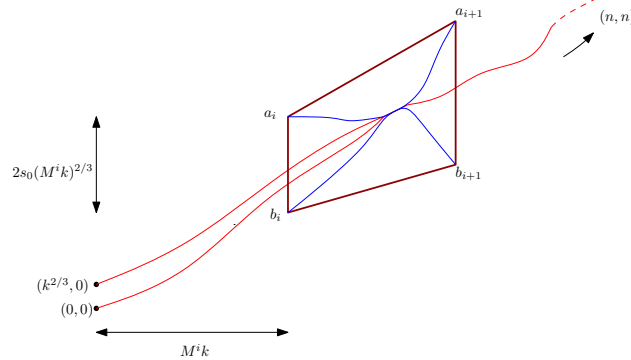


Figure 2.2: Merging of paths as in the proof of Theorem 1. Using Theorem 2.2.1 it follows that the geodesics  $\Gamma_1$  and  $\Gamma_2$  are very likely to pass between the points  $a_i$  and  $b_i$  for all  $i$  sufficiently large. Using Theorem 2.1.4 we show that with a positive probability they merge in one of those intervals. Proof of Theorem 2.1.4 is similar, except there we choose intervals growing doubly exponentially and the distance between the points  $a_i$  and  $b_i$  much larger. The argument proceeds by using Proposition 2.3.1 instead of Theorem 2.1.4 and Theorem 3 together with a union bound replacing Theorem 2.2.1.

The idea of the proofs of Theorem 2.1.4 and Theorem 1 is illustrated in Figure 2.2. It will be clear from our proof that Theorem 2.1.4 works for geodesics in any fixed direction bounded away from the co-ordinate axes directions. This, together with Corollary 2.2.4 implies the following corollary, which we state without proof.

**Corollary 2.3.2.** *Let  $L, m_0$  be two fixed positive constants and let  $n \gg k$ . Let  $\Gamma$  be the geodesic from  $(0, Lk^{2/3})$  to  $(n, m_0 n + Ln^{2/3})$  and  $\Gamma'$  be the geodesic from  $(0, -Lk^{2/3})$  to  $(n, m_0 n - Ln^{2/3})$  in the exponential LPP model. Let  $v_* = (v_{*,1}, v_{*,2})$  be a leftmost common point between  $\Gamma$  and  $\Gamma'$ . Then,*

$$\mathbb{P}(v_{*,1} > Rk) \leq CR^{-c},$$

for some positive constants  $C, c$  that depend on  $m_0, L$  but not on  $k, n$ .

To complete the proof of Theorem 1, it only remains to prove Proposition 2.3.1. This is done over the next two sections. Observe that by scaling of the process, it suffices to prove Proposition 2.3.1 for  $r = 1$ . For the next two sections we shall always be in this setting even though we might not explicitly mention it every time.

## 2.4 Favourable events

First we need to define a set of events that will be key to our proof. Some of these are similar to the events in Section 3 of [16]. We need to introduce some more notations before we can

define these events.

### More notations

Fix  $z \in \mathbb{N}$  and consider the set up as in Proposition 2.3.1 with  $r = 1$ . Define  $x = z^{3/2}$ . In this section, all the events are constructed for this fixed  $z$ . As there is no scope for confusion, we suppress the dependence of the events on  $z$ . Recall that  $\Gamma_0$  is the geodesic from  $a_1$  to  $a_2$  and  $\Gamma'_0$  be the geodesic from  $b_1$  to  $b_2$ .

Let  $\mathcal{P}(w, \ell, h, s)$  denote the parallelogram whose leftmost endpoint is  $(w, w - h)$ , and two sides are parallel to the diagonal and  $y$ -axis and are of length  $\sqrt{2}\ell$  and  $h + s$  respectively, i.e., whose endpoints are  $(w, w - h), (w + \ell, w + \ell - h), (w, w + s), (w + \ell, w + \ell + s)$ . Construct the barrier  $B$  at  $x$  of width  $x/10$  and height  $(4M + S)x^{2/3}$  as follows,

$$B = \mathcal{P}(x, x/10, 2Mx^{2/3}, (2M + S)x^{2/3}).$$

Let us denote the left wall of the barrier as  $L_1$  and the right wall  $L_2$ . Also let

$$\mathcal{Z} = \{(u, v) \in \mathbb{R}^2 : x \leq u \leq x + \frac{x}{10}\}$$

be the region bounded by the vertical lines at  $x$  and  $x + \frac{x}{10}$ . Define  $x' = (2 + \frac{1}{10})x$  and let  $L_3$  denote the line segment joining  $(x', x' - 2M(x')^{2/3})$  and  $(x', x' + (2M + S)(x')^{2/3})$ . See Figure 2.3 for an illustration of the above definitions.

### Choice of parameters

The construction of the favourable events will depend on a number of parameters. In the definitions that follow  $H, M, S$  will denote large positive constants to be chosen appropriately later (not depending on  $z$ ). The dependence among these constants are as follows.

1.  $H$  will denote a large absolute constant.
2.  $M$  will denote a large absolute constant.<sup>3</sup>
3.  $S$  is chosen sufficiently large depending on  $M$  and  $H$ .

With this preparation we can now define the favourable events, which are divided into four types. The first three are typical in the sense that they hold with probability close to 1 (for the appropriate choice of parameters) but the final one only occurs with probability bounded away from 0.

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<sup>3</sup>Note that the parameter  $M$  here and in subsequent sections is in no way related to the constant  $M$  used in the earlier Sections 2.2 and 2.3.

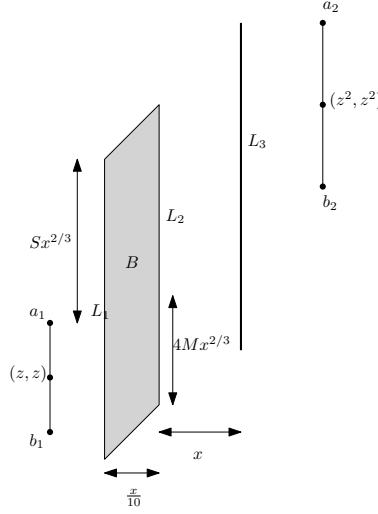


Figure 2.3: The basic elements of our construction, the barrier  $B$ , and the line segments  $L_1, L_2$  and  $L_3$ . Notice that  $x = z^{3/2}$  so the barrier is much closer to the left boundary than to the right one.

### 2.4.1 Wing condition

Fix some large absolute constant  $H$  to be chosen appropriately later. We say  $G$  holds if the following two conditions hold:

- (i) For all  $u \in L_1$ , the left wall of the barrier,

$$|\tilde{T}_{a_1, u}| \leq H\sqrt{S}x^{1/3}.$$

- (ii) For all  $u' \in L_2$ , and  $v \in L_3$ ,

$$|\tilde{T}_{u', v}| \leq H\sqrt{S}x^{1/3}.$$

The point of this condition is to ensure that the passage times to the left and the right of the barrier region behave typically. It follows from Lemma 7.3 in [16] that the event  $G$  holds with high probability, i.e., by choosing  $H$  a large absolute constant,

$$\mathbb{P}(G) \geq \frac{99}{100}.$$

Observe that  $G$  depends only on the configuration in  $\mathcal{Z}^c$ .

### 2.4.2 Typical path

Let  $m = \frac{11}{10}$ , so that the barrier ends at the vertical line  $y = mx$ , and recall  $x' = (m + 1)x$ . Let  $\gamma$  be an increasing path from  $b_1$  to  $b_2$ . Define,

$$\ell(\widetilde{\gamma[x, mx]}) := \ell(\gamma[x, mx]) - \mathbb{E}(T_{(x, \gamma(x)), (mx, \gamma(mx))}),$$

and

$$\ell(\widehat{\gamma[x, mx]}) := \ell(\gamma[x, mx]) - 2d((x, \gamma(x)), (mx, \gamma(mx))),$$

where  $\gamma[x, mx]$  is the part of  $\gamma$  between  $x$  and  $mx$  and  $\ell(\gamma)$  is the weight of  $\gamma$ . We say  $\gamma$  is **typical** at location  $x$ , if the weight of  $\gamma[x, mx]$  behave typically, and a series of geometric conditions hold ensuring  $\gamma$  passes through the bottom part of  $B$  and fluctuates at the typical transversal fluctuation scale of geodesics in the region between  $L_1$  and  $L_3$ . More concretely, we ask for the following conditions:

**Weight conditions:**

$$\left| \ell(\widehat{\gamma[x, mx]}) \right| \leq H\sqrt{M}x^{1/3}. \quad (2.3)$$

$$\left| \ell(\widetilde{\gamma[x, mx]}) \right| \leq H\sqrt{M}x^{1/3}. \quad (2.4)$$

**Geometric Conditions:**

$$x + Mx^{2/3} \geq \gamma(x) \geq x - Mx^{2/3}, \quad (2.5)$$

$$mx + M(mx)^{2/3} \geq \gamma(mx) \geq mx - M(mx)^{2/3}, \quad (2.6)$$

$$x' + M(x')^{2/3} \geq \gamma(x') \geq x' - M(x')^{2/3}. \quad (2.7)$$

$$\{(t, \gamma(t)) : x \leq t \leq mx\} \subseteq \mathcal{P}(x, x/10, 2Mx^{2/3}, 2Mx^{2/3}) =: B_0. \quad (2.8)$$

Note that conditions (2.3) and (2.4) involve  $\Pi_{\gamma[x, mx]}$ , the configuration on  $\gamma[x, mx]$ , and the rest are conditions on the geometric properties of the path  $\gamma$ . See Figure 2.4 for an illustration. We shall show later that geodesics are typical with high probability.

### 2.4.3 Path condition

Fix any increasing path  $\gamma$  from  $b_1$  to  $b_2$ . Our next favourable event asks that paths from  $a_1$  to  $a_2$  that have atypical transversal fluctuations in order to avoid crossing the barrier region will be not competitive with paths that simply cross the barrier region coinciding with  $\gamma$ .

Let  $T_{\gamma, x, mx}$  denote the weight of the best path from  $a_1$  to  $a_2$  that coincides with  $\gamma[x, mx]$  between  $x$  and  $mx$ . Let  $F_\gamma^1$  be the weight of the best path from  $a_1$  to  $a_2$  that is more than a distance of  $(2M + S)x^{2/3}$  above the diagonal at  $x$ , (i.e., at  $x$  is above the left boundary of  $B$ ) and stays above  $\gamma[x, mx]$  in  $[x, mx]$ . Let

$$A_\gamma^1 = \{F_\gamma^1 < T_{\gamma, x, mx} - \sqrt{S}x^{1/3}\}.$$

Similarly, let  $F_\gamma^2$  be the weight of the best path from  $a_1$  to  $a_2$  that is more than a distance of  $(2M + S)(mx)^{2/3}$  above the diagonal at  $mx$ , (i.e., passes above the right boundary of  $B$ ) and stays above  $\gamma[x, mx]$  in  $[x, mx]$ , and define

$$A_\gamma^2 = \{F_\gamma^2 < T_{\gamma, x, mx} - \sqrt{S}x^{1/3}\}.$$

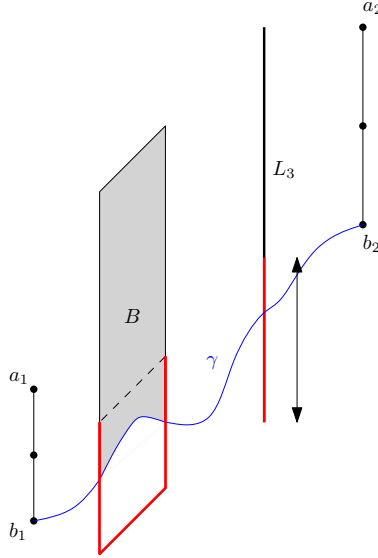


Figure 2.4: Typical paths and barrier condition: a path  $\gamma$  from  $b_1$  to  $b_2$  is typical if it has typical transversal fluctuations while crossing the barrier region and whose length restricted to the barrier region is typical. The barrier condition  $R_\gamma$  asserts that the region in the barrier above  $\gamma$  is really bad in the sense that any path crossing the barrier from left to right above  $\gamma$  and is disjoint with  $\gamma$  is much smaller than typical length.

Also let  $F_\gamma^3$  be the weight of the best path from  $a_1$  to  $a_2$  that is more than a distance of  $(2M + S)(x')^{2/3}$  above the diagonal at  $x'$ , (i.e., passes above  $L_3$ ) and stays above  $\gamma[x, mx]$  between  $[x, mx]$ , and define

$$A_\gamma^3 = \{F_\gamma^3 < T_{\gamma, x, mx} - \sqrt{S}x^{1/3}\}.$$

Define

$$A_\gamma = A_\gamma^1 \cap A_\gamma^2 \cap A_\gamma^3.$$

Observe that the event  $A_\gamma$  is decreasing on the configuration of  $\mathcal{Z} \setminus \{\gamma\}$  conditioned on the remaining configuration. See Figure 2.5.

Recall that  $\Gamma'_0$  is the geodesic from  $b_1$  to  $b_2$ . Define  $A_{\Gamma'_0}$  similarly with  $\gamma$  replaced by  $\Gamma'_0$ . We shall show later that  $A_{\Gamma'_0}$  is a high probability event.

#### 2.4.4 Barrier condition

So far all the events that we have described are events that typically hold. Our final favourable event is one that ensures any path crossing the barrier disjointly with  $\Gamma'_0$  will be penalised a lot. This is not a typical event but one that only holds with constant probability (independent of  $z$ ). Fix an increasing path  $\gamma$  from  $b_1$  to  $b_2$  satisfying the geometric conditions (2.5), (2.6) and (2.8). We say the barrier condition  $R_\gamma$  holds if:

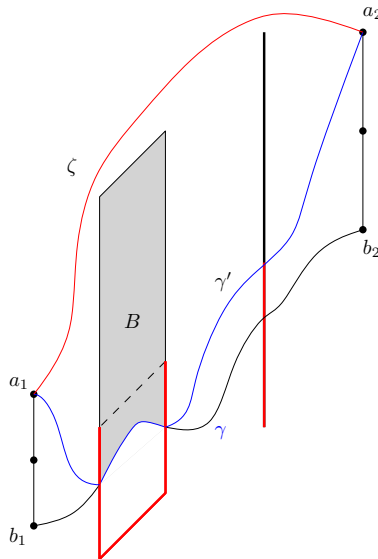


Figure 2.5: Path condition  $A_\gamma$ : it asserts that a path  $\zeta$  which passes above either wall of the barrier or the line segment  $L_3$  must be much smaller than the path  $\gamma'$  which is the longest path from  $a_1$  to  $a_2$  that agrees with  $\gamma$  along the barrier.

Any path from the left to the right wall of the barrier  $B$  that avoids  $\gamma$  and crosses the barrier is much shorter. That is, for all  $u \in L_1$  and  $u' \in L_2$ , such that  $(x, \gamma(x)) \preceq u$  and  $(mx, \gamma(mx)) \preceq u'$ ,

$$\tilde{T}_{u,u'}^\gamma \leq -S^4 x^{1/3}.$$

It follows from Lemma 8.3 of [16], that there exists a constant  $\beta > 0$  not depending on  $x$  and  $\gamma$ , (depending on  $M, S$ ), such that

$$\mathbb{P}(R_\gamma) \geq \beta > 0.$$

Observe that  $R_\gamma$  depends only on the configuration in  $\mathcal{Z} \setminus \{\gamma\}$  and is decreasing in  $\mathcal{Z} \setminus \{\gamma\}$ .

## 2.5 Forcing geodesics to merge using favorable events

In order to show that a collection of geodesics coalesce, we shall need the following lemmas about the events defined in the previous subsection. Recall that  $z \in \mathbb{N}$  is fixed, the set up is as given in Lemma 2.3.1 with  $r = 1$  and  $x = z^{3/2}$ . Now we consider the events described in the previous section. Then we have the following lemmas. The first lemma says that geodesics are *typical* with high probability.

**Lemma 2.5.1.** *Let  $\Gamma'_0$  be the geodesic from  $b_1$  to  $b_2$ . Then  $\Gamma'_0$  is **typical** with probability at least  $\frac{99}{100}$ .*

*Proof.* We first show that conditions (2.5), (2.6) and (2.7) each occur with probability at least  $\frac{999}{1000}$ . Enough to show one of them, the others are similar. Let  $\mathcal{S}$  be the line segment joining the two points  $b_1$  and  $b_2$ . Then at the point  $(x, x)$  on the diagonal,  $x \geq \mathcal{S}(x) \geq x - \frac{1}{4}x^{2/3}$  (where  $\mathcal{S}(x)$  is such that  $(x, \mathcal{S}(x)) \in \mathcal{S}$ ). Hence, by Corollary 2.2.4, (2.5) occurs with probability at least  $\frac{999}{1000}$  by choosing  $M$  large.

Next we show that conditions (2.3) and (2.4) hold with high probability. Let  $S_1$  be the line segment joining  $(x, x - Mx^{2/3})$  and  $(x, x + Mx^{2/3})$ , and  $S_2$  be the line segment joining  $(mx, mx - M(mx)^{2/3})$  and  $(mx, mx + M(mx)^{2/3})$ . Let  $A$  denote the event that

$$\sup_{u \in S_1, u' \in S_2} |\tilde{T}_{u,u'}| \leq \frac{H}{2} \sqrt{M} x^{1/3}.$$

That  $\mathbb{P}(A) \geq \frac{999}{1000}$  follows from Corollary 10.4 and Corollary 10.7 of [16] and hence condition (2.3) holds with high probability. Let  $A'$  be the event that  $|\Gamma'_0(mx) - \Gamma'_0(x)| \leq \frac{\sqrt{H}}{2} M^{1/6} x^{2/3}$ . Breaking  $S_1$  into subintervals of length  $\frac{\sqrt{H}}{2} M^{1/6} x^{2/3}$ , using Corollary 2.2.4 for the geodesics starting from each of the endpoints of the subintervals to  $b_2$ , and polymer ordering (Lemma 4.3.2) and union bound gives that  $\mathbb{P}(A') \geq \frac{999}{1000}$ . Observe that for any geodesic  $\gamma$  from  $u$  to  $u'$  with  $u \in S_1, u' \in S_2$ , such that  $|\ell(\gamma)| \leq \frac{H\sqrt{M}x^{1/3}}{2}$  and  $|\gamma(u') - \gamma(u)| \leq \frac{\sqrt{H}}{2} M^{1/6} x^{2/3}$ , one has  $|\widehat{\ell}(\gamma)| \leq H\sqrt{M}x^{1/3}$ . This, together with condition (2.5) gives that condition (2.4) occurs with probability at least  $\frac{997}{100}$ .

The only thing left to show is that condition (2.8) occurs with probability at least  $\frac{999}{1000}$ . Together with polymer ordering (Lemma 4.3.2) and conditions (2.5) and (2.6), it is enough to show that the geodesic joining  $(x, x - Mx^{2/3})$  and  $(mx, mx - M(mx)^{2/3})$ , and that joining  $(x, x + Mx^{2/3})$  and  $(mx, mx + M(mx)^{2/3})$  have *transversal fluctuation* at most  $\frac{M}{2}x^{2/3}$  with high probability. This follows from Theorem 11.1 of [16] by choosing  $M$  a large constant.  $\square$

The proof of the next lemma is similar to that of Theorem 3.

**Lemma 2.5.2.** *For any fixed increasing path  $\gamma$  from  $b_1$  to  $b_2$  that satisfies the geometric conditions (2.5), (2.6), (2.7) and (2.8),  $\mathbb{P}(A_\gamma | \gamma \text{ is typical}) \geq \frac{97}{100}$ .*

*Proof.* Observe that for an increasing path  $\gamma$  satisfying the geometric conditions in (2.5), (2.6), (2.7), (2.8), the condition that it is typical depends only on the configuration  $\Pi_{\{\gamma[x, mx]\}}$ . We show that  $\mathbb{P}(A_\gamma^1 | \gamma \text{ is typical}) \geq \frac{99}{100}$ . The bounds for  $A_\gamma^2$  and  $A_\gamma^3$  follow similarly and that for  $A_\gamma$  follows by taking a union bound. The proof of this is similar to what we did for the proof of Theorem 3.

See Figure 2.5. Let  $\zeta$  be the best path joining  $a_1$  and  $a_2$  that is more than a distance of  $(2M + S)x^{2/3}$  above the diagonal at  $x$  and is above  $\gamma[x, mx]$  in  $[x, mx]$ . Let  $\mathcal{W}$  be the straight line segment joining  $a_1$  and  $a_2$ . Choose  $\alpha = 2^{\frac{1}{6}}$ . For  $j \geq 0$ , let  $B'_j$  denote the event that  $\zeta(2^j x) - \mathcal{W}(2^j x) \geq (2M + S)((2\alpha)^j x)^{2/3}$  and  $\zeta(2^{j+1} x) - \mathcal{W}(2^{j+1} x) \leq (2M + S)((2\alpha)^{j+1} x)^{2/3}$ . It is enough to show that on each of these  $B'_j$ , the weight of the union of the maximal path joining  $a_1$  to  $(x, \gamma(x))$ ,  $\gamma[x, mx]$ , and the maximal path joining  $(mx, \gamma(mx))$

and  $(2^{j+1}x, \zeta(2^{j+1}x))$  is larger than the sum of the length of  $\zeta[z, 2^{j+1}x]$  and  $\sqrt{S}x^{1/3}$  with sufficiently high probability.

As before, define  $U_r$  as the line segment joining  $(2^jx, \mathcal{W}(2^jx) + (2M + S + r)((2\alpha)^jx)^{2/3})$  and  $(2^jx, \mathcal{W}(2^jx) + (2M + S + r + 1)((2\alpha)^jx)^{2/3})$  and  $V_r$  as the line segment joining  $(2^{j+1}x, \mathcal{W}(2^{j+1}x) + (2M + S - r)((2\alpha)^{j+1}x)^{2/3})$  and  $(2^{j+1}x, \mathcal{W}(2^{j+1}x) + (2M + S - r + 1)((2\alpha)^{j+1}x)^{2/3})$ . Note that  $r$  here is used as an index variable and, in particular is not related to the same symbol used in statement of Proposition 2.3.1. Also recall that  $S_1$  was the line segment joining  $(x, x - Mx^{2/3})$  and  $(x, x + Mx^{2/3})$  and  $S_2$  was the line segment joining  $(mx, mx - M(mx)^{2/3})$  and  $(mx, mx + M(mx)^{2/3})$ . Define

$$D_{x,r,r',j} = \sup_{u \in U_r, v \in V_{r'}, w_1 \in S_1, w_2 \in S_2} \left( T_{a_1, (2^jx, u)} + T_{(2^jx, u), (2^{j+1}x, v)} - T_{a_1, (x, w_1)} - T_{(mx, w_2), (2^{j+1}x, v)} \right)$$

and set

$$C_{j,r,r'} = \left\{ D_{x,r,r',j} \geq \ell(\gamma[x, mx]) - \sqrt{S}x^{1/3} \right\}.$$

Recall that  $\ell(\gamma)$  denotes the length of the path  $\gamma$ . Computing expectations, it is easy to see that for some constant  $c_1$  not depending on  $x, S, r, r', j$  (depending on  $M$ ), for all  $u \in U_r, v \in V_{r'}$ , (observe that  $\gamma(x) \in S_1, \gamma(mx) \in S_2$ ),

$$\begin{aligned} \mathbb{E}(T_{a_1, (2^jx, u)}) + \mathbb{E}(T_{(2^jx, u), (2^{j+1}x, v)}) &\leq \mathbb{E}(T_{a_1, (x, \gamma(x))}) + \mathbb{E}(T_{(x, \gamma(x)), (mx, \gamma(mx))}) \\ &\quad + \mathbb{E}(T_{(mx, \gamma(mx)), (2^{j+1}x, v)}) - c_1(S + r + r')\alpha^{\frac{2j}{3}}(2^jx)^{1/3}. \end{aligned}$$

Using the moderate deviation estimates for supremum and infimum of the lengths of a collection of geodesics given in Proposition 10.1 and Proposition 10.5 of [16] and the fact that  $\gamma$  is a typical path (hence condition (2.3) holds), this implies, choosing  $S$  large enough compared to  $M$ ,

$$\mathbb{P}(C_{j,r,r'}) \leq Ce^{-c(\frac{S}{2} + r + r')\alpha^{\frac{2j}{3}}}.$$

Summing over  $r, r', j$ , and choosing  $S$  large enough, gives the result.  $\square$

We point out that the argument in the proof of Lemma 2.5.2 is useful in other contexts also. We already know from Theorem 3 that the transversal fluctuation of a geodesic from  $\mathbf{0}$  to  $\mathbf{n}$  at  $r \ll n$  is  $O(r^{2/3})$ . The argument above shows the following stronger fact: any path having transversal fluctuation  $\gg r^{2/3}$  at scale  $r$  will typically be much shorter than the geodesic. See [11] for an application of such a result.

The next lemma states that  $A_{\Gamma'_0}$  is a high probability event.

**Lemma 2.5.3.** *Let  $\Gamma'_0$  be the geodesic from  $b_1$  to  $b_2$ . Then  $\mathbb{P}(A_{\Gamma'_0}) \geq \frac{9}{10}$ .*

*Proof.* Let  $U_1$  be the line segment joining  $(x, x - Mx^{2/3})$  and  $(x, x + Mx^{2/3})$ ,  $U_2$  be the line segment joining  $(mx, mx - M(mx)^{2/3})$  and  $(mx, mx + M(mx)^{2/3})$  and  $U_3$  be the line segment joining  $(x', x' - 2M(x')^{2/3})$  and  $(x', x' + 2M(x')^{2/3})$ .

Let  $D_1$  be the event that the geodesic from  $(mx, mx + M(mx)^{2/3})$  to  $a_2$  and the geodesic from  $(mx, mx - M(mx)^{2/3})$  to  $a_2$  are within  $2M(x')^{2/3}$  distance from the diagonal at  $x'$ . By Corollary 2.2.4 it follows that  $\mathbb{P}(D_1) \geq \frac{999}{1000}$ . This, together with polymer ordering, ensures that all geodesics from some point in  $U_2$  to  $a_2$  pass through  $U_3$  with probability at least  $\frac{999}{1000}$ .

Let  $D_2$  be the event such that the followings happen:

For all  $u_1 \in U_1$ ,

$$|\tilde{T}_{a_1, u_1}| \leq H\sqrt{M}x^{1/3}.$$

For all  $u_2 \in U_2$  and  $u_3 \in U_3$ ,

$$|\tilde{T}_{u_2, u_3}| \leq H\sqrt{M}x^{1/3}.$$

By Lemma 7.3 of [16], it follows that  $\mathbb{P}(D_2) \geq \frac{999}{1000}$ .

Observe that for any  $v_1, v_2 \in U_1$  and  $w_1, w_2 \in U_2, z \in U_3$

$$|\mathbb{E}(T_{a_1, v_1}) + \mathbb{E}(T_{v_1, w_1}) + \mathbb{E}(T_{w_1, z}) - (\mathbb{E}(T_{a_1, v_2}) + \mathbb{E}(T_{v_2, w_2}) + \mathbb{E}(T_{w_2, z}))| \leq cM^2x^{1/3}, \quad (2.9)$$

where  $c$  is some absolute positive constant.

Let  $\mathcal{Q}$  be the set of increasing paths from  $b_1$  to  $b_2$  that satisfy the geometric conditions (2.5), (2.6), (2.7), (2.8). Fix two paths  $\gamma_1, \gamma_2 \in \mathcal{Q}$ . Let  $\Gamma^{(1)}$  and  $\Gamma^{(2)}$  be the two best paths from  $a_1$  to  $a_2$  that coincide with  $\gamma_1[x, mx]$  and  $\gamma_2[x, mx]$  between  $x$  and  $mx$  respectively. Let  $v_1 = (x, \gamma_1(x)), v_2 = (x, \gamma_2(x)), w_1 = (mx, \gamma_1(mx)), w_2 = (mx, \gamma_2(mx)), z_1 = (x', \Gamma^{(1)}(x')), z_2 = (x', \Gamma^{(2)}(x'))$ . Since  $T_{w_2, a_2} \geq T_{w_2, z_1} + T_{z_1, a_2}$  and  $T_{w_1, a_2} \geq T_{w_1, z_2} + T_{z_2, a_2}$ , it is easy to see that, on  $D_1$ ,

$$|T_{\gamma_1, x, mx} - T_{\gamma_2, x, mx}| \leq \sup_{z \in U_3} |T_{a_1, v_1} + T_{v_1, w_1} + T_{w_1, z} - (T_{a_1, v_2} + T_{v_2, w_2} + T_{w_2, z})|.$$

If in addition, the configuration on  $\gamma_1[x, mx]$  and  $\gamma_2[x, mx]$  are such that  $\gamma_1, \gamma_2$  are *typical*, then on  $D_2$  one has, using (2.9),

$$\sup_{\gamma_1, \gamma_2 \text{ typical}} |T_{\gamma_1, x, mx} - T_{\gamma_2, x, mx}| \leq cM^2x^{1/3} + 6H\sqrt{M}x^{1/3}. \quad (2.10)$$

Next we use Lemma 2.5.2 to show that  $\bigcap_{\gamma \text{ typical}} \{F_\gamma^1 < T_{\gamma, x, mx} - \sqrt{S}x^{1/3}\}$  is a high probability event. Recall the region  $B_0$  defined in (2.8). It is easy to see using standard arguments that the geodesic from  $(x, (2M + S)x^{2/3})$  to  $a_2$  stays above the region  $B_0$  between  $x$  and  $mx$  with probability at least  $\frac{999}{1000}$ . On this event, and on  $D_1 \cap D_2$ , because of (2.10), it follows that (recall that  $S$  was chosen much larger compared to  $H, M$ ), if  $\{F_\gamma^1 < T_{\gamma, x, mx} - 2\sqrt{S}x^{1/3}\}$  holds for some  $\gamma$  *typical*, then,  $\bigcap_{\gamma \text{ typical}} \{F_\gamma^1 < T_{\gamma, x, mx} - \sqrt{S}x^{1/3}\}$  holds. Hence using Lemma 2.5.2,

$$\mathbb{P}\left(\bigcap_{\gamma \text{ typical}} \{F_\gamma^1 < T_{\gamma, x, mx} - \sqrt{S}x^{1/3}\}\right) \geq \frac{98}{100}. \quad (2.11)$$

Now we show that  $\mathbb{P}(A_{\Gamma'_0}^1) \geq \frac{97}{100}$ . The bounds for  $A_{\Gamma'_0}^2, A_{\Gamma'_0}^3$  follow similarly and that for  $A_{\Gamma'_0}$  follows by taking a union bound. It is easy to see that,

$$\mathbb{P}(A_{\Gamma'_0}^1) \geq \mathbb{P}\left(\bigcap_{\gamma \text{ typical}} \{F_\gamma^1 < T_{\gamma,x,mx} - \sqrt{S}x^{1/3}\} \cap \{\Gamma'_0 \text{ is typical}\}\right)$$

Since  $\mathbb{P}(\Gamma'_0 \text{ is typical}) \geq \frac{99}{100}$  by Lemma 2.5.1, it follows by using (2.11) and taking a union bound that

$$\mathbb{P}(A_{\Gamma'_0}^1) \geq \frac{97}{100},$$

completing the proof.  $\square$

Let  $\Gamma_0$  be the geodesic from  $a_1$  to  $a_2$ . The crux of the next lemma is that on the event that  $G, R_\gamma$  and  $A_\gamma$  occur and  $\gamma$  is *typical*,  $\Gamma_0$  merges with  $\gamma$ .

**Lemma 2.5.4.** *If  $\gamma$  is any fixed increasing path from  $b_1$  to  $b_2$ , then on the event  $G \cap R_\gamma \cap A_\gamma \cap \{\gamma \text{ is typical}\}$ ,  $\Gamma_0$ , the geodesic from  $a_1$  to  $a_2$ , meets  $\gamma$ .*

*Proof.* First observe that if  $\Gamma_0$  gets below  $\gamma$  at any point, it has to intersect  $\gamma$ . Also note that, on  $A_\gamma \cap \{\gamma \text{ is typical}\}$ , the maximal path  $\Gamma_0$  cannot hit the barrier above  $(2M + S)x^{2/3}$  distance from the diagonal at  $x$  or above  $(2M + S)(mx)^{2/3}$  distance from the diagonal at  $mx$  without hitting  $\gamma$ . Also  $\gamma$  is a typical path and hence hits the walls of  $B$ , if  $\Gamma_0(x)$  or  $\Gamma_0(mx)$  is below the walls of  $B$ , it must already intersect  $\gamma$ . Otherwise,  $\Gamma_0$  enters and exits through the left and right walls of  $B$ . We show that this cannot happen without hitting  $\gamma$ . See Figure 2.6.

Let the point on the left wall of  $B$  where  $\Gamma_0$  enters the barrier be  $u_1$  and the point on the right wall of  $B$  where  $\Gamma_0$  exits the barrier be  $u_2$ . Also the point where  $\Gamma_0$  intersects  $L_3$  be  $u_3$  ( $\Gamma_0$  must intersect  $L_3$  since on  $A_\gamma$  any path passing above  $L_3$  is worse than a path that merges with  $\gamma$ , and if it passes below  $L_3$ , it intersects with  $\gamma$ , as  $\gamma$  passes through  $L_3$ ). We compare the part of the path  $\Gamma_0$  till  $L_3$  with  $\Gamma_{a_1,(x,\gamma(x))} \cup \gamma[x, mx] \cup \Gamma_{(mx,\gamma(mx)),u_3}$  (by a minor abuse of notation we denote by  $\gamma[x, mx]$  the part of  $\gamma$  between  $(x, \gamma(x))$  and  $(mx, \gamma(mx))$ , it does not affect our calculations in any way). Hence enough to prove

$$\widehat{T}_{a_1,u_1} + \widehat{T}_{u_1,u_2}^\gamma + \widehat{T}_{u_2,u_3} \leq \widehat{T}_{a_1,(x,\gamma(x))} + \ell(\widehat{\gamma[x, mx]}) + \widehat{T}_{(mx,\gamma(mx)),u_3}. \quad (2.12)$$

This follows because on  $G \cap R_\gamma$ , for  $\gamma$  typical,

$$\widehat{T}_{a_1,u_1} \leq S^3 x^{1/3}, \widehat{T}_{u_1,u_2}^\gamma \leq -S^4 x^{1/3}, \widehat{T}_{u_2,u_3} \leq S^3 x^{1/3},$$

$$\widehat{T}_{a_1,(x,\gamma(x))} \geq -S^3 x^{1/3}, \ell(\widehat{\gamma[x, mx]}) \geq -H\sqrt{M}x^{1/3}, \widehat{T}_{(mx,\gamma(mx)),u_3} \geq -S^3 x^{1/3}.$$

Here we have used that since  $u_1 < (x, x + 2Sx^{2/3})$ , hence for some absolute constant  $c$ ,  $|\mathbb{E}T_{a_1,u_1} - 2d(a_1, u_1)| \leq cS^2 x^{1/3}$ . This, together with the fact that  $|\widetilde{T}_{a_1,u_1}| \leq H\sqrt{S}x^{1/3}$  because of the event  $G$ , one has that  $|\widehat{T}_{a_1,u_1}| \leq S^3 x^{1/3}$  by choosing  $S$  large. Similar arguments apply to  $\widehat{T}_{u_2,u_3}, \widehat{T}_{a_1,(x,\gamma(x))}$  and  $\widehat{T}_{(mx,\gamma(mx)),u_3}$ .

From here equation (2.12) follows by choosing  $S$  sufficiently large compared to  $M$  and  $H$ .  $\square$

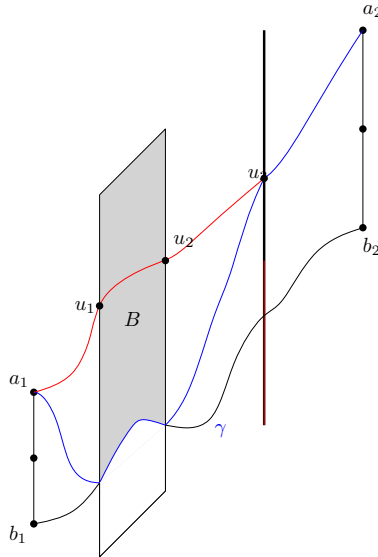


Figure 2.6: Merging of  $\Gamma_0$  with  $\gamma$  in the proof of Lemma 2.5.4. On  $A_\gamma$ , any path that passes above the barrier or the line  $L_3$  is uncompetitive, and if any path crosses the barrier and hits the line  $L_3$ , the red part of the path can be replaced by the blue path which on the good events and presence of the barrier has larger weight and also merges with  $\gamma$ .

### 2.5.1 Proof of Proposition 2.3.1

In this subsection we complete the proof of Proposition 2.3.1. As stated earlier, without loss of generality, we shall prove it for  $r = 1$ .

*Proof of Proposition 2.3.1.* Consider the events  $G, R_\gamma$  and the barrier  $B$  and the event  $A_{\Gamma'_0}$  as defined in the previous sections. The proof shall follow by conditioning on the lower path  $\Gamma'_0 = \gamma$ . We first define sets  $J_1, J_2, J_3$  of increasing paths  $\gamma$  from  $b_1$  to  $b_2$  together with the configuration  $\Pi_{\{\gamma\}}$  on  $\{\gamma\}$  such that it is very likely that  $\Gamma'_0 \in J_1 \cap J_2 \cap J_3$ .

Let  $J_1$  denote the set of all *typical* paths from  $b_1$  to  $b_2$ . Then Lemma 2.5.1 gives that  $\mathbb{P}(\Gamma'_0 \in J_1) \geq \frac{9}{10}$ .

Let  $J_2$  denote the set of all increasing paths  $\gamma$  from  $b_1$  to  $b_2$  and configurations  $\Pi_{\{\gamma\}}$  such that  $\mathbb{P}(G|\Gamma'_0 = \gamma) \geq \frac{9}{10}$ . Since  $\mathbb{P}(G) \geq \frac{99}{100}$ , one gets by Markov's inequality,

$$\mathbb{P}(\Gamma'_0 \in J_2) \geq \frac{9}{10}.$$

Let  $J_3$  denote the set of all increasing paths  $\gamma$  from  $b_1$  to  $b_2$  together with the configurations  $\Pi_{\{\gamma\}}$  such that  $\mathbb{P}(A_{\Gamma'_0}|\Gamma'_0 = \gamma) \geq \frac{2}{3}$ . Since by Lemma 2.5.3  $\mathbb{P}(A_{\Gamma'_0}) \geq \frac{9}{10}$ , by Markov's inequality,

$$\mathbb{P}(\Gamma'_0 \in J_3) \geq \frac{7}{10}.$$

Then by union bound,  $\mathbb{P}(\Gamma'_0 \in J_1 \cap J_2 \cap J_3) \geq \frac{1}{2}$ .

Fix a particular  $(\gamma, \Pi_\gamma) \in J_1 \cap J_2 \cap J_3$ .

Since

$$\mathbb{P}(A_\gamma | \Gamma'_0 = \gamma) = \mathbb{P}(A_{\Gamma'_0} | \Gamma'_0 = \gamma) \geq \frac{2}{3},$$

and  $\mathbb{P}(G | \Gamma'_0 = \gamma) \geq \frac{9}{10}$ , hence

$$\mathbb{P}(G \cap A_\gamma | \Gamma'_0 = \gamma) \geq \frac{1}{2}. \quad (2.13)$$

Also as  $R_\gamma$  is a decreasing event on the configuration of  $\mathcal{Z} \setminus \{\gamma\}$ , and is independent of the configuration in  $(\mathcal{Z} \setminus \{\gamma\})^c$ , and  $A_\gamma$  and  $\Gamma'_0 = \gamma$  are also decreasing in the configuration of  $\mathcal{Z} \setminus \{\gamma\}$ , by FKG inequality it follows that,

$$\mathbb{P}(R_\gamma \cap A_\gamma \cap \{\Gamma'_0 = \gamma\} | (\mathcal{Z} \setminus \{\gamma\})^c) \geq \mathbb{P}(R_\gamma) \mathbb{P}(A_\gamma \cap \{\Gamma'_0 = \gamma\} | (\mathcal{Z} \setminus \{\gamma\})^c).$$

As  $G$  is  $(\mathcal{Z})^c$  measurable,

$$\mathbb{P}(R_\gamma \cap A_\gamma \cap \{\Gamma'_0 = \gamma\} | G) \geq \mathbb{P}(R_\gamma) \mathbb{P}(A_\gamma \cap \{\Gamma'_0 = \gamma\} | G).$$

Hence,

$$\mathbb{P}(R_\gamma | G \cap A_\gamma \cap \{\Gamma'_0 = \gamma\}) \geq \mathbb{P}(R_\gamma) \geq \beta > 0.$$

This, together with (2.13), gives

$$\mathbb{P}(R_\gamma \cap G \cap A_\gamma | \Gamma'_0 = \gamma) \geq \frac{\beta}{2} =: \beta' > 0.$$

Now, it follows from Lemma 2.5.4 that on the event  $R_\gamma \cap G \cap A_\gamma \cap \{\gamma \text{ is typical}\}$ ,  $\Gamma_0$  meets  $\gamma$ . Hence for any fixed  $(\gamma, \Pi_\gamma) \in J_1 \cap J_2 \cap J_3$ ,

$$\mathbb{P}(\Gamma_0 \text{ meets } \gamma | \Gamma'_0 = \gamma) \geq \mathbb{P}(R_\gamma \cap G \cap A_\gamma | \Gamma'_0 = \gamma) \geq \beta' > 0,$$

where  $\beta' = \frac{\beta}{2}$  is an absolute positive constant not depending on the typical path  $\gamma$ . Hence, by integrating over all  $(\gamma, \Pi_\gamma) \in J_1 \cap J_2 \cap J_3$ ,

$$\mathbb{P}(\Gamma_0 \text{ meets } \Gamma'_0) \geq \beta' \mathbb{P}(\Gamma'_0 \in J_1 \cap J_2 \cap J_3) \geq \beta'/2 =: \alpha > 0.$$

Also observe that  $\alpha$  does not depend on  $z$ , this completes the proof.  $\square$

## 2.6 Optimal tail estimate for coalescence of semi-infinite geodesics

In this section we prove Theorem 2 for the semi-infinite geodesics. Before proceeding with the proof let us briefly recall the setting of the theorem. We had points  $v_3 = (k^{2/3}, -k^{2/3})$  and  $v_4 = (-k^{2/3}, k^{2/3})$ , and we denoted by  $\Gamma_{v_3}$  and  $\Gamma_{v_4}$  the semi-infinite geodesics started respectively from  $v_3$  and  $v_4$  in the direction  $(1, 1)$ , and  $v^* = (v_1^*, v_2^*)$  denoted the point of

coalescence of  $v_3$  and  $v_4$ . The distance to coalescence  $\text{dist}(v_3, v_4)$  was defined to be equal to  $v_1^* + v_2^*$ . As mentioned before to prove that  $\mathbb{P}(\text{dist}(v_3, v_4) > Rk) \asymp CR^{-2/3}$  we shall appeal to the translation invariance of the underlying passage time field.

Observe that it suffices to prove Theorem 2 for all sufficiently large  $k$ . Fix now  $k$  sufficiently large. Let us now identify the line  $\mathbb{L}$  with  $\mathbb{Z}$  via the identification  $i \mapsto u_i = \mathbf{0} + i(-1, 1)$ . We call  $u_i \in \mathbb{L}$  a  $k$ -**boundary point** if  $d(u_i, u_{i+1}) > k$ ; see Figure 2.7. Define a sequence  $\{X_i^{(k)}\}_{i \in \mathbb{Z}}$  by setting  $X_i = 1$  if  $u_i$  is a  $k$ -boundary point and 0 otherwise. Observe that translation invariance implies that this is a stationary sequence. The main step of the proof will be the following proposition.

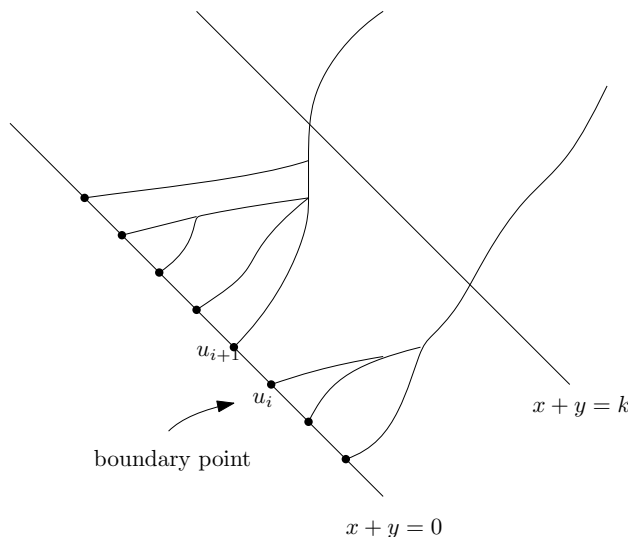


Figure 2.7: Definition of a  $k$ -boundary point. If semi-infinite geodesic starting from two neighbouring points on  $x + y = 0$  coalesce above the line  $x + y = k$  then one of them is a  $k$ -boundary point.

**Proposition 2.6.1.** *There exists  $C_1, C_2 > 0$  such that for each  $i \in \mathbb{Z}$  and for each  $k$ , we have*

$$\frac{C_2}{k^{2/3}} \leq \mathbb{P}(X_i^{(k)} = 1) \leq \frac{C_1}{k^{2/3}}.$$

We postpone the proof of Proposition 2.6.1 for now and prove the upper bound of Theorem 2 first, the lower bound of Theorem 2 will be proved at the end of this section.

*Proof of Theorem 2: upper bound.* Fix  $R > 1$ . Clearly, on  $\{\text{dist}(v_3, v_4) > Rk\}$ , there must exist  $i \in \llbracket -k^{2/3}, k^{2/3} \rrbracket$  such that  $u_i$  is an  $Rk$ -boundary point. It follows that

$$\mathbb{P}(\text{dist}(v_3, v_4) > Rk) = \mathbb{P}\left(\sum_{i=-k^{2/3}}^{k^{2/3}} X_i^{(Rk)} > 0\right) \leq \mathbb{E}\left[\sum_{i=-k^{2/3}}^{k^{2/3}} X_i^{(Rk)}\right] \leq 2k^{2/3} \frac{C_1}{(Rk)^{2/3}}$$

where the final inequality follows from Proposition 2.6.1. This completes the proof of the theorem.  $\square$

### 2.6.1 Proof of Proposition 2.6.1

We prove Proposition 2.6.1 in this subsection. This proof is essentially independent of the rest of the chapter, except we need to use a variant of Theorem 3, that also holds for the semi-infinite geodesics. We first record this statement.

**Proposition 2.6.2.** *For  $v$  on the line  $\mathbb{L} : x + y = 0$ , let  $f(v) = (f(v)_1, f(v)_2)$  denote the point the semi-infinite geodesic  $\Gamma_v$  started from  $v$  in the direction  $(1, 1)$  intersects the line  $x + y = k$ . For  $h > 0$ , let  $A_h = A_{h,k}$  denote the event that there exists a point  $v$  on  $\mathbb{L}$  between  $v_3$  and  $v_4$  such that  $|f(v)_1 - f(v)_2| \geq hk^{2/3}$ . Then there exists  $h_0 > 0, c > 0$  such that for all  $h > h_0$  and all sufficiently large  $k$  we have  $\mathbb{P}(A_h) \leq e^{-ch^2}$ .*

We shall not provide a detailed proof of this proposition, but let us indicate how one can obtain this result by arguing as in the proof of Theorem 3. Without loss of generality take  $v = \mathbf{0}$ . We need to upper bound  $\mathbb{P}(f(0)_1 - f(0)_2 \geq hk^{2/3})$ . Let  $\mathcal{L}$  denote the straight line  $y = (1 - k^{-1/3})x$ . By definition of  $\Gamma_{\mathbf{0}}$ , all but finitely many points on it lies to the left of  $\mathcal{L}$ , while  $f(v)$  lies to its right (for  $h$  sufficiently large) in the event  $f(0)_1 - f(0)_2 \geq hk^{2/3}$ . We can now use the strategy of Proof of Theorem 3 to show that it is unlikely for the path to cross the line  $\mathcal{L}$ , by checking at dyadically increasing scales. Here we need to use the observation that the proof of Theorem 3 works for paths in the direction other than  $(1, 1)$  as explained in Corollary 2.2.4. We omit the details.

The main input of this section however is the following from [13], a result obtained by the first and third author jointly with Christopher Hoffman. A slightly stronger version of the result is used in [13] to show the non-existence of bigeodesics in exponential LPP. We need some preparation to describe the result. Let  $\mathbb{L}_0$  denote the line segment joining  $(k^{2/3}, -k^{2/3})$  and  $(-k^{2/3}, k^{2/3})$ . For  $h \in \mathbb{N}$ , let  $\mathbb{L}'_h$  denote the line segment  $(k, k) + 2h\mathbb{L}_0$ . For points  $u, v \in \mathbb{L}_0$  we say  $u \leq v$  if  $v = u + i(-1, 1)$  for some  $i > 0$ , and similarly on  $\mathbb{L}'_h$ . For  $\ell \in \mathbb{N}$ , let  $\mathcal{C}_{\ell,h}$  denote the event that there exists points  $u_1 \leq u_2 \leq \dots \leq u_\ell$  on  $\mathbb{L}_0$ , and  $w_1 \leq w_2 \leq \dots \leq w_\ell$  on  $\mathbb{L}'_h$  such that the geodesics  $\Gamma_{u_i, w_i}$  are disjoint. We quote the following theorem from [13].

**Proposition 2.6.3** ([13, Corollary 2.7]). *There exists  $k_0, \ell_0 > 0$ , such that for all  $k > k_0, k^{0.01} > \ell > \ell_0$  and all  $h \leq \ell^{1/16}$  we have  $\mathbb{P}(\mathcal{C}_{\ell,h}) \leq e^{-c\ell^{1/4}}$  for some positive constant  $c$ .*

Observe that this proposition immediately implies that in the same set-up  $\mathbb{P}(\mathcal{C}_{\ell,h}) \leq e^{-c\ell^{1/4}} + e^{-ck^{0.001}}$  for all  $\ell > \ell_0$  and  $h \leq k^{0.0005}$ . The main idea of the proof of Proposition 2.6.3 is the following: if there are too many disjoint geodesics across a rectangle of size  $k \times k^{2/3}$ , there must be one which is constrained to be in a thin rectangle. The proof can be completed with using the by now well-known fact that paths restricted to thin rectangles

are unlikely to be competitive [12, 11], together with an application of the BK inequality. See [13] for the detailed argument, we shall omit this proof.

We shall now complete the proof of Proposition 2.6.1 using Proposition 2.6.3. First we need the following lemma.

**Lemma 2.6.4.** *There exists  $k_0, l_0$  such that for all  $k > k_0$ , and  $\ell > l_0$ , we have*

$$\mathbb{P} \left( \sum_{i=-k^{2/3}}^{k^{2/3}} X_i^{(k)} \geq \ell \right) \leq e^{-c\ell^{1/8}} + e^{-ck^{0.001}}.$$

*Proof.* Fix  $\ell$  sufficiently large and set  $h = \min\{\ell^{1/16}, k^{0.0005}\}$ . Now observe that on the event

$$\left\{ \sum_{i=-k^{2/3}}^{k^{2/3}} X_i^{(k)} \geq \ell \right\}$$

we must either have that the event  $A_h$  from Proposition 2.6.2 holds, or the event  $C_{\ell, h}$  from Proposition 2.6.3 holds. Observe that, by Proposition 2.6.2 the probability of the first event is bounded by  $e^{-c\ell^{1/8}} + e^{-ck^{0.001}}$ , and by Proposition 2.6.3 the probability of the second event is also bounded by  $e^{-c\ell^{1/8}} + e^{-ck^{0.001}}$ . The proof is completed by taking a union bound.  $\square$

We are now ready to prove Proposition 2.6.1.

*Proof of Proposition 2.6.1.* For the upper bound simply note that by translation invariance  $\mathbb{P}(X_i^{(k)} = 1)$  is independent of  $i$ . The proof is now completed by noting that  $\sum_{i=-k^{2/3}}^{k^{2/3}} X_i^{(k)} \leq 2k^{2/3}$  and Lemma 2.6.4 implies that  $\mathbb{E} \sum_{i=-k^{2/3}}^{k^{2/3}} X_i^{(k)} \leq C$  for some large constant  $C$  uniformly in all large  $k$ .

For the lower bound, observe the following. By the same argument as in the first part, it suffices to prove that there exists  $M$  sufficiently large such that with  $w_1 = (Mk^{2/3}, -Mk^{2/3})$  and  $w_2 = (-Mk^{2/3}, Mk^{2/3})$ , we have for all  $k > 0$

$$\mathbb{P}(\text{dist}(w_1, w_2) > k) \geq \frac{1}{3}, \tag{2.14}$$

which ensures that with probability bounded away from 0, there is at least one boundary point out of the  $2Mk^{2/3}$  between  $w_1$  and  $w_2$ . The existence of such an  $M$  follows from Proposition 2.6.2 by noticing that for  $M$  sufficiently large the semi-infinite geodesic from  $w_1$  in the direction  $(1, 1)$  hits the line  $x + y = k$  below the point  $(\frac{k}{2}, \frac{k}{2})$  with probability at least  $2/3$ , whereas the semi-infinite geodesic from  $w_1$  in the direction  $(1, 1)$  hits the line  $x + y = k$  above the point  $(\frac{k}{2}, \frac{k}{2})$  with probability at least  $2/3$ .  $\square$

**Remark 2.6.5.** *Observe that using the same argument as in the proof of the lower bound in the above proposition, it follows (with notations as in Theorem 2) that one has*

$\limsup_{k \rightarrow \infty} \mathbb{P}(d(v_3, v_4) \leq Rk) \leq e^{-c/R}$  for some constant  $c > 0$ , for  $R$  small. It recovers the lower bound on distance to coalescence obtained by [70], with a better quantitative estimate.

### 2.6.2 Lower bound in Theorem 2

It remains to prove the lower bound in Theorem 2. As mentioned before, this has already been proved by Pimentel [70, 69] using a duality formula but we provide a short alternative proof using our techniques.

*Proof of Theorem 2: lower bound.* Starting as in the proof of the upper bound, with same notations, we are required to lower bound  $\mathbb{P}\left(\sum_{i=-k^{2/3}}^{k^{2/3}} X_i^{(Rk)} > 0\right)$ . Let  $M$  be as in (2.14). By translation invariance it follows that

$$MR^{2/3}\mathbb{P}\left(\sum_{i=-k^{2/3}}^{k^{2/3}} X_i^{(Rk)} > 0\right) \geq \mathbb{P}(d(w'_1, w'_2) > Rk) > \frac{1}{3}$$

where  $w'_1 = (M(Rk)^{2/3}, -M(Rk)^{2/3})$  and  $w'_2 = (-M(Rk)^{2/3}, M(Rk)^{2/3})$ . The lower bound  $\mathbb{P}\left(\sum_{i=-k^{2/3}}^{k^{2/3}} X_i^{(Rk)} > 0\right) \geq \frac{1}{3M}R^{-2/3}$  is now immediate.  $\square$

An alternative proof of the above can also be obtained by using the second moment method.

## Chapter 3

# Invariant measures for TASEP with a slow bond

### 3.1 Introduction

A topic of contemporary interest in equilibrium and non-equilibrium statistical mechanics has been to understand how the macroscopic behaviour of a system changes if some local, microscopic defect of arbitrarily small strength is introduced. As described in the introduction chapter, a specific such model was introduced in the context of TASEP by Janowsky and Lebowitz [49, 48] who considered TASEP with a slow bond at the origin where a particle jumping from the origin jumps at some rate  $r < 1$ . It is easy to see that for  $r$  small this model, started from the step initial condition, has smaller asymptotic current; Janowsky and Lebowitz asked whether the same happens for an arbitrarily small strength of defect, i.e., values of  $r$  arbitrarily close to one. Over two decades, there were disagreements among physicists about what the answer should be with different groups predicting different answers, and this problem came to be known as the “slow bond problem”. As is typical for exactly solvable models, much of the detailed analysis of TASEP is non-robust, i.e., the analysis breaks down under minor modifications to the model. In particular, the study of the model with a slow bond is no longer facilitated by the exact formulae, and even the stationary distributions are non-explicit, making the study of the model much harder. This question was settled very recently in [16] where it was shown using a geometric approach together with the exactly solvable ingredients from the TASEP (without the slow bond) that the local current is restricted for any arbitrary small blockage parameter. That is, for any value of  $r < 1$ , it was established that the limiting current is strictly less than  $\frac{1}{4}$ , which is the corresponding value for usual TASEP.

Here, we develop further the geometric techniques introduced in [16] to study the stationary measures for TASEP with a slow bond. Following the works [49, 48] and some unpublished works by Bramson, the conjectural picture that emerged (modulo the affirmative answer to the slow bond problem which has now been established) is described in

Liggett's 1999 book [62, p. 307]. The distribution of usual TASEP started with the step initial condition converges to the invariant product Bernoulli measure with density  $\frac{1}{2}$ . The slowdown due to the slow bond implies that there is a long range effect near the origin where the region to the right of origin is sparser and there is a traffic jam to the left of the slow bond with particle density higher than a half. However, it was conjectured that as one moves far away from the origin, the distribution becomes close to a product measure albeit with a different density  $\rho < \frac{1}{2}$  to the right of the origin and  $\rho' > \frac{1}{2}$  to the left of the origin. Our contribution here is to establish this picture rigorously and thus answering Liggett's question described above; see Theorem 4, Theorem 3.1.4 and Corollary 3.1.3 below.

As in [16] our argument is also based on the connection between TASEP and directed last passage percolation (DLPP) on  $\mathbb{Z}^2$  with Exponential passage times, which will be recalled below in Subsection 3.1.3. We study the geometry of the geodesics (maximal paths) in the last passage percolation models corresponding to both TASEP and TASEP with a slow bond. We use the result from [16] to establish quantitative estimates about pinning of certain point-to-point geodesics in the slow bond model. This establishes certain correlation decay and mixing properties for the average occupation measures which implies the existence of a limiting invariant measure. The heart of the argument showing that this invariant measure is close to product measure far away from the origin is another analysis of the geometry of the geodesics in the Exponential directed last passage percolation model, together with a coupling between TASEP with a slow bond and a stationary TASEP. We use crucially a result about coalescence of geodesic in Exponential LPP, obtained in the companion paper [14] and discussed in the previous chapter.

We now move towards formal definitions and precise statement of our results.

### 3.1.1 Formal definitions and main result

TASEP is defined as a continuous time Markov process with the state space  $\{0, 1\}^{\mathbb{Z}}$ . Let  $\{\eta_t\}_{t \geq 0}$  denote the particle configuration at time  $t$ , i.e., for  $t \geq 0$  and  $x \in \mathbb{Z}$ , let  $\eta_t(x) = 1$  or  $0$  depending on whether there is a particle at time  $t$  on site  $x$  or not. Let  $\delta_x$  denote the particle configuration with a single particle at site  $x$ . For a particle configuration  $\eta = (\eta(x) : x \in \mathbb{Z})$ , denote by  $\eta^{x, x+1}$  the particle configuration  $\eta - \delta_x + \delta_{x+1}$ , i.e., where a particle has jumped from the site  $x$  to the site  $x + 1$ . TASEP dynamics defines a Markov process with the generator given by

$$\mathcal{L}f(\eta) = \sum_{x \in \mathbb{Z}} \eta(x)(1 - \eta(x + 1))(f(\eta^{x, x+1}) - f(\eta)).$$

Let  $\{P_t\}_{t \geq 0}$  denote the corresponding semigroup. A probability measure  $\nu$  on  $\{0, 1\}^{\mathbb{Z}}$  is called an *invariant measure* or *stationary measure* for TASEP if  $\mathbb{E}_\nu(f) = \mathbb{E}_\nu(P_t f)$  for all  $t \geq 0$  and for all bounded continuous functions  $f$  on the state space. Denoting the distribution of  $\eta_t$  when  $\eta_0$  is distributed according to  $\nu$  by  $\nu P_t$  the above says that for an invariant measure  $\nu$  one has  $\nu P_t = \nu$ . The invariant measures for TASEP can be characterised, see Section 3.1.2 below.

**TASEP with a slow bond at the origin**

We shall consider the TASEP dynamics in presence of the following microscopic defect: for a fixed  $r < 1$  consider the exclusion dynamics where every particle jumping out of the origin jumps at a slower rate  $r < 1$ . Formally this is a Markov process with the generator

$$\mathcal{L}^{(r)} f(\eta) = r\eta(0)(1 - \eta(1))(f(\eta^{0,1}) - f(\eta)) + \sum_{x \neq 0} \eta(x)(1 - \eta(x+1))(f(\eta^{x,x+1}) - f(\eta)).$$

Denote the corresponding semigroup by  $\{P_t^{(r)}\}_{t \geq 0}$ . For the rest of this chapter we shall treat  $r$  as a fixed quantity arbitrarily close to one. It turns out even a microscopic defect of arbitrarily small strength has a macroscopic effect to the system, see Section 3.1.2 for more details of this model. In particular, there is long range correlation near the origin and the invariant measures for this model, unlike usual TASEP, does not admit any explicit description. As mentioned above, following [49, 48] and unpublished works by Bramson, Liggett [62, p.307] described the conjectural behaviour for the invariant measures for TASEP with a slow bond at the origin. Our main result in this chapter is to confirm this conjecture. We now introduce notations and definitions necessary for stating our results.

The initial condition  $\eta_0 = \mathbf{1}_{(-\infty, 0]}$  (i.e., one particle each at all sites  $x \leq 0$  and no particles at sites  $x > 0$ ) is particularly important to study of TASEP and is called *step initial condition*. For  $0 < \rho < 1$ , let  $\nu_\rho$  denote the product Bernoulli measure on the space of particle configurations  $\{0, 1\}^{\mathbb{Z}}$  with density  $\rho$ ; i.e.,  $\nu_\rho(\eta(x) = 1) = \rho$  independently for all  $x \in \mathbb{Z}$ . For measures  $\nu$  on  $\{0, 1\}^{\mathbb{Z}}$ ,  $E \subseteq \mathbb{Z}$  and  $A \subseteq \{0, 1\}^E$  we shall set without loss of generality  $\nu(A) = \nu(A \times \{0, 1\}^{\mathbb{Z} \setminus E})$ . Also for  $x \in \mathbb{Z}$ , let  $A^x$  denote the subset of  $\{0, 1\}^{x+E}$  obtained by a co-ordinate wise translation by  $x$ . We shall need the following definition.

**Definition 3.1.1.** *A probability measure  $\nu$  on the configuration space  $\{0, 1\}^{\mathbb{Z}}$  is said to be asymptotically equivalent to  $\nu_\rho$  at  $\infty$  (resp. at  $-\infty$ ) if the following holds: for every finite  $E$  and a subset  $A$  of  $\{0, 1\}^E$  we have*

$$\nu(A^k) \rightarrow \nu_\rho(A)$$

as  $k \rightarrow \infty$  (resp.  $k \rightarrow -\infty$ ).

We are now ready to state our main result which solves part (a) of the sequence of questions about the invariant measures for TASEP with a slow bond in [62, p.307]. The other parts of the conjecture also follow from this work and have been outlined in this chapter, see Theorem 3.1.4 and Corollary 3.1.3 for the statements of parts (b) and (c) of the question respectively.

**Theorem 4.** *For every  $r < 1$ , there exists a measure  $\nu_*$  on  $\{0, 1\}^{\mathbb{Z}}$  and  $\rho < \frac{1}{2}$  both depending on  $r$  such that  $\nu_*$  is an invariant measure for the Markov process with generator  $\mathcal{L}^{(r)}$  and  $\nu_*$  is asymptotically equivalent to  $\nu_\rho$  (resp.  $\nu_{1-\rho}$ ) at  $\infty$  (resp.  $-\infty$ ). Furthermore, started from the step initial condition  $\eta_0$ , the process converges weakly to  $\nu_*$ , i.e.,  $\delta_{\eta_0} P_t^{(r)} \Rightarrow \nu_*$  as  $t \rightarrow \infty$ .*

For the rest of this chapter we shall keep  $r < 1$  fixed. The density  $\rho$  in the statement of the theorem is not an explicit function of  $r$ ; however, we can evaluate  $\rho$  explicitly in terms of the asymptotic current in the process with a slow bond; see Remark 3.1.2 below.

### 3.1.2 Background

As mentioned before, studying the invariant measures is crucial for understanding many particle systems such as TASEP especially at infinite volume. It is a well known fact that the set of all invariant measures is a compact convex subset of the set of all probability measures in the topology of weak convergence and hence it suffices to study the extremal invariant measures. In [59], Liggett identified the set of all invariant measures for TASEP, and showed that apart from a few trivial measures, the extremal invariant measures for TASEP are the Bernoulli product measures  $\{\nu_\rho : \rho \in (0, 1)\}$ . Notice that these are all translation invariant stationary measures for TASEP, and the measure  $\nu_\rho$  corresponds to the stationary current  $J = \rho(1 - \rho)$  where  $J$  denotes the rate at which particles cross a bond. Hence the maximum possible value of the stationary current is  $\frac{1}{4}$ . In particular, started with the step initial condition  $\eta_0 = \mathbf{1}_{(-\infty, 0]}$  TASEP converges in distribution to the stationary measure  $\nu_{\frac{1}{2}}$  (cf. Theorem 3.29 in part III of [62]). Rost [72] established the hydrodynamic density profile and asymptotic current for TASEP started with step initial condition. Let  $L_n$  denote time it takes for the particle started at  $-n$  particles to cross the origin. Rost [72] established

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}L_n}{n} = 4. \quad (3.1)$$

Observe that the reciprocal of the number on the right hand side above denotes the asymptotic rate at which particles cross the origin, which in this case matches the stationary current of the limiting measure  $\nu_{\frac{1}{2}}$ .

#### Further results

Let  $L_n^{(r)}$  denote the time it takes for the particle started at  $-n$  to cross the origin when the origin has been imputed with a slow bond; is  $\lim_{n \rightarrow \infty} \frac{L_n^{(r)}}{n}$  strictly larger than 4 for all values of  $r < 1$ , or there is a critical value  $r_c < 1$  below which this is observed? This question came to be known as the *slow bond problem* in the statistical physics literature. Recently, this question was settled in [16] using a geometric approach together with estimates coming from the exactly solvable nature of TASEP. They showed the following.

[[16], Theorem 2] For any  $r < 1$ , there exists  $\varepsilon = \varepsilon(r) > 0$  such that

$$\lim_{n \rightarrow \infty} \frac{L_n^{(r)}}{n} = 4 + \varepsilon. \quad (3.2)$$

This result does not readily yield any information about the invariant measures for TASEP with a slow bond. Because of the long range effect of the slow bond, it is expected that any invariant measure for this process must have complicated correlation structure near

the origin. Theorem 4 establishes that starting from the step initial condition the process converges to such an invariant measure  $\nu_*$  which, as conjectured, is asymptotically equivalent to product measures  $\nu_\rho$  at  $\infty$  and  $\nu_{1-\rho}$  at  $-\infty$  for some  $\rho = \rho(r)$  strictly smaller than  $\frac{1}{2}$ . Although we do not have an explicit formula for  $\rho$  in terms of  $r$ , it can be related to  $\varepsilon$  in a simple manner.

**Remark 3.1.2.** *Let  $\varepsilon = \varepsilon(r)$  be as in Theorem 3.2. Then  $\rho$  in Theorem 4 is given by the unique real number less than  $\frac{1}{2}$  satisfying*

$$\rho(1 - \rho) = \frac{1}{4 + \varepsilon}.$$

It is easy to see why the above should be true. Since the current at any site under the stationary measure  $\nu_*$  is the same; the current at the origin (equal to  $\frac{1}{4+\varepsilon}$  by Theorem 3.2) should be same as the current at some far away site both to the left and right. Recall that the stationary current under  $\nu_\rho$  is  $\rho(1 - \rho)$  and thus Theorem 4 suggests that  $\rho(1 - \rho) = \frac{1}{4+\varepsilon}$ . We shall see from our proof of Theorem 4 that this is indeed the case. Similar considerations give the following easy corollary of Theorem 3.2 answering part (c) of Liggett's question in [62, p.307].

**Corollary 3.1.3.** *Let  $\rho = \rho(r)$  be as above. For  $p > \rho$  there does not exist an invariant measure of the Markov process with generator  $\mathcal{L}^{(r)}$  that is equivalent to  $\nu_p$  at  $\infty$  (resp.  $\nu_{1-p}$  at  $-\infty$ ).*

Arguments similar to our proof of Theorem 4 can also be used to settle part (b) of Liggett's question in [62, p.307] where Liggett asks if for any  $p < \rho(r)$  (resp. for any  $p > 1 - \rho(r)$ ) there exists an invariant measure of  $\mathcal{L}^{(r)}$  that is equivalent to  $\nu_p$  at  $\pm\infty$  (resp.  $\nu_{1-p}$  at  $\pm\infty$ ). We show that such an invariant measure is obtained in the limit if the process is started from product  $\text{Ber}(p)$  stationary initial condition.

**Theorem 3.1.4.** *Let  $p < \rho(r)$  or  $p > 1 - \rho(r)$  be fixed. Then there exists an invariant measure  $\nu_p^*$  of the Markov process with generator  $\mathcal{L}^{(r)}$  such that  $\nu_p P_t^{(r)} \Rightarrow \nu_p^*$  and  $\nu_p^*$  is asymptotically equivalent to  $\nu_p$  at  $\pm\infty$ .*

Theorem 4, Theorem 3.1.4 and Corollary 3.1.3 answer all parts of Liggett's question. However we do not identify all invariant measures for TASEP with a slow bond. A natural question asked by Liggett [57] is whether or not the invariant measures given in Theorem 4 and Theorem 3.1.4 are the only nontrivial extremal invariant measures of the process. Another question of interest again pointed out by Liggett [57] is whether similar results hold for other translation invariant exclusion systems with positive drift, i.e., ASEP. Our techniques do not apply as there is no known simple polymer representation for ASEP, and even the question whether or not a slow bond of arbitrarily small strength affects the asymptotic current is open.

### 3.1.3 TASEP and last passage percolation

One can map TASEP on  $\mathbb{Z}$  into a directed last passage percolation model on  $\mathbb{Z}^2$  with i.i.d. Exponential weights, and much of the recent advances in understanding of TASEP has come from looking at the corresponding last passage percolation picture. For each vertex  $v \in \mathbb{Z}^2$  associate i.i.d. weight  $\xi_v$  distributed as  $\text{Exp}(1)$ . Define  $u \preceq v$  if  $u$  is co-ordinate wise smaller than  $v$  in  $\mathbb{Z}^2$ . For  $u \preceq v$  define the last passage time from  $u$  to  $v$ , denoted  $T_{u,v}$  by

$$T_{u,v} := \max_{\pi} \sum_{v' \in \pi} \xi_{v'}$$

where the maximum is taken over all up/right oriented paths from  $u$  to  $v$ . In particular, let  $T_n$  denote the passage time from  $(0,0)$  to  $(n,n)$ . One can couple the TASEP with the Exponential directed last passage percolation (DLPP) as follows. For  $v = (x,y) \in \mathbb{Z}^2$  let  $\xi_v$  be the waiting time for the  $(\min(x,y) + 1)$ -th jump at the site  $(x-y)$  (once there is a particle at  $x-y$  and the site  $x-y+1$  is empty). It is easy to see (see e.g. [73]) that under this coupling  $T_n$  is equal to the time it takes for  $n+1$  particles to jump out of the origin when TASEP starts with step initial condition, i.e., the time taken by the particle at  $-n$  to jump to site 1. Often, when there is no scope for confusion, we shall denote by  $T_{a,b}$  the last passage time from  $(0,0)$  to  $(a,b)$ . In this notation,  $T_{n+k,n}$  equals the time taken by the particle at  $-n$  to jump to site  $k+1$ .

Even when TASEP starts from some arbitrary initial condition, the above coupling can be used to describe jump times as last passage times, but the more general formula involves last passage time from a point to a set rather than last passage time between two points; see Section 3.5 for more details.

Observe that in the coupling described above the passage times of the vertices on the line  $x-y=i$  describes the weighting times for jumps at site  $i$ . Using this it is easy to translate the slow bond model to the last passage percolation framework. Indeed, we only need to modify the passage times on the diagonal line  $x=y$  by i.i.d.  $\text{Exp}(r)$  variables independently of the passage times of the other sites. We shall use  $T_n^{(r)}$  to denote the last passage times in this model. It is clear from the above discussion that  $T_n^{(r)}$  has the same distribution as  $L_n^{(r)}$ , where  $L_n^{(r)}$  is as in Theorem 3.2. For the rest of the chapter we shall be working mostly with the last passage picture, implicitly we shall always assume the aforementioned coupling with TASEP to move between TASEP and DLPP even though we might not explicitly mention it every time. We shall see later how statistics from the last passage percolation model can be interpreted in terms of the occupation measures of sites in TASEP, and provide information about invariant measures.

Studying the geometry of the geodesics will enable us to prove facts about the occupation measure of certain sites in the corresponding particle systems, and thus conclude Theorem 4. A crucial step in the arguments in this chapter would use the coalescence of geodesics starting from vertices that are close to each other. This result follows from Corollary 2.3.2 of the previous chapter with  $k=1$  and  $R=n$ . We record it here.

**Theorem 3.1.5** ([14], Corollary 3.2). *Let  $L, L', m_0 > 0$  be fixed constants. Let  $\Gamma$  be the geodesic from  $(0, L)$  to  $(n, m_0 n + L'n^{2/3})$  and  $\Gamma'$  be the geodesic from  $(0, -L)$  to  $(n, m_0 n - L'n^{2/3})$  in the Exponential LPP model. Let  $E$  be the event that  $\Gamma$  and  $\Gamma'$  meet. Then,*

$$\mathbb{P}(E) \geq 1 - Cn^{-c},$$

for some absolute positive constants  $C, c$  (depending only on  $L, L', m_0$  but not on  $n$ ).

### 3.1.4 Outline of the proofs

We provide a sketch of the main arguments in this subsection. The proof of Theorem 4 has two major components. The first part is devoted to establishing the existence of the limiting distribution of TASEP with a slow bond at the origin started from the step initial conditions. In the second part, we show that the limiting distribution is asymptotically equivalent to a product Bernoulli measures with different densities far away from the origin to the left and to the right. Throughout the rest of the chapter we shall work with a fixed  $r < 1$  and  $\varepsilon = \varepsilon(r)$  given by Theorem 3.2.

#### A basic observation connecting last passage times and occupation measures

The basic observation underlying all the arguments regarding invariant measures in this chapter can be simply stated in the following manner. The amount of time the particle starting at  $-n$  spends at 0 is  $T_{n,n} - T_{n-1,n}$  (recall our convention that  $T_{a,b}$  is the last passage time from  $(0, 0)$  to  $(a, b)$  when there is no scope for confusion). In the same vein, the total time the site 0 is occupied between  $T_n$  and  $T_{n+k}$  is determined by the pairwise differences  $T_{i,i} - T_{i-1,i}$  for  $i \in \{n+1, n+2, \dots, n+k\}$ . More generally for states  $I = [-b, b] \cap \mathbb{Z}$ , where  $b \in \mathbb{N}$  and  $A \subseteq \{0, 1\}^I$ , using the coupling between TASEP and last passage percolation, it is not difficult to see that the occupation measure at sites  $I$  between  $T_n$  to  $T_{n+k}$ , i.e.,  $A \mapsto \int_{T_n}^{T_{n+k}} \mathbf{1}(\eta_t(I) \in A) dt$  is a function of the pairwise differences of the last passage times  $T_{0,v}$  for a set of vertices  $v$  around the diagonal between  $(n, n)$  and  $(n+k, n+k)$ . All our arguments about the limiting distribution in this chapter are motivated by the above basic observation and the idea that the average occupation measure should be close to the limiting distribution as  $n$  and  $k$  becomes large.

#### Convergence to a limiting measure

The idea of showing that such a limit to the average occupation measure exists is as follows. Using Theorem 3.2 we can show that in the reinforced last passage percolation model, the geodesics (from  $\mathbf{0}$  to  $\mathbf{n}$ , say where  $\mathbf{n} := (n, n)$  and  $\mathbf{0}$  denotes the origin) are pinned to the diagonal and the typical fluctuation of the paths away from the diagonal is  $O(1)$ . This in turn implies that for some fixed  $k$  and  $n \gg k$ , all the geodesics from  $\mathbf{0}$  to points near the diagonal between  $\mathbf{n}$  and  $\mathbf{n} + \mathbf{k}$  merge together at some point on the diagonal near  $\mathbf{n}$  with overwhelming probability. This indicates that the pairwise difference of the passage

times is determined by the individual passage times near the region on the diagonal between  $\mathbf{n}$  and  $\mathbf{n} + \mathbf{k}$ . Using this localisation into disjoint boxes (with independent passage time configuration), the average occupation measure over a large time can be approximated by an average of i.i.d. random measures. A law of large numbers ensure that the average occupation measures converge to a measure. To show that the process, started from a step initial condition converges weakly to this distribution requires a comparison between average occupation measure during a random interval with the distribution at a fixed time and this is done via a smoothing argument using local limit theorems.

### Occupation measures far away from origin

The second part of the the argument, i.e., to show that the limiting measure is asymptotically equivalent to a product Bernoulli measure far away from the origin, is more involved. Let us restrict to the measure far away to the right of origin. Consider a fixed length interval  $[k, k + L]$  for  $k \gg 1$ . From the above discussion it follows that the average occupation measure at some random time interval after a large time will be determined by the pairwise differences  $T_{\mathbf{0},v_i} - T_{\mathbf{0},v_j}$  for the vertices  $v_i, v_j$  in some square of bounded size around the vertex  $(n + k, n)$  for  $n \gg k$ . Let us first try to understand the geodesics  $\Gamma_{\mathbf{0},v_i}$  from  $\mathbf{0}$  to  $v_i$ , and in particular the geodesic from  $(0, 0)$  to  $(n + k, n)$ .

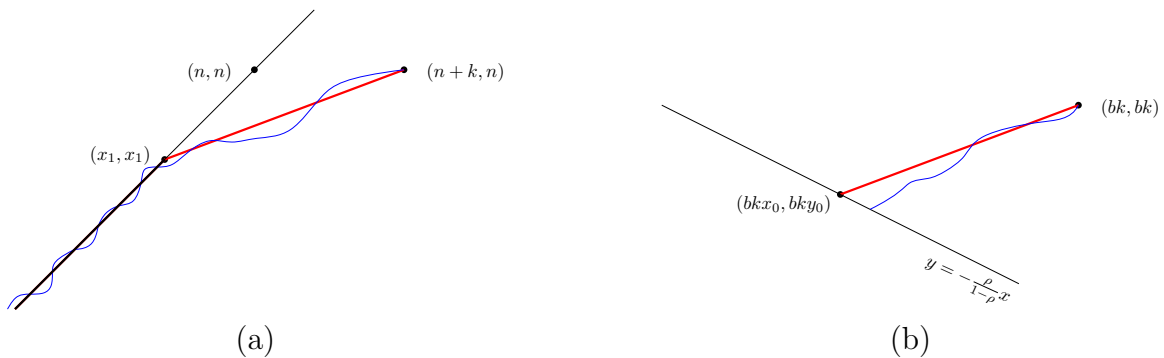


Figure 3.1: (a) First order behaviour of  $\Gamma_{n+k,n}$ , this remains pinned to the diagonal until point  $(x_1, x_1)$  then follows a path that in the the first order is a straight line to  $(n + k, n)$ . (b) Point to line geodesic to the line  $y = -\frac{\rho}{1-\rho}x$ . This corresponds to TASEP with a product  $\text{Ber}(\rho)$  initial condition. The geodesic in first order looks like a straight line. Our proof will compare the second part of the geodesic  $\Gamma_{n+k,n}$  to the point to line geodesic in (b). We choose  $\rho$  such that the slopes match.

It is easy to see that the geodesic should be pinned to the diagonal until about  $O(k)$  distance from  $(n, n)$  and then should approximately look like a geodesic in the unconstrained model to  $(n + k, n)$ . The approximate location until which this pinning occurs can be computed by a first order analysis using Theorem 2.1.2 and Theorem 3.2 (Theorem 2.1.2 implies that  $\mathbb{E}T_{x,y} \approx (\sqrt{x} + \sqrt{y})^2$  in the first order). These estimates yield that the last

hitting point of the diagonal should be close to the point  $(x_1, x_1)$  where  $x_1$  maximises the function  $(4 + \varepsilon)x + (\sqrt{n+k-x} + \sqrt{n-x})^2$ . An easy optimization gives

$$n - x_1 = k \frac{(\sqrt{4 + \varepsilon} - \sqrt{\varepsilon})^2}{4\sqrt{\varepsilon(4 + \varepsilon)}}. \quad (3.3)$$

Hence, the slope of the line joining  $(n+k, n)$  to  $(x_1, x_1)$  is

$$\left( \frac{\sqrt{4 + \varepsilon} - \sqrt{\varepsilon}}{\sqrt{4 + \varepsilon} + \sqrt{\varepsilon}} \right)^2, \quad (3.4)$$

and so the unpinned part of the geodesic  $\Gamma_{0, (n+k, n)}$  is approximately a geodesic in an environment of i.i.d. Exponentials along a direction with the slope given by the above expression. We shall show the following. *For  $k \gg 1$ , with probability close to 1 the geodesics from all the vertices on a square of bounded size to 0, coalesce before reaching the diagonal near  $(x_1, x_1)$ .* This follows from Theorem 3.1.5. This in turn will imply that for large  $k$ , on a set with probability close to 1, that the pairwise differences between those passage time will be locally determined by the i.i.d. exponential random environment in some box of size  $\ll k$  around the point  $(n+k, n)$ .

The final observation above will allow us to couple TASEP with a slow bond together with a stationary TASEP with product  $\text{Ber}(\rho)$  (for some appropriately chosen  $\rho$ ) stationary distribution, so that with probability close to 1 (under the coupling), the average occupation measure on the interval  $[k, k+L]$  (here  $L$  is fixed) during some late and large interval of time for the slow bond TASEP, will be equal to the average occupation measure of the interval  $[0, L]$  during an interval of same time. Hence the occupation measures will be close in total variation distance. Due to the stationarity of the latter process the latter occupation measure is close to product  $\text{Ber}(\rho)$ , and we shall be done by taking appropriate limits.

### A coupling with a stationary TASEP

Towards constructing the coupling described above we use the correspondence between a stationary TASEP and Exponential last passage percolation. It is well known that jump times in stationary TASEP corresponds to last passage times in point-to-set last passage percolation in i.i.d. Exponential environment just as the jump times in TASEP with step initial condition corresponds to point-to-point passage times. See Section 3.5.1 for a precise definition; roughly the following is true. There exists a random curve  $S$  (a function of the realisation of the stationary initial condition), such that the jump times of the stationary TASEP correspond to the last passage time from  $S$  to  $v$  for vertices  $v \in \mathbb{Z}^2$  (naturally the last passage time from  $S$  to  $v$  means the maximum passage time of all paths that start somewhere in  $S$  and end at  $v$ ). It is standard, that for a product  $\text{Ber}(\rho)$  initial condition the curve  $S$  is well approximated by the line  $\mathbb{L}$  with the equation

$$y = -\frac{\rho}{1 - \rho}x.$$

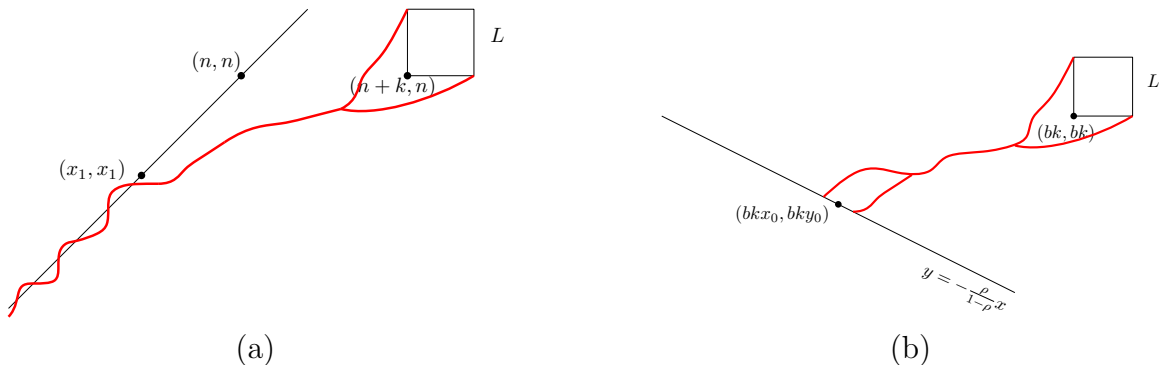


Figure 3.2: We shall construct a coupling between the two systems: (a) TASEP with a slow bond, and (b) Stationary TASEP with product  $\text{Ber}(\rho)$  initial condition. Roughly the coupling will assign the same individual vertex weights to both the systems upto a translation that takes the marked square in (a) to the marked square in (b). Using coalescence of geodesics, we shall show that on a large probability event for  $n \gg k \gg 1$ , the pairwise difference of passage times to the points in the marked square, will be determined locally in both the systems, and hence will be identical under the coupling.

At this point we perform another first order calculation to determine the approximate slope for geodesics from  $\mathbb{L}$  to  $(n, n)$ . Such a geodesic should hit the line  $\mathbb{L}$  close to the point  $(nx_0, ny_0)$  for  $(x_0, y_0)$  which maximizes  $(\sqrt{n - nx} + \sqrt{n - ny})^2$  for all  $(x, y) \in \mathcal{L}$ . An easy calculation gives that the approximate slope of the geodesic is  $\left(\frac{\rho}{1-\rho}\right)^2$ . To construct a successful coupling, one needs to match this slope to the one in (3.4). Solving the equation one gets

$$\rho = \frac{\sqrt{4 + \varepsilon} - \sqrt{\varepsilon}}{2\sqrt{4 + \varepsilon}}. \tag{3.5}$$

Observe that  $\rho < \frac{1}{2}$  and  $\rho(1 - \rho) = \frac{1}{4 + \varepsilon}$  as expected. Now there is a natural way to couple the two processes. Recall the point  $(x_1, x_1)$ . Choose  $b > 0$  and a translation of  $\mathbb{Z}^2$  that takes  $(bk, bk)$  to  $(n + k, n)$  and  $(bkx_0, bky_0)$  to  $(x_1, x_1)$ . Coupling the environment below the diagonal in the LPP corresponding to the slow bond model with the environment in the LPP of stationary TASEP (naturally under the above translation which takes  $\mathbb{L}$  to a parallel line passing through  $(x_1, x_1)$ ) and using the coalescence result Theorem 3.1.5 would give the required result under this coupling. (This is morally correct, even though not technically entirely precise. The formal argument is slightly more involved because it has to take care of the effect of the reinforced diagonals; see Section 3.5 for details).

**Remark 3.1.6.** *One can redo the same argument far away to the left of the origin. For an interval  $[-k - L, -k]$  with  $k \gg 1$  one finds that the appropriate density  $\rho'$  for the stationary*

TASEP to be coupled with the slow bond TASEP is given by

$$\rho' = \frac{\sqrt{4 + \varepsilon} + \sqrt{\varepsilon}}{2\sqrt{4 + \varepsilon}}. \tag{3.6}$$

and  $\rho'$  is indeed equal to  $1 - \rho$  as it should be.

### 3.1.5 Notations

For easy reference purpose, let us collect here a number of notations, some of which have already been introduced, that we shall use throughout the remainder of this chapter. Define the partial order  $\preceq$  on  $\mathbb{Z}^2$  by  $u = (x, y) \preceq u' = (x', y')$  if  $x \leq x'$ , and  $y \leq y'$ . For  $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^2$  with  $\mathbf{a} \preceq \mathbf{b}$ , let  $\Gamma_{\mathbf{a}, \mathbf{b}}^{(r)}$  denote the geodesic from  $\mathbf{a}$  to  $\mathbf{b}$  in the reinforced model when the passage times on the diagonal have been changed to i.i.d.  $\text{Exp}(r)$  variables. Also when  $\mathbf{a} = 0$ ,  $\Gamma_{\mathbf{0}, \mathbf{b}}^{(r)}$  is simply denoted as  $\Gamma_{\mathbf{b}}^{(r)}$ , and for  $\mathbf{b} = (b, b)$ ,  $\Gamma_{\mathbf{b}}^{(r)}$  is denoted simply as  $\Gamma_b^{(r)}$ . We shall drop the superscript  $r$  when there is no scope for confusion. For the usual exponential DLPP, i.e. when  $r = 1$ , we shall abuse the notation and denote the corresponding geodesics by  $\Gamma_{\mathbf{a}, \mathbf{b}}^0, \Gamma_{\mathbf{b}}^0, \Gamma_b^0$ . We shall also denote by  $T_{\mathbf{a}, \mathbf{b}}$  (resp.  $T_{a, b}^0$ ) the weight of the geodesic  $\Gamma_{\mathbf{a}, \mathbf{b}}$  (resp.  $\Gamma_{\mathbf{a}, \mathbf{b}}^0$ ).<sup>1</sup>

For  $u \preceq u'$  in  $\mathbb{Z}^2$ , let  $\text{Box}(u, u')$  denote the rectangle with bottom left corner  $u$  and top right corner  $u'$ . For an increasing path  $\gamma$  and  $\ell \in \mathbb{Z}$ ,  $\gamma(\ell) \in \mathbb{Z}$  will denote the maximum number such that  $(\ell, \gamma(\ell)) \in \gamma$  and  $\gamma^{-1}(\ell) \in \mathbb{Z}$  be the maximum number such that  $(\gamma^{-1}(\ell), \ell) \in \gamma$ .

As we shall be working on  $\mathbb{Z}^2$ , often we use the notation  $[[\cdot, \cdot]]$  for discrete intervals, i.e.,  $[[a, b]]$  shall denote  $[a, b] \cap \mathbb{Z}$ . In the various theorems and lemmas, the values of the constants  $C, C', c, c'$  appearing in the bounds change from one line to the next, and will be chosen small or large locally.

### 3.1.6 Organization of the chapter

The remainder of this chapter is organized as follows. Section 3.2 develops the geometric properties of the geodesics in the slow bond model, in particular the diffusive fluctuations of

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<sup>1</sup>In certain settings we shall work with the following modified passage times without explicitly mentioning so. For an increasing path  $\gamma$  from  $v_1$  to  $v_2$  let us denote the passage time of  $\gamma$  by

$$\ell(\gamma) = \sum_{v \in \gamma \setminus \{v_2\}} \xi_v;$$

Observe that this is a little different from the usual definition of passage time as we exclude the final vertex while adding weights. This is done for convenience as our definition allows  $\ell(\gamma) = \ell(\gamma_1) + \ell(\gamma_2)$  where  $\gamma$  is the concatenation of  $\gamma_1$  and  $\gamma_2$ . As the difference between the two definitions is minor while considering last passage times between far away points, all our results will be valid for both our and the usual definition of LPP.

the geodesics and localisation and coalescence of geodesics near the diagonal are shown in this section for the reinforced model. Section 3.3 is devoted to constructing a candidate for the invariant measure in the slow bond TASEP by passing to the limit of average occupation measures. That this measure is the limiting measure of the slow bond TASEP started from step initial condition is shown in Section 3.4. Finally, Section 3.5 deals with the coupling between the slow bond TASEP and the stationary TASEP that ultimately leads to the proof of Theorem 4. We finish off the section by providing a sketch of the argument for Theorem 3.1.4. The proofs of a few technical lemmas used in Sections 3.4 and 3.5 have been relegated to Section 3.6. Additionally, a Central Limit Theorem for the passage times in slow bond TASEP is provided in Section 3.7 as this has not been not directly used in the chapter.

## 3.2 Geodesics in slow bond model

Consider the Exponential last passage percolation model corresponding to TASEP with a slow bond at the origin that rings at rate  $r < 1$ . We shall work with a fixed  $r < 1$  throughout the section and  $\varepsilon$  will be as in Theorem 3.2. We shall refer to this as the slow bond model when there is no scope for confusion. It follows easily by comparing (3.1) and Theorem 3.2 that in this model the geodesic is *pinned* to the diagonal, i.e., the expected number of times the geodesic  $\Gamma_n$  between  $\mathbf{0} = (0, 0)$  to  $\mathbf{n} = (n, n)$  (as there is no scope of confusion we shall suppress the dependence of  $\Gamma$  on  $r$ ) hits the reinforced diagonal line is linear in  $n$ . In this section we establish stronger geometric properties of those geodesics. Indeed we shall show that the typical distance between two consecutive points on  $\Gamma_n$  that are on the diagonal is  $O(1)$  and also the transversal distance of  $\Gamma_n$  from the diagonal at a typical point is also  $O(1)$ . We begin with the following easy lemma.

**Lemma 3.2.1.** *There exists absolute positive constants  $m_0, c$  (depending only on  $r$ ) such that for any  $m \in \mathbb{N}$ ,  $m \geq m_0$ , the probability that  $\Gamma_m$  does not touch the diagonal between  $\mathbf{0}$  and  $\mathbf{m}$  is at most  $e^{-cm^{2/3}}$ .*

The proof follows easily by comparing the lengths of the geodesics in the reinforced and unreinforced environments.

*Proof.* Consider the coupling between the slow bond model and the (unreinforced) DLPP where the passage times at all vertices not on the diagonal are same, and those on the diagonal are replaced by i.i.d.  $\text{Exp}(r)$  variables independent of all other passage times. Then it is easy to see that if  $\Gamma_m$  avoids the diagonal between  $(0, 0)$  and  $(m, m)$ , then it is the maximal path in the unreinforced environment between  $(0, 0)$  and  $(m, m)$  that never touches the diagonal in between, and hence its length is at most the length of the geodesic in the unreinforced environment. Hence,

$$\mathbb{P}(\Gamma_m \text{ avoids diagonal}) \leq \mathbb{P}(T_m \leq T_m^0) \leq \mathbb{P}\left(T_m^0 > \left(4 + \frac{\varepsilon}{2}\right)m\right) + \mathbb{P}\left(T_m < \left(4 + \frac{\varepsilon}{2}\right)m\right).$$

Now by moderate deviation estimate in Theorem 2.1.2, for  $m \geq m_0$ ,

$$\mathbb{P}\left(T_m^0 > \left(4 + \frac{\varepsilon}{2}\right)m\right) \leq e^{-c_2 m^{2/3}},$$

where  $c_2$  is a constant depending only on  $\varepsilon$ . In order to bound the probability that the geodesic in the reinforced environment is not too short, first choose  $M$  large enough so that  $\mathbb{E}\left(\frac{T_M}{M}\right) > 4 + \frac{3\varepsilon}{4}$ . Then because of super-additivity of the path lengths,  $\frac{T_{nM}}{nM} \geq \frac{X_1 + \dots + X_n}{n} =: \bar{X}$  where  $X_i := \frac{T_{(i-1)M, iM}}{M}$  are i.i.d. random variables, each having the same distribution as that of  $X_1 = \frac{T_M}{M}$ . Let  $[m/M] = n$ , and  $m$  is large enough so that  $\frac{4+\varepsilon/2}{n} < \varepsilon/8$ , then

$$\mathbb{P}\left(T_m < \left(4 + \frac{\varepsilon}{2}\right)m\right) \leq \mathbb{P}\left(\frac{T_{nM}}{nM} < \left(4 + \frac{\varepsilon}{2}\right)\frac{n+1}{n}\right) < \mathbb{P}\left(|\bar{X} - \mathbb{E}X_1| > \frac{\varepsilon}{8}\right) \leq e^{-c'n} \leq e^{-cm}.$$

where  $c, c'$  are constants depending only  $\varepsilon$  and  $r$ . The last inequality follows as it is easy too see that for a fixed  $M$ ,  $T_M \preceq_{ST} \text{Gamma}(M^2, r)$  which has exponential tails where  $\preceq_{ST}$  denotes stochastic domination.  $\square$

We remark that the exponent here is not optimal. One can prove an upper bound of  $e^{-cm}$  by using large deviation estimates from [50] instead of Theorem 2.1.2, but this is sufficient for our purposes.

The following proposition controls the transversal fluctuation of the geodesics  $\Gamma_n$ . Recall that for  $\ell \in \mathbb{Z}$ ,  $\Gamma(\ell) \in \mathbb{Z}$  is the maximum number such that  $(\ell, \Gamma(\ell)) \in \Gamma$  and  $\Gamma^{-1}(\ell) \in \mathbb{Z}$  be the maximum number such that  $(\Gamma^{-1}(\ell), \ell) \in \Gamma$ .

**Proposition 3.2.2.** *For  $h \in \llbracket 0, n \rrbracket$ , we have for all  $n \geq m > m_0$ , for some absolute positive constants  $m_0, c$ ,*

$$\begin{aligned} \mathbb{P}(|\Gamma_n(h) - h| \geq m) &\leq e^{-cm^{1/2}}, \text{ and} \\ \mathbb{P}(|\Gamma_n^{-1}(h) - h| > m) &\leq e^{-cm^{1/2}}. \end{aligned}$$

*Proof.* For the purpose of this proof we drop the subscript  $n$  from  $\Gamma_n$ . First note that if  $|\Gamma(h) - h| > m$ , then  $h \geq m$  or  $n - h \geq m$ . If  $B$  is the event that the geodesic  $\Gamma$  hits the diagonal at  $(i, i)$  and returns to the diagonal again at  $(j, j)$  with  $|j - i| \geq s$ , then applying previous lemma 3.2.1,

$$\mathbb{P}(B) \leq \sum_{j=s}^{\infty} e^{-cj^{2/3}} \leq e^{-c's^{7/12}}.$$

Hence if  $\Gamma(h) > h + m$ , by summing up over all positions where  $\Gamma$  touches the diagonal for the last time before  $(h, h)$ , one has,

$$\mathbb{P}[\Gamma(h) - h > m] \leq \sum_{i=0}^h e^{-c'(i+m)^{7/12}} \leq \sum_{i=m}^{\infty} e^{-c'i^{7/12}} \leq e^{-cm^{1/2}}.$$

Similar arguments work for the events  $\{\Gamma(h) < h - m\}$  and  $\{|\Gamma^{-1}(h) - h| > m\}$ .  $\square$

Notice that it is not hard to establish using similar arguments that for pairs of points not far away from the diagonal, the geodesic between them also has  $O(1)$  transversal fluctuation from the diagonal at a typical point. We state this below without a proof.

**Corollary 3.2.3.** *Let  $h \in \llbracket 0, n \rrbracket$  and  $t = o(h)$  and  $t' = o(n - h)$  and  $\Gamma = \Gamma_{(0,t),(n,n+t')}$ . Then there exist absolute positive constants  $n_0, h_0, m_0, c$  such that for all  $n \geq n_0, h \geq h_0, m \geq m_0$ ,*

$$\mathbb{P}[|\Gamma(h) - h| \geq m] \leq e^{-cm^{1/2}}, \text{ and}$$

$$\mathbb{P}(|\Gamma^{-1}(h) - h| > m) \leq e^{-cm^{1/2}}.$$

Our next result will establish something stronger. We shall show that typically geodesic between every pair of points, one of which is close to  $\mathbf{0}$  and the other close to  $\mathbf{m}$ , meet the diagonal *simultaneously*.

**Theorem 3.2.4.** *Fix  $0 < \alpha < 1$ . Let  $L_1$  be the line segment joining  $(0, -m^\alpha)$  to  $(0, m^\alpha)$ . Similarly  $L_2$  be the line segment joining  $(m, m - m^\alpha)$  to  $(m, m + m^\alpha)$ . Let  $\mathcal{E}$  denote the event that there exists  $u \in \llbracket 0, m \rrbracket$  such that  $(u, u) \in \Gamma_{a,b}$  for all  $a \in L_1 \cap \mathbb{Z}^2, b \in L_2 \cap \mathbb{Z}^2$ . Then there exist some absolute positive constants  $m_0, c$  such that for all  $m \geq m_0, \mathbb{P}(\mathcal{E}) \geq 1 - e^{-cm^\ell}$  where  $\ell = \min\{\frac{1-\alpha}{2}, \frac{\alpha}{2}\}$ .*

We emphasize again that in this Theorem 3.2.4 as well as in the preceding lemmas, we have been very liberal about the exponents, and have not always attempted to find the best possible exponents in the bounds, as long as they suffice for our purpose.

We shall need a few lemmas to prove Theorem 3.2.4. The following lemma is basic and was stated in [16], we restate it here without proof.

**Lemma 3.2.5** ([16], Lemma 11.2, Polymer Ordering:). *Consider points  $a = (a_1, a_2), a' = (a_1, a_3), b = (b_1, b_2), b' = (b_1, b_3)$  such that  $a_1 < b_1$  and  $a_2 \leq a_3 \leq b_2 \leq b_3$ . Then we have  $\Gamma_{a,b}(x) \leq \Gamma_{a',b'}(x)$  for all  $x \in [a_1, b_1]$ .*

The next lemma shows that two geodesics between pairs of points not far from the diagonal have a positive probability to pass through the midpoint of the diagonal. Define  $\alpha' = \frac{\alpha+1}{2}$ . Clearly  $\alpha < \alpha' < 1$ .

**Lemma 3.2.6.** *Let  $a_1 = m^\alpha, \Gamma_1 = \Gamma_{(0,-m^\alpha),(a_1,a_1-m^\alpha)}, \Gamma_2 = \Gamma_{(0,m^\alpha),(a_1,a_1+m^\alpha)}$  and  $v := (\frac{a_1}{2}, \frac{a_1}{2})$ . Let  $F_1$  be the event that  $v \in \Gamma_1 \cap \Gamma_2$ . Then there exists some absolute positive constant  $\delta$  such that  $\mathbb{P}(F_1) \geq \delta$ .*

The idea is as follows. There is a positive probability that the exponential random variable  $\xi_v$  at  $v$  is large, and all other random variables that lie in a large but constant sized box around  $v$  are small. As Corollary 3.2.3 says that  $\Gamma_1$  and  $\Gamma_2$  are very likely to be in close proximity to  $v$ , they have a positive probability to pass through  $v$ . Formally, we do the following.

*Proof of Lemma 3.2.6.* Define

$$\mathcal{B}_1 = \text{Box}\left(\left(\frac{a_1}{2}, \frac{a_1}{2} - C\right), \left(\frac{a_1}{2} + C, \frac{a_1}{2}\right)\right);$$

and

$$\mathcal{B}_2 = \text{Box}\left(\left(\frac{a_1}{2} - C, \frac{a_1}{2}\right), \left(\frac{a_1}{2}, \frac{a_1}{2} + C\right)\right)$$

as squares of side length  $C$  with a common vertex  $v$ . Here  $C$  is an absolute constant to be chosen appropriately later. See Figure 3.3 (a). Define the following events

$$D_1 = \left\{ \sum_{x \in \mathcal{B}_1, x \neq v} \xi_x < 2C^2, \sum_{x \in \mathcal{B}_2, x \neq v} \xi_x < 2C^2 \right\};$$

$$D_2 = \{ |\Gamma_j(v) - v| \leq C, |\Gamma_j^{-1}(v) - v| \leq C \text{ for } j = 1, 2 \}.$$

Note  $D_1$  is the intersection of two independent high probability events as sum of  $C^2 - 1$  many i.i.d. exponential random variables is less than  $2C^2$  with high probability for large enough  $C$ . Also from Corollary 3.2.3, it follows that  $D_2$  is the intersection of two events with high probability, hence can be made to occur with arbitrarily high probability by choosing  $C$  large (note that  $m^\alpha = o(a_1)$ ). Hence choose  $C$  large enough so that  $\mathbb{P}(D_1) \geq \frac{3}{4}$  and  $\mathbb{P}(D_2) \geq \frac{3}{4}$ , so that  $\mathbb{P}(D_1 \cap D_2) \geq \frac{1}{2}$ .

Notice that both the events  $\{\xi_v > 2C^2\}$  and  $D_2$  are increasing in the value at  $v$  given the configuration on  $\mathbb{R}^2 \setminus \{v\}$ . Hence by the FKG inequality and the fact that  $\{\xi_v > 2C^2\}$  and  $D_1$  are independent, it follows that,

$$\mathbb{P}(\xi_v > 2C^2 | D_1 \cap D_2) \geq \mathbb{P}(\xi_v > 2C^2 | D_1) = \mathbb{P}(\xi_v > 2C^2) = e^{-2rC^2}.$$

Hence

$$\mathbb{P}(\{\xi_v > 2C^2\} \cap D_1 \cap D_2) \geq \frac{e^{-2rC^2}}{2} =: \delta.$$

We claim that on  $\{\xi_v > 2C^2\} \cap D_1 \cap D_2$ , both  $\Gamma_1$  and  $\Gamma_2$  pass through  $v$ . To see this, define  $z_1$  to be the point where  $\Gamma_1$  enters one of the boxes  $\mathcal{B}_1$  or  $\mathcal{B}_2$  and  $w_1$  denote the point where it leaves the box, similarly define  $z_2$  and  $w_2$  as the box entry and exit of  $\Gamma_2$ . Since  $D_2$  holds, we can join  $z_1$  and  $w_1$  to  $v$  by line segments and get an alternate increasing path that equals  $\Gamma_1$  everywhere else, and inside the box it goes from  $z_1$  to  $v$  to  $w_1$  in straight lines. Call this new path  $\Gamma'_1$  (see Figure 3.3 (a)). Because of the events  $\{\xi_v > 2C^2\}$  and  $D_1$ , the weight of  $\Gamma'_1$  is more than that of  $\Gamma_1$ , unless  $\Gamma'_1 = \Gamma_1$ . Thus on  $\{\xi_v > 2C^2\} \cap D_1 \cap D_2$ ,  $\Gamma_1$  passes through  $v$ . A similar argument applies to  $\Gamma_2$ . Hence

$$\mathbb{P}(F_1) = \mathbb{P}(v \in \Gamma_1 \cap \Gamma_2) \geq \mathbb{P}(\{\xi_v > 2C^2\} \cap D_1 \cap D_2) \geq \delta > 0.$$

□

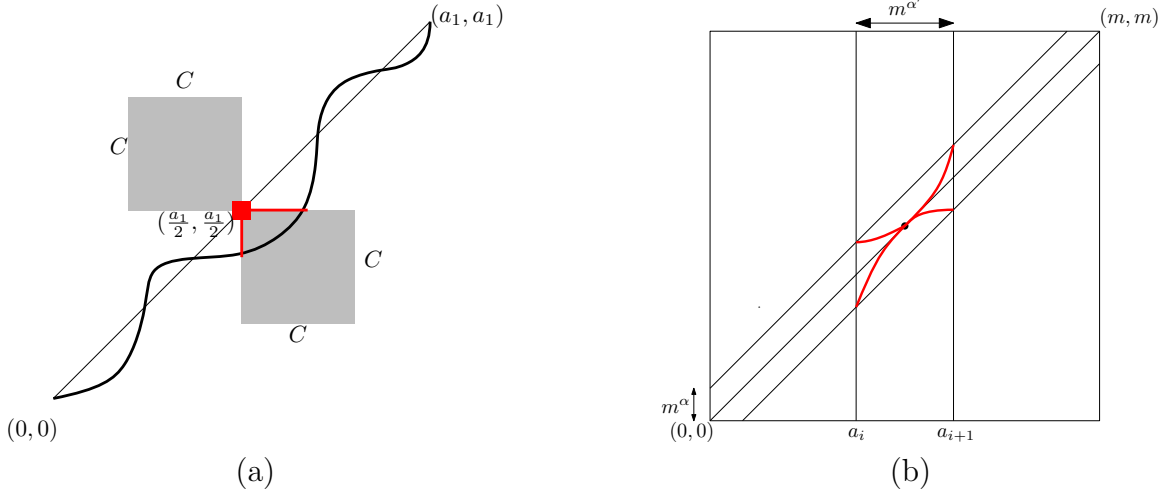


Figure 3.3: (a) Proof of Lemma 3.2.6: on the positive probability event that the path passes close to  $(\frac{a_1}{2}, \frac{a_1}{2})$ , the weight of that vertex is abnormally large, and that of the nearby vertices are typical the path passes through  $(\frac{a_1}{2}, \frac{a_1}{2})$ . (b) Proof of Theorem 3.2.4: there is a positive probability that all the paths between  $a_i$  and  $a_{i+1}$  coalesce at the midpoint on the diagonal.

Now we prove Theorem 3.2.4. Recall that  $\alpha' = \frac{\alpha+1}{2}$ . The idea is to break up the diagonal into intervals of lengths  $m^{\alpha'}$ . Because of Proposition 3.2.2, we know that all the geodesics stay close to the diagonal at each of the endpoints of these intervals. The above Lemma 3.2.6 together with polymer ordering ensures that all these paths meet at the midpoints of each interval with positive probability; see Figure 3.3 (b). Because of independence in each interval, the theorem follows. This kind of argument is very crucial and has been repeated throughout the chapter.

*Proof of Theorem 3.2.4.* We formalize the above idea. Let  $\tilde{\Gamma}_1 = \Gamma_{(0, m^{-\alpha}), (m, m - m^{-\alpha})}$  and  $\tilde{\Gamma}_2 = \Gamma_{(0, m^\alpha), (m, m + m^\alpha)}$ . Notice that, if there exists  $u \in \llbracket 0, m \rrbracket$  such that  $u \in \tilde{\Gamma}_1 \cap \tilde{\Gamma}_2$ , then because of polymer ordering as stated in Lemma 4.3.2,  $u \in \Gamma_{a,b}$  for all  $a \in L_1 \cap \mathbb{Z}$ ,  $b \in L_2 \cap \mathbb{Z}$ . Define  $n = m^{1-\alpha'} = m^{\frac{1-\alpha}{2}}$  and

$$a_i = im^{\alpha'} \text{ for } i = 0, 1, 2, \dots, n.$$

Define the event  $A$  as

$$A = \left\{ |\tilde{\Gamma}_j(a_i) - a_i| \leq m^\alpha \text{ for all } j = 1, 2; i = 0, 1, 2, \dots, n \right\}.$$

Then from Proposition 3.2.2 it follows by taking union bound,

$$\mathbb{P}(A) \geq 1 - 2m^{\frac{1-\alpha}{2}} e^{-cm^{\frac{\alpha}{2}}} \geq 1 - e^{-c'm^{\frac{\alpha}{2}}}.$$

Also for  $i = 1, 2, \dots, n$  define the events  $F_i$  as

$$F_i = \left\{ \left( \frac{a_{i-1} + a_i}{2}, \frac{a_{i-1} + a_i}{2} \right) \in \Gamma_{(a_{i-1}, a_{i-1} + m^\alpha), (a_i, a_i + m^\alpha)} \cap \Gamma_{(a_{i-1}, a_{i-1} - m^\alpha), (a_i, a_i - m^\alpha)} \right\}.$$

Again due to polymer ordering, it is easy to see that,

$$\mathcal{E} \supseteq A \cap \left\{ \bigcup_{i=1}^n F_i \right\}$$

As the  $F_i$ 's are i.i.d. and  $\mathbb{P}(F_1) \geq \delta$  by Lemma 3.2.6, hence,

$$\mathbb{P}(\mathcal{E}^c) \leq \mathbb{P}(\cap_i F_i^c) + \mathbb{P}(A^c) \leq \prod_i \mathbb{P}(F_i^c) + e^{-c'm^{\frac{\alpha}{2}}} \leq (1 - \delta)^n + e^{-c'm^{\frac{\alpha}{2}}} \leq e^{-cm^\ell},$$

where  $\ell = \min\{\frac{1-\alpha}{2}, \frac{\alpha}{2}\}$ . □

The following corollary follows easily from Theorem 3.2.4. It says that a collection of geodesics whose starting points are close to each other and so are their endpoints, has a high probability to meet the diagonal simultaneously.

**Corollary 3.2.7.** *Fix  $0 < \alpha < 1$  and  $K > 0$ . Let  $U_1$  be the parallelogram whose four vertices are  $(0, m^\alpha)$ ,  $(0, -m^\alpha)$ ,  $(m^K, m^K + m^\alpha)$ ,  $(m^K, m^K - m^\alpha)$ . Similarly, define  $U_2$  as the parallelogram with vertices  $(m^K + m, m^K + m + m^\alpha)$ ,  $(m^K + m, m^K + m - m^\alpha)$ ,  $(2m^K + m, 2m^K + m + m^\alpha)$  and  $(2m^K + m, 2m^K + m - m^\alpha)$ . Let  $\mathcal{A}$  denote the event that there exists  $u \in \llbracket m^K, m^K + m \rrbracket$  such that  $(u, u) \in \Gamma_{a,b}$  for all  $a \in U_1 \cap \mathbb{Z}^2, b \in U_2 \cap \mathbb{Z}^2$ . Then there exists constants  $m_0, c$  depending only on  $K$ , such that for all  $m \geq m_0$ ,  $\mathbb{P}(\mathcal{A}) \geq 1 - e^{-cm^\ell}$  where  $\ell = \min\{\frac{1-\alpha}{2}, \frac{\alpha}{2}\}$ .*

*Proof.* Let  $L_1$  be the line segment joining  $(m^K + \frac{m}{3}, m^K + \frac{m}{3} + m^\alpha)$  and  $(m^K + \frac{m}{3}, m^K + \frac{m}{3} - m^\alpha)$ . Similarly let  $L_2$  be the line segment joining  $(m^K + \frac{2m}{3}, m^K + \frac{2m}{3} + m^\alpha)$  and  $(m^K + \frac{2m}{3}, m^K + \frac{2m}{3} - m^\alpha)$ . Let  $\mathcal{B}$  denote the event for all  $a \in U_1 \cap \mathbb{Z}^2, b \in U_2 \cap \mathbb{Z}^2, \Gamma_{a,b}(m^K + \frac{m}{3}) \in L_1$  and  $\Gamma_{a,b}(m^K + \frac{2m}{3}) \in L_2$ . Using Corollary 3.2.3 and union bound, it is easy to see that,

$$\mathbb{P}(\mathcal{B}) \geq 1 - 4m^{2K} e^{-cm^{\alpha/2}} \geq 1 - e^{-c'm^{\alpha/2}}.$$

Hence applying Theorem 3.2.4 to all geodesics from  $L_1$  to  $L_2$ , one has the result. □

Similarly, the following corollary is immediate. We omit the proof.

**Corollary 3.2.8.** *Fix  $0 < \alpha < 1$ . Let  $0 < a < b < L$  such that  $|b - a| = m$  and  $a \geq n, L - b \geq n$  and  $n \geq m$ . Let  $E_1$  be the line segment joining  $(0, -n^\alpha)$  and  $(0, n^\alpha)$ . Let  $E_2$  be the line segment joining  $(L, L - n^\alpha)$  and  $(L, L + n^\alpha)$ . Let  $B_1$  be the line segment joining  $(a, a - m^\alpha)$  and  $(a, a + m^\alpha)$ , and  $B_2$  be the segment joining  $(b, b - m^\alpha)$  and  $(b, b + m^\alpha)$ . Let  $\mathcal{E}$  be the event that there exists  $u \in \llbracket a, b \rrbracket$  such that  $(u, u) \in \Gamma_{i,j}$  for all  $i \in E_1, j \in E_2$  and all  $i \in B_1, j \in B_2$ . Then there exists absolute positive constant  $c$  such that  $\mathbb{P}(\mathcal{E}) \geq 1 - e^{-cm^\ell}$  where  $\ell = \min\{\frac{1-\alpha}{2}, \frac{\alpha}{2}\}$ .*

### 3.2.1 Subdiffusive fluctuations of the last passage time

Unlike the TASEP where the passage times  $L_n$  is of the order  $n^{1/3}$ , in presence of a slow bond the passage times  $T_n$  (note we suppress the dependence on  $r$ ) show diffusive fluctuation. This is a consequence of the path getting pinned to the diagonal at a constant rate; and using Theorem 3.2.4 one can argue that  $T_n$  can be approximated by partial sums of a stationary process. Using this, and the mixing properties guaranteed by Theorem 3.2.4, it is possible to prove a central limit theorem for  $T_n$ . Although such a result is interesting, it is not crucial for our purposes in this chapter. We shall often want to compare best paths in the reinforced environment (i.e., the slow bond model with the diagonals boosted) with paths that do not use the diagonal. Typically the paths that use the diagonal will be larger, and to quantify this we would need concentration bounds for  $|T_n - (4 + \varepsilon)n|$ . This will be done in two steps (a) control on the difference between  $\mathbb{E}T_n$  and  $(4 + \varepsilon)n$  and (b) concentration of  $T_n$  around its mean. A proof of the central limit theorem is provided in Section 3.7.

We first start with the following lemma.

**Lemma 3.2.9.** *There exists an absolute constant  $K > 0$  such that*

$$\mathbb{E}T_n \geq (4 + \varepsilon)n - K.$$

Note that due to superadditivity,  $\mathbb{E}T_n \leq (4 + \varepsilon)n$  always holds. The main idea in the proof of this lemma is that if  $0 < a < b < L$ , then since the geodesic  $\Gamma_L$  is close to the diagonal at  $(a, a)$  and  $(b, b)$ , the part of  $\Gamma_L$  falling between the lines  $x = a$  and  $x = b$  is close in length to that of the geodesic  $\Gamma_{(a,a),(b,b)}$ . This gives a way to compare geodesics between intervals of different lengths.

*Proof.* For any  $n \in \mathbb{N}$  and  $0 < a < b < n$ , define  $T_n(a, b)$  as the weight of the part of the geodesic  $\Gamma_n$  that lies between the vertical lines  $x = a$  and  $x = b$ . That is,  $T_n(a, b) = \sum_{(x_i, y_i) \in \Gamma_n: a \leq x_i \leq b} \xi_{(x_i, y_i)}$ . Then we claim that there exists absolute positive constants  $M_0, K$  such that for any  $M \geq M_0$ , any  $n \in \mathbb{N}$  and any  $i \in \llbracket 0, M - 1 \rrbracket$ ,

$$|\mathbb{E}(T_{nM}(in, (i + 1)n)) - \mathbb{E}(T_{in, (i+1)n})| \leq K.$$

To see this fix any  $m \in \mathbb{N}$ , and define the event  $E$  that  $\Gamma_{nM}$  and  $\Gamma_{in, (i+1)n}$  meet together on the diagonal between  $\llbracket in, in + m^{1/3} \rrbracket$  and again between  $\llbracket (i + 1)n - m^{1/3}, (i + 1)n \rrbracket$ , and let  $F$  be the event that  $\Gamma_{nM}$  passes through the line segment joining  $(in, in - m^{1/6})$  and  $(in, in + m^{1/6})$ , and again through the line segment joining  $((i + 1)n, (i + 1)n - m^{1/6})$  and  $((i + 1)n, (i + 1)n + m^{1/6})$ . Let  $Y \sim \text{Gamma}(2m^{2/3}, r)$  is the sum of  $2m^{2/3}$  many i.i.d.  $\text{Exp}(r)$  random variables. Then using Proposition 3.2.2 and Theorem 3.2.4,

$$\begin{aligned} & \mathbb{P}(|T_{nM}(in, (i + 1)n) - T_{in, (i+1)n}| \geq m) \\ & \leq \mathbb{P}(\{|T_{nM}(in, (i + 1)n) - T_{in, (i+1)n}| \geq m\} \cap E \cap F) + \mathbb{P}(E^c \cap F) + \mathbb{P}(F^c) \\ & \leq 2\mathbb{P}(Y \geq m) + C'e^{-cm^{1/12}} + 2C'e^{-cm^{1/12}} \\ & \leq 2C'e^{-cm} + C'e^{-cm^{1/12}} + C'e^{-cm^{1/12}} \leq C''e^{-c'm^{1/12}}. \end{aligned}$$

Hence, summing over all  $m \in \mathbb{N}$ , we have, for all  $n, M, i$ , there exists some absolute positive constant  $K$  such that,

$$|\mathbb{E}(T_{nM}(in, (i+1)n)) - \mathbb{E}(T_{\mathbf{in},(i+1)\mathbf{n}})| \leq \mathbb{E}|T_{nM}(in, (i+1)n) - T_{\mathbf{in},(i+1)\mathbf{n}}| \leq K. \quad (3.7)$$

As  $T_{\mathbf{in},(i+1)\mathbf{n}} \stackrel{d}{=} T_n$ , hence, adding up (3.7) over all  $i \in \llbracket 0, M-1 \rrbracket$ ,

$$|\mathbb{E}(T_{nM}) - M\mathbb{E}(T_n)| \leq KM.$$

That is,

$$\frac{\mathbb{E}(T_{nM})}{nM} \leq \frac{M\mathbb{E}(T_n)}{nM} + \frac{KM}{nM} = \frac{\mathbb{E}T_n}{n} + \frac{K}{n}.$$

Hence, keeping  $n$  fixed, and taking  $M \rightarrow \infty$ ,

$$4 + \varepsilon \leq \frac{\mathbb{E}T_n}{n} + \frac{K}{n}.$$

Hence, for all  $n$ ,  $\mathbb{E}T_n \geq (4 + \varepsilon)n - K$ . □

For the second ingredient, while it is possible to prove a concentration at scale  $n^{1/2}$ , a tail bound at scale  $n^{1/2+o(1)}$  is more standard and much easier to prove. We state, without proof the following result which can be proved using standard Martingale techniques with some truncation (cf. the proof of Theorem 3.11 in [54]). This will be sufficient for our purpose.

**Lemma 3.2.10.** *Fix  $\delta > 0$ . Then there exist absolute positive constants  $C', c$  such that*

$$\mathbb{P}(|T_n - \mathbb{E}T_n| \geq n^{1/2+\delta}) \leq C' e^{-cn^{\delta/2}}.$$

Lemma 3.2.9 and Lemma 3.2.10 imply the following proposition that will be useful later. As discussed earlier in the introduction, in order to look at the limiting distribution away from the origin, we will have to consider geodesics from  $(0, 0)$  to  $(n+k, n)$  for  $k$  small compared to  $n$ . And the geodesic from  $(0, 0)$  to  $(n+k, n)$  is expected to hit the diagonal for the last time near the point  $(x_1, x_1)$ , such that  $x_1$  maximizes the quantity  $r(x) := (4 + \varepsilon)x + (\sqrt{n+k-x} + \sqrt{n-x})^2$ . This is made precise in the following proposition. As before, we have not been very strict about the correct order of the exponents here.

**Proposition 3.2.11.** *Let  $n \geq k^{8/7}$ , and  $\Gamma_{n+k,n}$  be the geodesic from  $(0, 0)$  to  $(n+k, n)$  in the reinforced environment. Let  $(X, X)$  be the last point on the diagonal that lies on  $\Gamma_{n+k,n}$ . Let  $x_1 \in \llbracket 0, n \rrbracket$  be the point that maximizes the quantity  $r(x)$  above. Then there exist constants  $C', c$  such that*

$$\mathbb{P}(|X - x_1| \geq k^{3/5}) \leq C' e^{-ck^{1/20}}.$$

*Proof.* Let  $v_1 = n - k^{8/7}$  and  $\Gamma_1$  be the geodesic from  $v = (v_1, v_1)$  to  $(n+k, n)$ . Let  $\mathcal{E}$  be the event that there exists  $u \in \llbracket n - \frac{k^{8/7}}{2}, n \rrbracket$  such that  $(u, u) \in \Gamma_1 \cap \Gamma_{n+k,n}$ . Then from Corollary 3.2.8,  $\mathbb{P}(\mathcal{E}) \geq 1 - e^{-c'k^{1/14}}$ . As  $\Gamma_1$  and  $\Gamma_{n+k,n}$  have the last endpoint common, hence once

they meet they coincide till  $(n + k, n)$ . Thus on  $\mathcal{E}$ , the last point on the diagonal for both  $\Gamma_1$  and  $\Gamma_{n+k,n}$  are same. Hence enough to find the point where  $\Gamma_1$  last meets the diagonal.

To this end, first note that from (3.3) in the introduction it follows that  $n - x_1 = ck$  for some constant  $c$ . Let  $\Gamma_2$  be the union of the two geodesics  $\Gamma_{2,1}$  from  $v$  to  $(x_1, x_1)$  and the geodesic  $\Gamma_{2,2}$  from  $(x_1, x_1)$  to  $(n + k, n)$  that avoids the diagonal. Let  $A$  be the event that  $\Gamma_1$  touches the diagonal for the last time at some point  $(x_2, x_2)$  with  $x_2 \in \llbracket n - \frac{k^{8/7}}{2}, n \rrbracket$  and  $|x_2 - x_1| \geq k^{3/5}$ . Then

$$\mathbb{P}(|X - x_1| \geq k^{3/5}) \leq \mathbb{P}(A) + \mathbb{P}(\mathcal{E}^c) \leq \mathbb{P}(A) + e^{-c'k^{1/14}}.$$

Let  $H = \{(x, y) \in \mathbb{R}^2 : y \geq x\}$  denote the region in  $\mathbb{R}^2$  that lies on or above the diagonal line  $x = y$ , and for  $z_1, z_2 \in \mathbb{R}^2$  with  $z_2 \geq z_1$  and  $z_1, z_2 \in (\bar{H}^c)$ , let  $T_{z_1, z_2}^H$  denote the weight of the geodesic from vertex  $z_1$  to  $z_2$  that does not pass through the region  $H$  (except possibly at the endpoints). Then clearly

$$A \subseteq \bigcup_{x_2: |x_2 - x_1| \geq k^{3/5}} \{T_{v, (x_2, x_2)} + T_{(x_2, x_2), (n+k, n)}^H \geq T_{v, (x_1, x_1)} + T_{(x_1, x_1), (n+k, n)}^H\}.$$

Calculating expectations using Lemma 3.2.9, we get for any such  $x_2$ , there exists some constant  $\alpha$  such that

$$\mathbb{E}(T_{v, (x_2, x_2)}) + \mathbb{E}(T_{(x_2, x_2), (n+k, n)}^0) \leq \mathbb{E}(T_{v, (x_1, x_1)}) + \mathbb{E}(T_{(x_1, x_1), (n+k, n)}^0) - \alpha k^{3/5}.$$

Since  $|v_1 - x_1| \sim k^{8/7}$ , by Lemma 3.2.10 it follows that  $\mathbb{P}(|T_{v, (x_1, x_1)} - \mathbb{E}(T_{v, (x_1, x_1)})| \geq \frac{\alpha}{4} k^{3/5}) \leq C' s^{-ck^{1/20}}$ . Also applying Proposition 12.2 from [16], one immediately gets that

$$\mathbb{P}(|T_{(x_1, x_1), (n+k, n)}^H - \mathbb{E}(T_{(x_1, x_1), (n+k, n)}^0)| \geq \frac{\alpha}{4} k^{3/5}) \leq e^{-ck^{4/45}}.$$

Since  $x_2 - v_1 \geq \frac{k^{8/7}}{2}$ , similar calculations for the fluctuation of  $T_{v, (x_2, x_2)} + T_{(x_2, x_2), (n+k, n)}^H$  around  $\mathbb{E}(T_{v, (x_2, x_2)}) + \mathbb{E}(T_{(x_2, x_2), (n+k, n)}^0)$  and union bound over all  $x_2 \in \llbracket n - \frac{k^{8/7}}{2}, n \rrbracket$  give the result.  $\square$

### 3.3 Constructing a candidate for invariant measure

In this section we construct a candidate for invariant measure for TASEP with a slow bond. The construction of the measure is fairly intuitive. Fix a finite interval  $[-b, b]$  around the origin. Recall that  $T_n$  is the last passage time between  $(0, 0)$  and  $(n, n)$  (i.e.,  $T_n$  is the time that  $(n+1)$ st particle crosses the origin). Now for  $n \gg k \gg 1$  consider the average occupation measure of sites in  $[-b, b]$  between  $T_n$  and  $T_{n+k}$ . Because the geodesics are localized around the diagonal it turns out using the correspondence between TASEP and DLPP that the occupation measures are approximately determined by the weight configuration on a small

box around the diagonal between  $(n, n)$  and  $(n+k, n+k)$ . Moreover using the independence of the configurations of such disjoint boxes and Theorem 3.2.4 one can construct such a sequence of occupation measures that are Cauchy. One gets an the candidate measure passing to a limit.

Recall that  $\eta_t = \eta_t^{(r)}$  is the configuration of TASEP with a slow bond started from the step initial condition, i.e.,  $\eta_t(i) = 0$  or  $1$  according to whether the site  $i$  is vacant or occupied at time  $t$ . Also for an interval  $I \subseteq \mathbb{Z}$  let  $\eta_t(I)$  denote the configuration restricted to  $I$ . Our main theorem in this section is the following.

**Theorem 3.3.1.** *There exists a measure  $Q$  on  $\{0, 1\}^{\mathbb{Z}}$  with the following property: Fix  $b \in \mathbb{N}$  and  $\delta > 0$ . Set  $I = [-b, b]$  and let  $Q_I$  denote the restriction of  $Q$  to  $I$ . Then there exist constants  $k_0, c > 0$  depending only on  $b$ , such that for all  $k \geq k_0$ ,*

$$\sup_n \mathbb{P} \left( \sup_{A \subseteq \{0,1\}^I} \left| \frac{1}{(4+\varepsilon)k} \int_{T_n}^{T_{n+k}} \mathbf{1}(\eta_t(I) \in A) dt - Q_I(A) \right| > \delta \right) < e^{-c\delta k^{1/13}}.$$

The main step of the proof of Theorem 3.3.1 is to establish that the sequence of average occupation measures as in the statement of the Theorem is almost surely Cauchy. To this end we have the following proposition.

**Proposition 3.3.2.** *Fix  $b \in \mathbb{N}$  and  $\delta > 0$ , set  $I = [-b, b]$  and fix  $A \subseteq \{0, 1\}^I$ . Then there exist absolute constants  $k_0, c > 0$  such that for all  $k \geq k_0$ ,*

$$\sup_{n \in \mathbb{N}, \alpha \in [\frac{3}{2}, 3]} \mathbb{P} \left( \left| \frac{1}{k^\alpha} \int_{T_n}^{T_{n+k^\alpha}} \mathbf{1}(\eta_t(I) \in A) dt - \frac{1}{k} \int_{T_n}^{T_{n+k}} \mathbf{1}(\eta_t(I) \in A) dt \right| > \delta \right) < e^{-c\delta k^{1/12}}.$$

In the proof, we shall need the following parallelogram. For  $s > b$ ,  $s, b \in \mathbb{N}$ , let  $U_{s,r,b}$  denote a parallelogram with endpoints  $(s-b-1, s)$ ,  $(s, s-b-1)$ ,  $(s+r+b, s+r)$ ,  $(s+r, s+r+b)$ .

Recall that for sites  $j \in [-b, b]$ , the length of  $T_{s+j,s}$  gives the time taken by a particle at  $-s$  to first visit site  $j+1$ . Hence, for any fixed  $A \subseteq \{0, 1\}^I$ ,  $\int_{T_s}^{T_{s+r}} \mathbf{1}(\eta_t(I) \in A) dt$ , the total occupation time at these sites corresponding to the states defined by  $A$  between times  $T_s$  and  $T_{s+r}$  is a function of the pairwise differences in the lengths of the geodesics starting from  $(0, 0)$  and ending in  $U_{s,r,b}$ . We state this without proof in the next lemma.

**Lemma 3.3.3.** *For  $s, r, b \in \mathbb{N}$ ,  $s > b$ , and  $I = [-b, b]$  and  $A \subseteq \{0, 1\}^I$ , let  $\Theta_{s,r,b} = \{T_{i,j} : (i, j) \in U_{s,r,b} \cap \mathbb{Z}^2\}$  denote the set of lengths of all geodesics starting from  $(0, 0)$  and ending in  $U_{s,r,b}$ . Then there exists a function  $f = f_{r,b,A} : \mathbb{R}_+^{|\Theta_{s,r,b}|} \mapsto \mathbb{R}_+$  such that for any  $c \in \mathbb{R}_+$ ,  $\mathbf{x} \in \mathbb{R}_+^{|\Theta_{s,r,b}|}$ ,  $f(\mathbf{x} + c) = f(\mathbf{x})$  and*

$$\frac{1}{r} \int_{T_s}^{T_{s+r}} \mathbf{1}(\eta_t(I) \in A) dt = f(\Theta).$$

Also the function  $f$  does not depend on the location  $s$ .

We apply this lemma to prove Proposition 3.3.2. Define  $\mathcal{B}(s, r, b)$  as the  $\text{Box}((s - b - 1, s - b - 1), (s + r + b, s + r + b))$  of size  $r + 2b + 1$ . Clearly  $U_{s,r,b} \subseteq \mathcal{B}(s, r, b)$ . The idea here is to break the  $k^\alpha$ -sized box at  $(n, n)$  into  $k^{\alpha-1}$ -many  $k$ -sized boxes, leaving sufficient amount of gap between each box, and use a renewal argument and a law of large numbers to get the required result. More formally, we do the following.

*Proof of Proposition 3.3.2.* Fix  $n \in \mathbb{N}$  and  $\alpha \in [\frac{3}{2}, 3]$  and let  $B := \mathcal{B}(n, k^\alpha, b)$ . Let  $r_k$  be the largest integer such that  $r_k(k + k^{1/3}) + k^{1/3} \leq k^\alpha$ . Clearly  $r_k \sim k^{\alpha-1}$ . Define

$$a_i = n + (i + 1)k^{1/3} + ik, \quad \text{for } i = 0, 1, 2, \dots, r_k.$$

Define the boxes

$$B_i = \mathcal{B}(a_i, k, b) \text{ for } i = 0, 1, 2, \dots, r_k,$$

and parallelograms

$$U_i = U_{a_i, k, b} \subseteq B_i \text{ for } i = 0, 1, 2, \dots, r_k.$$

Let  $k^{1/3} > 4b$ . Then each of these  $k$ -sized boxes  $B_i$  are separated by a distance of at least  $k^{1/3}/2$ . Let  $p_i = a_i - b - 1$  and  $q_i = a_i + k + b$ , so that  $(p_i, p_i)$  and  $(q_i, q_i)$  are the endpoints of the box  $B_i$ . Define  $q_{-1} = n$ . See Figure 3.4.

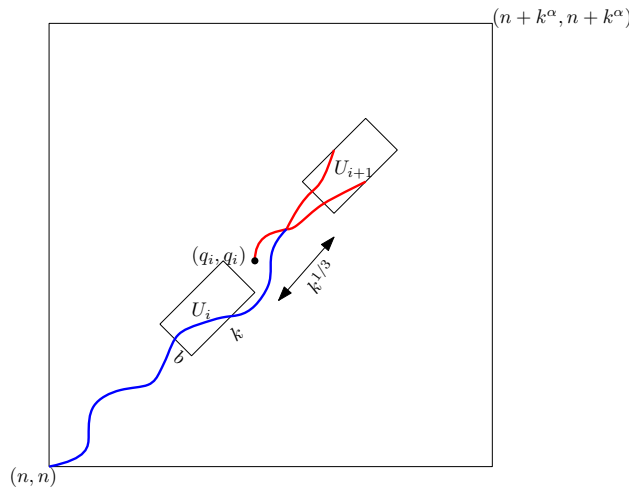


Figure 3.4: Using the fact that geodesics are localised near the diagonal it follows that on a high probability event the pairwise difference of passage times from the origin to vertices in  $U_{i+1}$  is same as the pairwise differences of passage times from the vertex  $(q_i, q_i)$  to vertices in  $U_{i+1}$ ; the latter collection is an i.i.d. sequence.

Now fix a particular  $i \in \{0, 1, 2, \dots, r_k\}$ . Let  $E_i$  denote the event that all the geodesics from  $(0, 0)$  to all points in  $U_i$  and all the geodesics from  $(q_{i-1}, q_{i-1})$  to  $U_i$  meet the diagonal simultaneously between  $[[q_{i-1}, p_i]]$ . Then by Corollary 3.2.7,

$$\mathbb{P}(E_i) \geq 1 - e^{-ck^{1/12}}.$$

Let  $E := \cap_{i=0}^{r_k} E_i$ . Then

$$\mathbb{P}(E) \geq 1 - k^\alpha e^{-ck^{1/12}} \geq 1 - e^{-c'k^{1/12}},$$

for all large enough  $k$ . Define,

$$\Theta_i^0 = \{T_{(0,0),(u,v)} : (u, v) \in U_i \cap \mathbb{Z}^2\}, \text{ for } i = 0, 1, 2, \dots, r_k,$$

$$\Theta_i^q = \{T_{(q_{i-1}, q_{i-1}), (u,v)} : (u, v) \in U_i \cap \mathbb{Z}^2\}, \text{ for } i = 0, 1, 2, \dots, r_k.$$

Then on the event  $E$ , for all  $i \in \llbracket 0, r_k \rrbracket$ ,  $T_{(0,0),(u,v)} - T_{(q_{i-1}, q_{i-1}), (u,v)} = T_{(0,0),(u',v')} - T_{(q_{i-1}, q_{i-1}), (u',v')}$  for all  $(u, v), (u', v') \in U_i$ . Using Lemma 3.3.3, there exists  $f = f_{k,b,A}$  such that  $\frac{1}{k} \int_{T_{a_i}}^{T_{a_i+k}} \mathbf{1}(\eta_t(I) \in A) dt = f(\Theta_i^0)$ . Using the property of translation invariance of  $f$ , on  $E$ , we have,

$$\frac{1}{k} \int_{T_{a_i}}^{T_{a_i+k}} \mathbf{1}(\eta_t(I) \in A) dt = f(\Theta_i^0) = f(\Theta_i^q).$$

Define  $Y_i := f(\Theta_i^q)$  for  $i = 0, 1, 2, \dots, r_k$ . Clearly,  $Y_i$ s are independent and identically distributed. The rest of the argument is standard and uses Chernoff bounds.

First note that

$$\mathbb{P} \left( \left| \frac{1}{k^\alpha} \int_{T_n}^{T_{n+k^\alpha}} \mathbf{1}(\eta_t(I) \in A) dt - \frac{1}{r_k} \sum_{i=0}^{r_k} \frac{1}{k} \int_{T_{a_i}}^{T_{a_i+k}} \mathbf{1}(\eta_t(I) \in A) dt \right| \geq \frac{\delta}{8} \right) \leq e^{-ck^{1/12}}.$$

Indeed, in order to show that  $\frac{1}{k^\alpha} \sum_{i=0}^{r_k-1} \int_{T_{a_i+k}}^{T_{a_{i+1}}} \mathbf{1}(\eta_t(I) \in A) dt$  is small, enough to show that  $\frac{1}{k^\alpha} \sum_{i=0}^{r_k-1} (T_{a_{i+1}} - T_{a_i+k})$  is small. Let  $F$  denote the event that for each  $i$ , the geodesics  $\Gamma_{a_{i+1}}$  and  $\Gamma_{a_i+k}$  meet together on the diagonal in the interval  $\llbracket a_i + k - k^{1/3}, a_{i+1} \rrbracket$ . Then from Corollary 3.2.8,  $\mathbb{P}(F) \geq 1 - e^{-ck^{1/12}}$ . On  $F$ ,

$$T_{a_{i+1}} - T_{a_i+k} \leq T_{(a_i+k-k^{1/3}, a_i+k-k^{1/3}), (a_{i+1}, a_{i+1})}.$$

As  $a_{i+1} - (a_i + k - k^{1/3}) = 2k^{1/3}$ ,  $T_{(a_i+k-k^{1/3}, a_i+k-k^{1/3}), (a_{i+1}, a_{i+1})} \stackrel{d}{=} T_{2k^{1/3}}$ . Hence, by union bound,

$$\mathbb{P} \left( \frac{1}{k^\alpha} \sum_{i=0}^{r_k-1} (T_{a_{i+1}} - T_{a_i+k}) \geq \delta \right) \leq k^{\alpha-1} \mathbb{P} \left( \frac{1}{k} T_{2k^{1/3}} \geq \delta \right).$$

It is easy to see that the right hand side is exponentially small. Similarly one can bound the other terms.

Hence it is enough to get an upper bound to

$$\mathbb{P} \left( \left| \frac{1}{r_k} \sum_{i=0}^{r_k} \frac{1}{k} \int_{T_{a_i}}^{T_{a_i+k}} \mathbf{1}(\eta_t(I) \in A) dt - \frac{1}{k} \int_{T_n}^{T_{n+k}} \mathbf{1}(\eta_t(I) \in A) dt \right| > \delta/4 \right).$$

Now note that,

$$\begin{aligned}
 & \mathbb{P} \left( \left| \frac{1}{r_k} \sum_{i=0}^{r_k} \frac{1}{k} \int_{T_{a_i}}^{T_{a_i+k}} \mathbf{1}(\eta_t(I) \in A) dt - \mathbb{E} \left[ \frac{1}{k} \int_{T_n}^{T_{n+k}} \mathbf{1}(\eta_t(I) \in A) dt \right] \right| > \delta/8 \right) \\
 & \leq \mathbb{P} \left( \left\{ \left| \frac{1}{r_k} \sum_{i=0}^{r_k} Y_i - \mathbb{E}(Y_0) \right| > \delta/8 \right\} \cap E \right) + e^{-ck^{1/12}} \\
 & \leq \mathbb{P} \left( \left| \frac{1}{r_k} \sum_{i=0}^{r_k} Y_i - \mathbb{E}(Y_0) \right| > \delta/8 \right) + e^{-ck^{1/12}},
 \end{aligned}$$

where  $Y_i = f(\Theta_i^q)$ ,  $i = 0, 1, \dots, r_k$  are i.i.d. having the same distribution as that of the occupation density  $\frac{1}{k} \int_{T_{k^{1/3-b}}}^{T_{k^{1/3-b+k}}} \mathbf{1}(\eta_t(I) \in A) dt$ , as discussed earlier. Also,

$$|Y_i| \stackrel{d}{=} \left| \frac{1}{k} \int_{T_{k^{\beta-b}}}^{T_{k^{\beta-b+k}}} \mathbf{1}(\eta_t(I) \in A) dt \right| \leq \left| \frac{1}{k} (T_{k^{\beta-b+k}} - T_{k^{\beta-b}}) \right| \leq \frac{1}{k} T_{k+k^{\beta}} \leq \frac{2T_{k+k^{\beta}}}{k+k^{\beta}},$$

and as  $\frac{T_{k+k^{\beta}}}{k+k^{\beta}}$  is a subexponential random variable (we can take the same parameters for this subexponential random variable for all values of  $k$ , as when  $m$  increases one gets better tail bounds for  $\frac{T_m}{m}$ ), one gets

$$\mathbb{P} \left( \left| \frac{1}{r_k} \sum_{i=0}^{r_k} Y_i - \mathbb{E}(Y_0) \right| > \delta/8 \right) \leq e^{-c\delta r_k} \leq e^{-c\delta\sqrt{k}/2}$$

for all  $\alpha \in [\frac{3}{2}, 3]$ . Hence, the only thing left to bound is

$$\mathbb{P} \left( \left| \frac{1}{k} \int_{T_n}^{T_{n+k}} \mathbf{1}(\eta_t(I) \in A) dt - \mathbb{E} \left[ \frac{1}{k} \int_{T_n}^{T_{n+k}} \mathbf{1}(\eta_t(I) \in A) dt \right] \right| > \delta/8 \right).$$

To this end, we follow the exact same procedure as we just did. We consider the  $\text{Box}((n-b-1, n-b-1), (n+k+b, n+k+b))$  and break it into a number of boxes of size  $k^{1/2}$  each, leaving a gap of size  $k^{1/3}$  between any two boxes. Define the event  $E'$  parallel to the event  $E$ , such that on  $E'$ ,  $\frac{1}{k} \int_{T_n}^{T_{n+k}} \mathbf{1}(\eta_t(I) \in A) dt$  can be written as a sum of independent and identically distributed random variables. Using exactly similar arguments, we have

$$\mathbb{P} \left( \left| \frac{1}{k} \int_{T_n}^{T_{n+k}} \mathbf{1}(\eta_t(I) \in A) dt - \mathbb{E} \left[ \frac{1}{k} \int_{T_n}^{T_{n+k}} \mathbf{1}(\eta_t(I) \in A) dt \right] \right| > \delta/8 \right) \leq e^{-c\delta k^{1/12}}.$$

This completes the proof.  $\square$

Now we are in a position to prove Theorem 3.3.1.

*Proof of Theorem 3.3.1.* Fix  $A \subseteq \{0, 1\}^I$  and  $n \in \mathbb{N}$ . Using  $\alpha = 2$  in Proposition 3.3.2, we have,

$$\mathbb{P} \left( \left| \frac{1}{k^2} \int_{T_n}^{T_{n+k^2}} \mathbf{1}(\eta_t(I) \in A) dt - \frac{1}{k} \int_{T_n}^{T_{n+k}} \mathbf{1}(\eta_t(I) \in A) dt \right| > \delta \right) < e^{-c\delta k^{1/12}}.$$

Choosing  $k = 2^{2^m}$  and  $\delta = 1/2^m$ , for all  $m$  sufficiently large, one has,

$$\mathbb{P} \left( \left| \frac{1}{2^{2^{m+1}}} \int_{T_n}^{T_{n+2^{2^{m+1}}}} \mathbf{1}(\eta_t(I) \in A) dt - \frac{1}{2^{2^m}} \int_{T_n}^{T_{n+2^{2^m}}} \mathbf{1}(\eta_t(I) \in A) dt \right| > \frac{1}{2^m} \right) < e^{-\frac{c}{2^m} 2^{2^m/12}}. \quad (3.8)$$

As the probabilities are summable in  $m$ , the sequence  $\left\{ \frac{1}{2^{2^m}} \int_{T_n}^{T_{n+2^{2^m}}} \mathbf{1}(\eta_t(I) \in A) dt \right\}_m$  is Cauchy almost surely, and hence converges almost surely, the limiting random variable is degenerate by Kolmogorov zero-one law. As  $\frac{T_k}{k} \rightarrow (4 + \varepsilon)$  a.s. as  $k \rightarrow \infty$ , it is easy to see that

$$\frac{1}{(4 + \varepsilon)2^{2^m}} \int_{T_n}^{T_{n+2^{2^m}}} \mathbf{1}(\eta_t(I) \in A) dt \xrightarrow{\text{a.s.}} Q_I(A),$$

for some probability measure  $Q_I$  on  $\{0, 1\}^I$ . (It is easy to see that the limit does not depend on  $n$ ). It is not hard to see that  $Q_I$  s form a consistent system of probability measures, and hence define a unique probability measure  $Q$  on  $\{0, 1\}^{\mathbb{Z}}$  such that  $Q_I$  is the projection of  $Q$  on  $I$ .

Also, for any fixed  $m$  large enough, by summing up the probabilities in (3.8), for all  $r > m$ ,

$$\begin{aligned} & \mathbb{P} \left( \left| \frac{1}{2^{2^m}} \int_{T_n}^{T_{n+2^{2^m}}} \mathbf{1}(\eta_t(I) \in A) dt - (4 + \varepsilon)Q_I(A) \right| > \delta \right) \\ & \leq \mathbb{P} \left( \sup_{r > m} \left| \frac{1}{2^{2^m}} \int_{T_n}^{T_{n+2^{2^m}}} \mathbf{1}(\eta_t(I) \in A) dt - \frac{1}{2^{2^r}} \int_{T_n}^{T_{n+2^{2^r}}} \mathbf{1}(\eta_t(I) \in A) dt \right| > \delta \right) \\ & \leq \sum_{r=m}^{\infty} \mathbb{P} \left( \left| \frac{1}{2^{2^r}} \int_{T_n}^{T_{n+2^{2^r}}} \mathbf{1}(\eta_t(I) \in A) dt - \frac{1}{2^{2^{r+1}}} \int_{T_n}^{T_{n+2^{2^{r+1}}}} \mathbf{1}(\eta_t(I) \in A) dt \right| > \frac{\delta}{2^r} \right) \\ & \leq \sum_{r=m}^{\infty} e^{-\frac{c\delta}{2^r} 2^{2^r/12}} \leq 2e^{-c\delta 2^{2^m/13}}. \end{aligned} \quad (3.9)$$

Now, for any  $2^{2^m} \leq k \leq 2^{2^{m+1}}$ , there exists some  $\alpha \in [\frac{3}{2}, 3]$ , such that  $k^\alpha = 2^{2^{m+1}}$  or  $k^\alpha = 2^{2^{m+2}}$ . (If  $k^\alpha = 2^{2^{m+1}}$  for some  $\alpha \in [\frac{3}{2}, 2]$  then we are done, else  $k^\alpha = 2^{2^{m+1}}$  for some  $\alpha \in (1, \frac{3}{2})$ , and then  $k^{2\alpha} = 2^{2^{m+2}}$  where  $2\alpha \in (2, 3)$ ). Thus, combining the bounds in Proposition 3.3.2 and that in (3.9), we get,

$$\mathbb{P} \left( \left| \frac{1}{(4 + \varepsilon)k} \int_{T_n}^{T_{n+k}} \mathbf{1}(\eta_t(I) \in A) dt - Q_I(A) \right| > \delta \right) < e^{-c\delta k^{1/13}}. \quad (3.10)$$

As this holds for all  $A \subseteq \{0, 1\}^I$ , and  $b \in \mathbb{N}$  is fixed, using (3.10) for all  $2^{2b+1}$  subsets  $A$ , and union bound,

$$\mathbb{P} \left( \sup_{A \subseteq \{0, 1\}^I} \left| \frac{1}{(4 + \varepsilon)k} \int_{T_n}^{T_{n+k}} \mathbf{1}(\eta_t(I) \in A) dt - Q_I(A) \right| > \delta \right) < e^{-c\delta k^{1/13}}.$$

As this holds for all  $n \in \mathbb{N}$ , the result follows.  $\square$

### 3.4 Convergence to invariant measure

In this section we shall establish that started from the step initial condition TASEP with a slow bond at the origin converges weakly to the measure  $Q$ . It suffices to prove the following theorem for convergence of finite dimensional distributions.

**Theorem 3.4.1.** *For any fixed  $b \in \mathbb{N}$ ,  $I = [-b, b]$  and  $A \subseteq \{0, 1\}^I$ ,*

$$\mathbb{P}(\eta_t(I) \in A) \rightarrow Q_I(A)$$

as  $t \rightarrow \infty$ .

The idea of the proof goes as follows. First we observe that the configuration of  $\eta_{T_n+s}(I)$  at time  $T_n + s$  does not depend on the exact value of the passage time  $T_n$ , but the amount of overshoot of the different passage times from  $T_n$ , and is thus roughly independent of the passage times near the origin. Also conditioning on all the exponential random variables except at a number of sufficiently spaced vertices on the diagonal near the origin, one can argue that the effects of these vertex weights on  $T_n$  are roughly independent. Hence, a local limit theorem suggests that the conditional distribution of  $T_n$  is close to Gaussian. Owing to the flatness of the Gaussian density, one can approximate  $t - T_n$  by the uniform distribution, and thus reduce  $\eta_t(I) = \eta_{T_n+(t-T_n)}(I)$  to the average occupation measure over suitable intervals. From here one can resort to Theorem 3.3.1 to get the convergence to  $Q_I$ .

For the proof of Theorem 3.4.1 we shall need a few lemmas. The following lemma is basic and follows easily from Theorem 3.3.1.

**Lemma 3.4.2.** *Fix  $b \in \mathbb{N}$ ,  $I = [-b, b]$  and  $A \subseteq \{0, 1\}^I$  and  $n \in \mathbb{N}$  and any  $\delta, \alpha > 0$ . Then, for any random variable  $U$  such that  $\mathbb{P}(U \in [\frac{n^2}{3}, 2n^2]) \geq 1 - \delta$ , and  $s \geq \alpha n^{1/12}$ ,*

$$\mathbb{E} \left| \frac{1}{s} \int_{T_n+U}^{T_n+U+s} \mathbf{1}(\eta_t(I) \in A) dt - Q_I(A) \right| \leq 2\delta + e^{-c\delta s^{1/30}}.$$

*Proof.* Observe that from Theorem 3.3.1 with  $A = \{0, 1\}^I$ , it immediately follows,

$$\sup_n \mathbb{P} \left( \left| \frac{T_{n+k} - T_n}{(4 + \varepsilon)k} - 1 \right| > \delta \right) < e^{-c\delta k^{1/13}}.$$

This, together with the statement of Theorem 3.3.1 gives

$$\mathbb{P} \left( \left| \frac{1}{T_{n+k} - T_n} \int_{T_n}^{T_{n+k}} \mathbf{1}(\eta_t(I) \in A) dt - Q_I(A) \right| > \delta \right) < e^{-c\delta k^{1/13}}.$$

Taking (polynomial in  $n$  number of) union bounds over  $m, \ell$  such that  $T_m \leq T_n + U < T_{m+1}$  and  $T_{m+\ell} \leq T_n + U + s < T_{m+\ell+1}$ , along with the bounds for  $T_m/m$ , this implies

$$\mathbb{P} \left( \left\{ U \in \left[ \frac{n^2}{3}, 2n^2 \right] \right\} \cap \left\{ \left| \frac{1}{s} \int_{T_n+U}^{T_n+U+s} \mathbf{1}(\eta_t(I) \in A) dt - Q_I(A) \right| > \delta \right\} \right) < e^{-c\delta s^{1/30}}.$$

Hence,

$$\begin{aligned} & \mathbb{E} \left| \frac{1}{s} \int_{T_n+U}^{T_n+U+s} \mathbf{1}(\eta_t(I) \in A) dt - Q_I(A) \right| \\ & \leq \delta + \mathbb{P} \left( \left| \frac{1}{s} \int_{T_n+U}^{T_n+U+s} \mathbf{1}(\eta_t(I) \in A) dt - Q_I(A) \right| > \delta \right) \leq 2\delta + e^{-c\delta s^{1/30}}. \end{aligned}$$

□

For the remainder of this section, we shall need the following notations. Fix  $n \in \mathbb{N}$  and define

$$a_i = in^{1/6}, \text{ for } i = 0, 1, 2, \dots, n^{1/6} + 1.$$

Let  $\mathcal{F}_n = \sigma\{\xi_{i,j} : (i,j) \notin \{(a_m, a_m) : m = 1, 2, \dots, n^{1/6}\}\}$  denote the  $\sigma$ -field generated by all the vertex weights except at the locations  $(a_i, a_i)$ . Also let

$$\mathcal{G}_n = \sigma\{\xi_{i,j} : i \in [0, 2n^{1/3}], j \in [0, 2n^{1/3}]\}.$$

For  $n \in \mathbb{N}$ , define

$$G'_n = T_{n,n} - \mathbb{E}(T_{n,n} | \mathcal{F}_n).$$

With these notations, we are in a position to state the next lemma which is required to prove Theorem 3.4.1.

**Lemma 3.4.3.** *In the above set up, there exists a  $\mathcal{G}_n$ -measurable random variable  $G_n$  such that  $\mathbb{P}(G'_n = G_n) \geq 1 - e^{-cn^{1/24}}$ . Moreover  $G_n$  is a sum of  $n^{1/6}$  many i.i.d. random variables with a non lattice distribution and having mean 0 and variance  $\tau^2 \in [a, b]$  for some absolute positive constants  $a, b$ .*

The argument is standard and almost mimics that in the proof of Proposition 3.3.2.

*Proof.* For any region  $B \in \mathbb{R}^2$ , let  $\Gamma(B) := \Gamma \cap B$  denote the part of  $\Gamma$  inside the region  $B$ , and its weight as  $T(B)$ . Consider the points

$$p_i := \frac{a_{i-1} + a_i}{2}, \text{ for } i = 1, 2, \dots, n^{1/6} + 1.$$

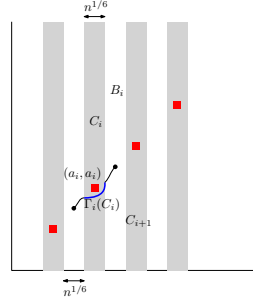


Figure 3.5: The vertices marked in red are the ones not revealed.  $T_i(C_i)$  are i.i.d. random variables whose sum we use to capture the fluctuation in  $T_n$  after all the remaining vertices are revealed.

Further define

$$m_i = \frac{p_i + a_i}{2}, \text{ for } i = 1, 2, \dots, n^{1/6} + 1, \text{ and } n_i = \frac{a_i + p_{i+1}}{2}, \text{ for } i = 1, 2, \dots, n^{1/6}.$$

The geodesic  $\Gamma_{(a_i+1, a_i+1), (a_{i+1}-1, a_{i+1}-1)}$  from  $(a_i + 1, a_i + 1)$  to  $(a_{i+1} - 1, a_{i+1} - 1)$  and the geodesic  $\Gamma_{n,n}$  meet together on the diagonal between  $a_i$  and  $n_i$  and again between  $m_{i+1}$  and  $a_{i+1}$ , and hence coincides between  $n_i$  and  $m_{i+1}$ , with probability atleast  $1 - e^{-cn^{1/24}}$  by Corollary 3.2.8. Let

$$B_i = \{(x, y) : n_i \leq x \leq m_{i+1}\} \text{ for } i = 1, 2, \dots, n^{1/6},$$

$$D = \{(x, y) : x \geq 2n^{1/3}\},$$

$$C_i = \{(x, y) : m_i \leq x \leq n_i\} \text{ for } i = 1, 2, \dots, n^{1/6}.$$

Note that, of these, only the sets  $C_i$  s contain the unrevealed locations  $(a_i, a_i)$ . Since for all  $i$ , the geodesics  $\Gamma_{(a_i+1, a_i+1), (a_{i+1}-1, a_{i+1}-1)}$  are  $\mathcal{F}_n$  measurable, hence

$$\mathbb{P}\left(T_{n,n}(B_i) - \mathbb{E}(T_{n,n}(B_i)) | \mathcal{F}_n = 0 \text{ for all } i\right) \geq 1 - n^{1/6} e^{-cn^{1/24}}.$$

A similar argument shows that  $T_{n,n}(D) - \mathbb{E}(T_{n,n}(D)) | \mathcal{F}_n = 0$  with probability atleast  $1 - e^{-cn^{1/24}}$ .

Let  $\Gamma_i := \Gamma_{(p_i, p_i), (p_{i+1}, p_{i+1})}$  and  $T_i(C_i)$  denote the weight of  $\Gamma_i \cap C_i$ . Then, repeating similar calculations,  $\Gamma_i$  and  $\Gamma_{n,n}$  coincide inside  $C_i$  for all  $i = 1, 2, \dots, n^{1/6}$  with probability atleast  $1 - n^{1/6} e^{-cn^{1/24}}$ . Define

$$G_n = \sum_i^{n^{1/6}} (T_i(C_i) - \mathbb{E}(T_i(C_i)) | \mathcal{F}_n). \quad (3.11)$$

Clearly  $G_n$  is  $\mathcal{G}_n$ -measurable and it follows that

$$\mathbb{P}(G'_n = G_n) \geq 1 - e^{-cn^{1/24}}.$$

Observe that  $T_i(C_i) - \mathbb{E}(T_i(C_i)|\mathcal{F}_n)$ ,  $i = 1, 2, \dots, n^{1/6}$  are i.i.d. mean zero random variables with a non lattice distribution. That they have bounded variance follows from Lemma 3.6.1. This completes the proof of Lemma 3.4.3.  $\square$

Now we begin the proof of Theorem 3.4.1.

*Proof of Theorem 3.4.1.* Fix  $t \geq 0$  sufficiently large. Fix  $n = \lfloor \sqrt{t} \rfloor$ . Fix any  $\delta > 0$ . Recall that  $G_n$  was defined in (3.11). Let  $M$  be a large enough constant such that  $-Mn^{1/12} \leq G_n \leq Mn^{1/12}$  with probability at least  $1 - \delta$ .

Define

$$h(g) := \eta_{(T_n+t-\mathbb{E}(T_n|\mathcal{F}_n)-g)}(I).$$

Then by Lemma 3.4.3,

$$\mathbb{P}(h(G_n) = \eta_t(I)) \geq 1 - e^{-cn^{1/24}}.$$

Now observe that if  $\mathbb{E}(T_n|\mathcal{F}_n) \leq \frac{n^2}{2}$ , and  $-Mn^{1/12} \leq g \leq Mn^{1/12}$ , then  $\frac{n^2}{3} \leq t - \mathbb{E}(T_n|\mathcal{F}_n) - g \leq 2n^2$ . Also for any  $0 \leq c \leq T_{2n^2} - T_n$ ,  $\eta_{T_n+c}(I)$  is a function of the differences  $T_{x,y} - T_n$  where  $(x, y) \in D$  where  $D$  is the  $2b$  sized strip along the diagonal in  $\text{Box}((n - b, n - b), (2n^2 + b, 2n^2 + b))$ . Let  $E$  be the event that there exists  $u \in \llbracket n^{1/2}, n - b \rrbracket$  such that  $(u, u) \in \bigcap_{(x,y) \in D} \Gamma_{\mathbf{0},(x,y)} \bigcap_{(x,y) \in D} \Gamma_{(n^{1/2}, n^{1/2}), (x,y)}$ . Then union bound and Corollary 3.2.7 imply that  $\mathbb{P}(E) \geq 1 - e^{-cn^{1/8}}$ . On  $E$ ,  $\eta_{T_n+c}(I)$  is a function of the differences  $T_{(n^{1/2}, n^{1/2}), (x,y)} - T_{(n^{1/2}, n^{1/2}), (n,n)}$  which is  $(\mathcal{G}_n)^c$  measurable. Then on the event that  $\mathbb{E}(T_n|\mathcal{F}_n) \leq \frac{n^2}{2}$ , there exists a function  $h'$  which is  $(\mathcal{G}_n)^c$  measurable, such that for each  $g \in [-Mn^{1/12}, Mn^{1/12}]$ ,

$$\mathbb{P}(h(g) = h'(g + \mathbb{E}(T_n|\mathcal{F}_n))) \geq 1 - e^{-cn^{1/8}}.$$

Then,

$$\begin{aligned} & \mathbb{P}(\eta_t(I) \in A) \\ &= \mathbb{P}(h(G_n) \in A) + R_n \\ &= \mathbb{E}(\mathbb{P}(h(G_n) \in A)|\mathcal{F}_n) + R_n \\ &= \mathbb{E} \left( \int \mathbb{P}(h(g) \in A|\mathcal{F}_n, G_n = g) f_{G_n|\mathcal{F}_n}(g) dg \right) + R_n \\ &= \mathbb{E} \left( \int \mathbb{P}(h'(g + \mathbb{E}(T_n|\mathcal{F}_n)) \in A|\mathcal{F}_n, G_n = g) f_{G_n|\mathcal{F}_n}(g) dg \right) + R'_n \\ &= \mathbb{E} \left( \int \mathbb{P}(h'(g + E(T_n|\mathcal{F}_n)) \in A|\mathcal{F}_n) f_{G_n|\mathcal{F}_n}(g) dg \right) + R'_n \\ &= \int \mathbb{E}(\mathbb{P}(h'(g + E(T_n|\mathcal{F}_n)) \in A|\mathcal{F}_n)) f_{G_n}(g) dg + R'_n \\ &= \int \mathbb{P}(h(g) \in A) f_{G_n}(g) dg + R''_n, \end{aligned}$$

where  $|R_n|, |R'_n|, |R''_n| \leq 3e^{-n^{1/24}}$ , by interchanging the integral and expectation, and the fact that given  $\mathcal{F}_n, h'$  and  $G_n$  are conditionally independent.

Fix  $A \subseteq \{0, 1\}^I$ . Since by Lemma 3.6.2,  $\psi(g) = \mathbb{P}(h(g) \in A)$  is uniformly continuous, choose  $h > 0$  such that  $\sup_{|g-g'| \leq h} |\psi(g) - \psi(g')| \leq \delta$ . For this  $h > 0$ , applying local central limit theorem to  $G_n$ , due to Lemma 3.4.3, we have,

$$\mathbb{P}(a \leq G_n \leq a+h) = h\phi_n(a) + o(1/n^{1/12}),$$

where the error term  $o(1/n^{1/12})$  is uniform in  $a$ , and  $\phi_n$  denotes the density of  $N(0, \tau^2 n^{1/6})$  distribution, where  $\tau$  is bounded. Then,

$$\left| \int \mathbb{P}(h(g) \in A) f_{G_n}(g) dg - \int \mathbb{P}(h(g) \in A) \phi_n(g) dg \right| \leq 4\delta + o(1).$$

Now,

$$\int \mathbb{P}(h(g) \in A) \phi_n(g) dg = \int \mathbb{P}(\eta_{T_n+U-g}(I) \in A) \phi_n(g) dg = \mathbb{E} \left( \int \mathbf{1}(\eta_{T_n+U-g}(I) \in A) \phi_n(g) dg \right),$$

where  $U = t - \mathbb{E}(T_n | \mathcal{F}_n) \in [\frac{n^2}{2}, n^2]$  with probability at least  $1 - \delta$ . Now, if  $\phi$  denotes the density of  $Z \sim N(0, \tau^2)$ , then get  $R$  large such that  $\mathbb{P}(|Z| \geq R) \leq \delta$ . Also let  $\beta$  be the modulus of continuity of the Gaussian density corresponding to this  $\delta$ . Divide  $[-R, R]$  into points  $a_1 = -R, a_2, \dots, a_r = R$  such that  $|a_i - a_{i+1}| = \beta$  (so that  $\sup_{x \in [a_i, a_{i+1}]} |\phi(x) - \phi(a_i)| \leq \delta$ ). Then if  $b_i := n^{1/12} a_i$ , then  $b_{i+1} - b_i = n^{1/12} \beta$  and,

$$\sup_{x \in [b_i, b_{i+1}]} |\phi_n(x) - \phi_n(b_i)| \leq \frac{\delta}{n^{1/12}}.$$

Hence, using Lemma 3.4.2, one gets,

$$\begin{aligned} & \left| \mathbb{E} \left( \int \mathbf{1}(\eta_{T_n+U-g}(I) \in A) \phi_n(g) dg \right) - Q_I(A) \right| \\ & \leq \left| \mathbb{E} \left( \sum_{i=1}^{r-1} \int_{b_i}^{b_{i+1}} \mathbf{1}(\eta_{T_n+U-g}(I) \in A) \phi_n(g) dg \right) - Q_I(A) \right| + \delta \\ & \leq \left| \mathbb{E} \left( \sum_{i=1}^{r-1} \int_{b_i}^{b_{i+1}} \mathbf{1}(\eta_{T_n+U-g}(I) \in A) \phi_n(b_i) dg \right) - Q_I(A) \right| + 2R\delta + \delta \\ & = \left| \sum_{i=1}^{r-1} \phi_n(b_i) (b_{i+1} - b_i) \mathbb{E} \left( \frac{1}{b_{i+1} - b_i} \int_{T_n+U-b_{i+1}}^{T_n+U-b_i} \mathbf{1}(\eta_s(I) \in A) ds \right) - Q_I(A) \right| + 2R\delta + \delta \\ & \leq \sum_{i=1}^{r-1} \phi_n(b_i) (b_{i+1} - b_i) \left( 2\delta + e^{-c\delta(b_{i+1}-b_i)^{1/30}} \right) + Q_I(A) \left| \sum_{i=1}^{r-1} \phi_n(b_i) (b_{i+1} - b_i) - 1 \right| + 2R\delta + \delta \\ & = \sum_{i=1}^{r-1} \phi_n(b_i) (b_{i+1} - b_i) \left( 2\delta + e^{-c\delta(n^{1/12}\beta)^{1/30}} \right) + 4R\delta + 2\delta \\ & \leq \left( 2\delta + e^{-c\delta(n^{1/12}\beta)^{1/30}} \right) + 2R\delta \left( 2\delta + e^{-c\delta(n^{1/12}\beta)^{1/30}} \right) + 4R\delta + 2\delta. \end{aligned}$$

Now let  $n \rightarrow \infty$ , and then  $\delta \rightarrow 0$ , and note that  $R\delta \rightarrow 0$  as  $R \sim \sqrt{\log(1/\delta)} \ll \frac{1}{\sqrt{\delta}}$  as  $\delta \rightarrow 0$ .  $\square$

### 3.5 Coupling TASEP with a slow bond with a stationary TASEP

We complete the proof of Theorem 4 in this section. Recall  $\rho$  from Remark 3.1.2. Now fix  $L \in \mathbb{N}$  and set  $I = [0, L]$ . For  $k \in \mathbb{N}$ , consider the interval  $k + I$ . We shall define a coupling between the stationary TASEP with density  $\rho$  (i.e., with product  $\text{Ber}(\rho)$  particle configuration) and the TASEP with a slow bond started from the step initial condition. We shall show that under this coupling for all  $k$  sufficiently large, the asymptotic occupation measure for  $k + I$  in the slow bond model is with probability close to one equal to the occupation measure of  $I$  in the stationary TASEP with density  $\rho$ . This implies the total variation distance between the two measures is small. By stationarity one of them is close to product  $\text{Ber}(\rho)$ , and hence the other must be so too. This will establish that the limiting stationary measure  $\nu_*$  of the slow bond process is asymptotically equivalent to  $\nu_\rho$  at  $\infty$ . By an identical argument one can establish asymptotic equivalence to  $\nu_{1-\rho}$  at  $-\infty$ . The crux of the argument will be to show that the coupling works with large probability and to show that we first need to consider the last passage percolation formulation of TASEP with an arbitrary initial condition, in particular a stationary one.

#### 3.5.1 Last passage percolation and stationary TASEP

The correspondence we described between TASEP started from step initial condition extends to arbitrary initial condition as follows. Let  $\eta \in \{0, 1\}^{\mathbb{Z}}$  be an arbitrary particle configuration. Let  $S_\eta$  be a bi-infinite connected subset of  $\mathbb{Z}^2$  (an injective image of  $\mathbb{Z}$ ) defined recursively. Define  $F : \mathbb{Z} \rightarrow \mathbb{Z}^2$  as follows: set  $F(0) = (0, 0)$ . For  $i > 0$ , set  $F(i) = F(i-1) + (0, -1)$  if  $\eta(i) = 1$  and set  $F(i) = F(i-1) + (1, 0)$  otherwise. For  $i < 0$ , set  $F(i) = F(i+1) + (0, 1)$  if  $\eta(i+1) = 1$  and  $F(i) = F(i+1) + (-1, 0)$  otherwise. Let  $S_\eta = F(\mathbb{Z})$ . Clearly  $S_\eta$  is a connected subset of  $\mathbb{Z}^2$  that divides  $\mathbb{Z}^2$  into two connected components (see Figure 3.6) one of which (let us call it  $R_\eta$ ) contains the whole positive quadrant. Also set  $R = R_\eta \cup S_\eta$ . Fix  $\ell \in \mathbb{Z}$ ; let  $a \in \mathbb{Z}$  be the smallest number such that  $(a + \ell, a) \in R$ . Consider the following coupling between TASEP with initial condition  $\eta$  and Last Passage Percolation with i.i.d. exponential weights by setting  $\xi_{a+j+\ell, a+j}$  (the edge weight at vertex  $(a + j + \ell, a + j)$ ) to be equal to the waiting time for the  $(j + 1)$ -th jump at site  $\ell$ . The following standard result gives the correspondence between last passage times and jump times under this coupling. We omit the proof, see e.g. [8].

**Proposition 3.5.1.** *Let  $\eta \in \{0, 1\}^{\mathbb{Z}}$  be fixed and consider the coupling described above. Fix  $\ell \in \mathbb{Z}$  and let  $v_n = (n + \ell, n)$ . Then the time taken for the  $n$ -th particle to left of the origin*

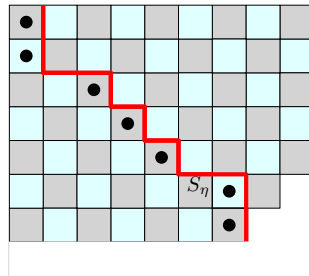


Figure 3.6: Correspondence between Last Passage Percolation and TASEP with general initial condition. The red line in the figure depicts a part of the boundary  $S_\eta$  between  $R_\eta$  and  $\mathbb{Z}^2 \setminus R_\eta$  for a part of the configuration  $\eta$  given by 110101010011. Jump times in TASEP started with initial condition  $\eta$  corresponds to last passage times from  $S_\eta$  to various vertices of  $\mathbb{Z}^2$ .

to jump through site  $\ell$  is equal to

$$\sup_{v \in S_\eta} T_{v, v_n}$$

where  $T_{v, v_n}$  denotes the usual last passage time between  $v$  and  $v_n$ .

Observe that in case  $\eta = \mathbf{1}_{(-\infty, 0]}$ , i.e., for step initial condition the set  $S_\eta$  is just the boundary of the positive quadrant of  $\mathbb{Z}^2$  and hence point to line (or general set  $S_\eta$ ) passage time reduce to point to origin passage time in that case and hence this result is consistent with the previous correspondence between TASEP and LPP that we quoted.

Let us now specialise to the stationary  $\text{Ber}(\rho)$  initial condition, i.e., in  $\eta$  each site is occupied with probability  $\rho$  independent of the others. Clearly in such a case the gap between two consecutive particles is a geometric random variable with mean  $\frac{1}{\rho} - 1$ . Hence in this case  $S_\eta$  is a random staircase curve passing through the origin that has horizontal steps of length that are distributed as i.i.d.  $\text{Geom}(\rho)$  and two consecutive horizontal steps (can also be of length 0) are separated by vertical steps of unit length. By a large deviation estimate on geometric random variables this random line  $S_\eta$  can be approximated by the deterministic line  $\mathbb{L}$  given by

$$y = -\frac{\rho}{1 - \rho}x;$$

so for our purposes we can approximate  $S_\eta$  with  $\mathcal{L}$  and consider the corresponding last passage times to  $\mathbb{L}$ . Before making a precise statement, let us first consider the last passage percolation to the deterministic line  $\mathcal{L}$ . Fix  $\ell \in \mathbb{Z}$  and consider the geodesic from  $\mathbb{L}$  to  $(n + \ell, n)$  for  $n$  large. If  $n \gg \ell$ , this geodesic is quite close to the geodesic from  $\mathbb{L}$  to  $(n, n)$ . Comparing the first order of the length of the geodesic from  $v$  to  $(n, n)$  for different  $v \in \mathbb{L}$  it is not too hard to see that the first order distance, i.e.,  $(\sqrt{n - x} + \sqrt{n - y})^2$  for all  $(x, y) \in \mathcal{L}$  is maximized at  $(nx_0, ny_0) \in \mathbb{L}$  where  $(x_0, y_0)$  is given by

$$(x_0, y_0) = \left( -\frac{1-2\rho}{\rho}, \frac{1-2\rho}{1-\rho} \right). \tag{3.12}$$

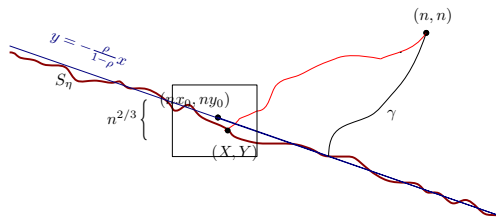


Figure 3.7: Proposition 3.5.2: we compare the best path to  $S_\eta$  from  $(n, n)$  to a path  $\gamma$  that intersects  $S_\eta$  far from  $(nx_0, ny_0)$ . Because of large deviation estimates, at that point  $S_\eta$  is not too far from the line  $y = -\frac{\rho}{1-\rho}x$ . By computing expectation we show that that expected length of  $\gamma$  is less than that of the best geodesic by an amount larger than the natural fluctuation scale.

Note that  $(nx_0, ny_0) \neq (0, 0)$  whenever  $\rho \neq \frac{1}{2}$ . It follows from this that the point where the geodesic from  $(n + \ell, n)$  to  $\mathbb{L}$  hits  $\mathbb{L}$  should be at distance  $O(n^{2/3})$  from  $(nx_0, ny_0)$ . In fact the same remains true for the geodesic from  $(n + \ell, n)$  to the random curve  $S_\eta$ . More precisely we have the following proposition.

**Proposition 3.5.2.** *Fix  $\rho \in (0, 1), \rho \neq \frac{1}{2}$ . Let  $(x_0, y_0)$  be the point on the line  $\mathbb{L}$  given by (3.12) and let  $\ell \in \mathbb{Z}$  be fixed. Let  $\Gamma_{n+\ell, n}^S$  be the geodesic from  $(n + \ell, n)$  to the random line  $S_\eta$ . Let  $(X_\ell, Y_\ell) \in S_\eta$  be such that  $\Gamma_{n+\ell, n}^S = \Gamma_{(X_\ell, Y_\ell), (n+\ell, n)}^0$ , where  $\Gamma_{u, v}^0$  is the point to point geodesic from  $u$  to  $v$  in the usual Exponential DLPP. Then given any  $\delta > 0$  however small, there exists  $M = M(\delta, \ell)$  such that,*

$$\mathbb{P}(\|(X, Y) - (nx_0, ny_0)\| \geq Mn^{2/3}) \leq \delta.$$

The proof of this proposition follows from a computation balancing expectation and fluctuation and using Theorem 2.1.2 to bound the tails. We shall omit the proof. The argument is by now standard and has been used a number of times in bounding transversal fluctuation for geodesics in various polymer models in KPZ universality class; see e.g. [16, Theorem 11.1]. In the setting of point-to-line last passage percolation this was considered in the very recent preprint [33]. Indeed, Lemma 4.3 of [33] shows that the geodesic  $(n + \ell, n)$  to any point on  $\mathcal{L}$  that is more than a distance of  $Mn^{2/3}$  from  $(nx_0, ny_0)$  is smaller than the geodesic to  $(nx_0, ny_0)$  with probability close to one. Also, by suitably applying Chernoff bound, one can show that  $|S_\eta(v) - L(v)| = O(n^{1/2})$  for  $|v| = O(n)$  with high probability, implying the proposition.

### 3.5.2 Convergence to product Bernoulli measure

We shall complete the proof of Theorem 4 now. Recall the invariant measure  $Q$  constructed in Section 3.3 and also recall the definition of  $\rho < \frac{1}{2}$  from Remark 3.1.2. For an interval  $[a, b]$  let  $Q_{[a,b]}$  be the projection of  $Q$  onto the coordinates in  $[a, b]$ . It suffices to prove that for each fixed  $L \in \mathbb{N}$

$$Q_{[k, k+L]}(\beta_0, \dots, \beta_L) \xrightarrow{k \rightarrow \infty} \rho^{\sum_i \beta_i} (1 - \rho)^{L+1 - \sum_i \beta_i}, \quad \text{and} \quad (3.13)$$

$$Q_{[-k-L, -k]}(\beta_0, \dots, \beta_L) \xrightarrow{k \rightarrow \infty} \rho^{L+1 - \sum_i \beta_i} (1 - \rho)^{\sum_i \beta_i} \quad (3.14)$$

for every  $\beta = (\beta_0, \dots, \beta_L) \in \{0, 1\}^{L+1}$ . This will complete the proof of Theorem 4 with  $Q = \nu_*$ . We shall only prove (3.13) and the second equation will follow from an identical proof.

We first set up the following notations. Here  $k' = ck$  is a constant multiple of  $k$  and  $n \gg k$ . The dependence among the various parameters is summarised below.

1.  $L$  denotes a fixed constant.
2.  $\delta > 0$  will denote some predefined quantity however small.
3.  $R$  denotes a sufficiently large constant, to be chosen appropriately later, depending only on  $L, \delta$ .
4.  $k$  is chosen large enough depending on  $L, \delta, R$ .
5.  $s$  is chosen large enough depending on  $k, L, \delta, R$ .
6.  $n \geq s$ .

Fix  $\rho < \frac{1}{2}$  from Remark 3.1.2. We shall call the LPP model corresponding to the stationary TASEP with product  $\text{Ber}(\rho)$  configuration as considered in the previous subsection the stationary model, the geodesics from  $(m + \ell, m)$  to the random line  $S_\eta$  as  $\Gamma_{m+\ell, m}^S$  and its weight as  $T_{m+\ell, m}^S$ .

Recall that  $x_1 = n - a(\varepsilon)k$  was defined for the reinforced model in Proposition 3.2.11 where  $a(\varepsilon)$  was calculated in (3.3) in the Introduction. Also  $(x_0, y_0)$  was defined in the stationary model in (3.12). Fix  $L, R \in \mathbb{N}$ . Define  $k'$  such that

$$k' - k'x_0 = k - (R + L).$$

Let  $v$  be the vertex where the line joining  $(x_1, x_1)$  to  $(n + k, n)$  intersect the vertical line  $x = n + R + L$ , i.e.,  $v = (v_1, v_2) = (n + R + L, n - \frac{a(\varepsilon)(k-R-L)}{a(\varepsilon)+1})$ , and for the reinforced model define

$$B_R := \text{Box}(v, (n + k, n)).$$

Also, for the stationary model, define

$$B_S := \text{Box}((k'x_0, k'y_0), (k', k')).$$

Since  $\rho$  is chosen such that the slopes of the line joining  $(x_1, x_1)$  to  $(n+k, n)$  match that of the line joining  $(k'x_0, k'y_0)$  to  $(k', k')$ , hence, the dimensions of the two boxes,  $B_S$  in the stationary model, and  $B_R$  in the reinforced model are same.

We consider  $2L + R$ -sized boxes at the vertices  $(k', k')$  and  $(n+k, n)$  of the two boxes  $B_S$  and  $B_R$ . To be precise, consider the box  $\text{Box}((k' - L, k' - L), (k' + R + L, k' + R + L))$  in the stationary model and let  $D_S$  be the  $L$  sized strip along and above the diagonal of this box, i.e.,  $D_S$  is the quadrilateral with endpoints  $(k' - L, k')$ ,  $(k', k')$ ,  $(k' + R, k' + R)$ ,  $(k' + R, k' + R + L)$ . Similarly consider the box  $\text{Box}((n+k - L, n - L), (n+k + R + L, n + R + L))$  in the reinforced model, and let  $D_R$  be the  $L$  sized strip along and above the diagonal of this box.

Also slightly enlarge the two boxes  $B_S$  and  $B_R$ , so that  $B_S^2 := \text{Box}((k'x_0, k'y_0), (k' + R + L, k' + R + L))$ , and  $B_R^2 := \text{Box}(v, (n+k + R + L, n + R + L))$ . Observe that both the boxes  $B_S^2$  and  $B_R^2$  have i.i.d.  $\text{Exp}(1)$  random variables at each interior vertex. See Figure 3.8.

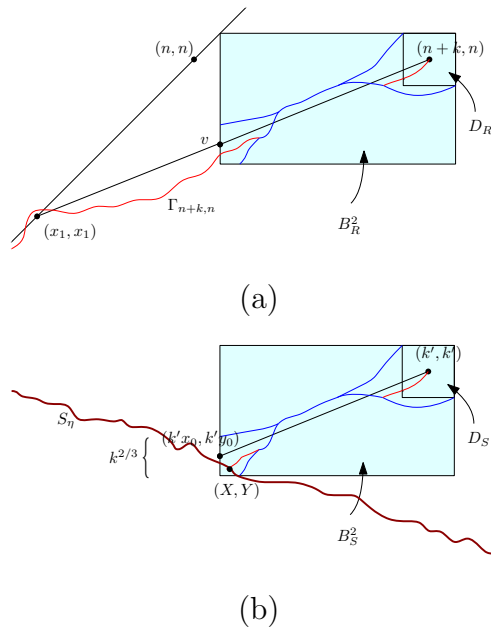


Figure 3.8: Coupling between TASEP with a slow bond in (a) and Stationary TASEP with density  $\rho$  in (b). The point  $v$  in (a) corresponds to the point  $(k'x_0, k'y_0)$  in (b) and the point  $(n+k, n)$  in (a) corresponds to the point  $(k', k')$  in (b). We couple the two systems so the the configuration of passage times in the box  $B_R^2$  in (a) is identical (upto translation) to that in  $B_S^2$  in (b). We use the fact that the geodesics  $\Gamma_{0,u}$  for all  $u$  in  $D_R$  pass through some point close to  $v$ , and the geodesics  $\Gamma_u^S$  for all  $u \in D_S$  pass through some point close to  $(k'x_0, k'y_0)$  and the coalescence result Theorem 3.1.5 to argue that the pairwise difference of those geodesics are identical in the two systems with probability close to one if  $k$  is large.

We have the following basic lemma that relates the expected occupation measures in the reinforced and stationary models.

**Lemma 3.5.3.** *Fix  $L, R \in \mathbb{N}$  and consider the above set up. Let  $\mathcal{H}_S$  be the event there exists some vertex  $u$  such that  $u \in \bigcap_{(x,y) \in D_S} \Gamma_{x,y}^S \bigcap_{(x,y) \in D_S} \Gamma_{(k'x_0, k'y_0), (x,y)}^0$ . Similarly let  $\mathcal{H}_R$  be the event that there exists some vertex  $u'$  such that  $u' \in \bigcap_{(x,y) \in D_R} \Gamma_{(0,0), (x,y)} \bigcap_{(x,y) \in D_R} \Gamma_{v, (x,y)}$ . Then on the event  $\mathcal{H}_S \cap \mathcal{H}_R$ , under the coupling of all exponential random variables at the corresponding vertices in the two boxes  $B_S^2$  and  $B_R^2$ , for any  $A \subseteq \{0, 1\}^{L+1}$ ,*

$$\frac{1}{R} \int_{T_{\mathbf{0}, (n+k, n)}}^{T_{\mathbf{0}, (n+k+R, n+R)}} \mathbf{1}(\eta_t[k, k+L] \in A) dt = \frac{1}{R} \int_{T_{k', k'}}^{T_{k'+R, k'+R}} \mathbf{1}(\eta_t^S[1, L+1] \in A) dt, \quad (3.15)$$

where  $\eta_t$  is the configuration of the TASEP with a slow bond started from the step initial condition and  $\eta_t^S$  is the configuration of the stationary TASEP with density  $\rho$ , and  $\eta_t[k, k+L]$  and  $\eta_t^S[1, L+1]$  are the configurations restricted to  $[k, k+L]$  and  $[1, L+1]$ .

*Proof.* Let  $A_S = \{\Gamma_{x,y}^S : (x,y) \in D_S\}$ ,  $A_S^0 = \{\Gamma_{(k'x_0, k'y_0), (x,y)}^0 : (x,y) \in D_S\}$ ,  $A_R = \{\Gamma_{(0,0), (x,y)} : (x,y) \in D_R\}$ ,  $A_R^0 = \{\Gamma_{v, (x,y)} : (x,y) \in D_R\}$ . By Lemma 3.3.3 it is easy to see that the occupation density  $\frac{1}{R} \int_{T_{k', k'}}^{T_{k'+R, k'+R}} \mathbf{1}(\eta_t^S[1, L+1] \in A) dt = f(A_S)$  for some function  $f$  such that  $f(\mathbf{x} + c) = f(\mathbf{x})$ . Hence, on  $\mathcal{H}_S$ , since the differences in the lengths of the maximal paths in the set  $A_S$  are the same as the differences in the lengths of the corresponding maximal paths starting from  $(k'x_0, k'y_0)$ ,

$$\frac{1}{R} \int_{T_{k', k'}}^{T_{k'+R, k'+R}} \mathbf{1}(\eta_t^S[1, L+1] \in A) dt = f(A_S^0).$$

Similarly, on  $\mathcal{H}_R$ ,

$$\frac{1}{R} \int_{T_{\mathbf{0}, (n+k, n)}}^{T_{\mathbf{0}, (n+k+R, n+R)}} \mathbf{1}(\eta_t[k, k+L] \in A) dt = f(A_R^0).$$

Since under the coupling,  $f(A_S^0) = f(A_R^0)$ , the result follows.  $\square$

The next proposition says that the expected occupation measures in the reinforced and stationary models are close.

**Proposition 3.5.4.** *Fix  $L, R \in \mathbb{N}$  and  $A \subseteq \{0, 1\}^{L+1}$  and  $\delta > 0$ . Let*

*$Z_R = \frac{1}{R} \int_{T_{\mathbf{0}, (n+k, n)}}^{T_{\mathbf{0}, (n+k+R, n+R)}} \mathbf{1}(\eta_t[k, k+L] \in A) dt$  and  $Z_S = \frac{1}{R} \int_{T_{k', k'}}^{T_{k'+R, k'+R}} \mathbf{1}(\eta_t^S[1, L+1] \in A) dt$  be the occupation densities in the reinforced and stationary models. Then there exist positive constants  $C, c$  depending only on  $\delta, L, R$  (and not on  $k$ ) such that*

$$|\mathbb{E}Z_R - \mathbb{E}Z_S| \leq \sqrt{\frac{C}{k^c}} + \delta.$$

*Proof.* First observe that, following similar arguments as in Lemma 3.6.3, the random variables  $Z_S$  and  $Z_R$ , which are bounded by  $\frac{T_{k'+R, k'+R}^S - T_{k', k'}^S}{R}$  and  $\frac{T_{\mathbf{0}, (n+k+R, n+R)} - T_{\mathbf{0}, (n+k, n)}}{R}$ , are  $\mathcal{L}^2$

bounded. Let  $C^*$  be the sum of squares of these two  $\mathcal{L}^2$  bounds, so that  $C^*$  is an absolute positive constant, not depending on  $k, R, \delta$ .

Recall the events  $\mathcal{H}_S, \mathcal{H}_R$  defined in Lemma 3.5.3. Let  $U_S$  be the line segment joining  $(k'x_0 + M(k')^{2/3}, k'y_0 - 2M(k')^{2/3})$  and  $(k'x_0 + M(k')^{2/3}, k'y_0 + 2M(k')^{2/3})$  for some fixed  $M = M(\delta)$ . Let  $E_S$  be the event that the geodesics in  $A_S := \{\Gamma_{x,y}^S : (x, y) \in D_S\}$  and the geodesics in  $A_S^0 = \{\Gamma_{(k'x_0, k'y_0), (x, y)}^0 : (x, y) \in D_S\}$  pass through the line segment  $U_S$ . Then Proposition 3.5.2 and Theorem 3 together with polymer ordering (Lemma 4.3.2) imply that, for the given  $\delta > 0$ , one can choose  $M = M(\delta)$  large enough such that

$$\mathbb{P}(E_S) \geq 1 - \frac{\delta}{2C^*}.$$

Let  $F_S$  denote the event that the geodesic from  $(k'x_0 + M(k')^{2/3}, k'y_0 - 2M(k')^{2/3})$  to  $(k' + L + R, k' - L)$  and the geodesic from  $(k'x_0 + M(k')^{2/3}, k'y_0 + 2M(k')^{2/3})$  to  $(k' - L, k' + L + R)$  meet together. Then it follows from Theorem 3.1.5 that

$$\mathbb{P}(F_S) \geq 1 - \frac{C'}{k^{c'}},$$

for some  $C', c'$  depending on  $M$  and hence on  $\delta$ , but not on  $k$ . Due to polymer ordering,

$$\mathbb{P}(\mathcal{H}_S) \geq \mathbb{P}(E_S \cap F_S) \geq 1 - \frac{\delta}{2C^*} - \frac{C'}{k^{c'}},$$

Now we consider the reinforced model. Recall that  $v = (v_1, v_2) = (n + R + L, n - \frac{a(\varepsilon)(k-R-L)}{a(\varepsilon)+1})$ . Let  $\Gamma_1$  be the geodesic from  $(x_1 + k^{3/5}, x_1 + k^{3/5})$  to  $(n + k - L, n + L + R)$  that avoids the diagonal line segment joining  $(0, 0)$  to  $(n + R + L, n + R + L)$ . Also let  $\Gamma_2$  be the geodesic from  $(x_1 - k^{3/5}, x_1 - k^{3/5})$  to  $(n + k + L + R, n - L)$  that avoids the diagonal line segment joining  $(0, 0)$  to  $(n + R + L, n + R + L)$ . Then we claim that one can choose  $M = M(\delta)$  large enough such that

$$\mathbb{P}(\{|\Gamma_1(n + R + L) - v_2| \leq Mk^{2/3}\} \cap \{|\Gamma_2(n + R + L) - v_2| \leq Mk^{2/3}\}) \geq 1 - \frac{\delta}{2C^*}. \quad (3.16)$$

To see this, define  $\Gamma'_1$  to be the geodesics from  $(x_1 + k^{3/5}, x_1 + k^{3/5})$  to  $(n + k - L, n + L + R)$  with the diagonal line segment joining  $(0, 0)$  to  $(n + R + L, n + R + L)$  not reinforced (i.e. corresponding to the usual exponential DLPP). Observe that  $\Gamma_1$  can never be above  $\Gamma'_1$  and by Theorem 3, one can choose  $M(\delta)$  such that  $\mathbb{P}(\Gamma'_1(n + R + L) \leq v_2 + Mk^{2/3}) \geq 1 - \frac{\delta}{4C^*}$ . Also, if  $\Gamma_2(n + R + L) \leq v_2 - Mk^{2/3}$ , then for any such  $u_2 \leq v_2 - Mk^{2/3}$ , there exists some constant  $\alpha$  such that for  $u = (n + R + L, u_2)$ ,

$$\mathbb{E}(T_{(x_1 - k^{3/5}, x_1 - k^{3/5}), u}^0) + \mathbb{E}(T_{u, (n + k + L + R, n - L)}^0) \leq \mathbb{E}(T_{(x_1 - k^{3/5}, x_1 - k^{3/5}), (n + k + L + R, n - L)}^0) - \alpha Mk^{1/3}.$$

Using Proposition 12.2 of [16] for fluctuations of constrained paths, and moderate deviation estimates of supremum and infimum of geodesic lengths in Proposition 10.1 and 10.5 of [16],

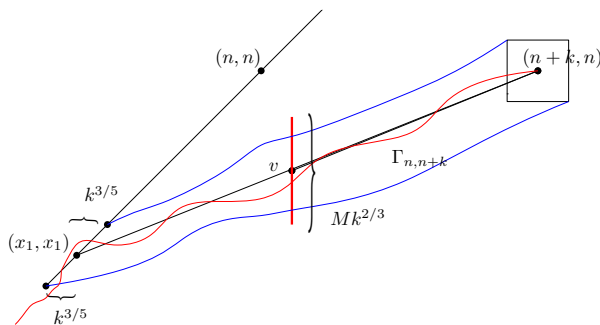


Figure 3.9: On the high probability event that  $\Gamma_{n+k,n}$  leaves the diagonal between  $x_1 - k^{3/5}$  and  $x_1 + k^{3/5}$ , it is sandwiched between the two geodesic marked in blue in the figure. By local path regularity estimate both of the blue paths are likely to intersect the vertical line through  $v_2$  at a distance no more than  $Mk^{2/3}$  for some large constant  $M$ , and hence the same is true for  $\Gamma_{n+k,n}$  with probability close to one.

and using standard arguments, one gets  $\mathbb{P}(\Gamma_2(n) \geq v_2 - Mk^{2/3}) \geq 1 - \frac{\delta}{4C^*}$  by choosing  $M$  large.

Let  $U_R$  be the line segment joining  $(n + R + L, v_2 - Mk^{2/3})$  and  $(n + R + L, v_2 + Mk^{2/3})$ . Let  $E_R$  be the event that the geodesics in  $A_R = \{\Gamma_{(0,0),(x,y)} : (x,y) \in D_R\}$  pass through the line segment  $U_R$ . Then Proposition 3.2.11 and equation (3.16) and polymer ordering imply

$$\mathbb{P}(E_R) \geq 1 - \frac{\delta}{2C^*} - C'e^{-ck^{1/20}}.$$

Let  $F_R$  denote the event that the geodesic from  $(n + R + L, v_2 - Mk^{2/3})$  to  $(n + k + L + R, n - L)$  and the geodesic from  $(n + R + L, v_2 + Mk^{2/3})$  to  $(n + k - L, n + L + R)$  meet together. Then it follows from Theorem 3.1.5 that there exists constants  $C', c'$  depending on  $M$  and hence on  $\delta$  such that

$$\mathbb{P}(F_R) \geq 1 - \frac{C'}{k^{c'}}.$$

Hence, again by polymer ordering,

$$\mathbb{P}(\mathcal{H}_R) \geq \mathbb{P}(E_R \cap F_R) \geq 1 - \frac{C'}{k^{c'}} - \frac{\delta}{2C^*}.$$

Let  $Z = Z_R - Z_S$  under the coupling in Lemma 3.5.3. Hence, due to Lemma 3.5.3 and using Cauchy-Schwarz inequality,

$$|\mathbb{E}Z_R - \mathbb{E}Z_S| \leq \mathbb{E}|Z| = \mathbb{E}|Z|\mathbf{1}_{(\mathcal{H}_S \cap \mathcal{H}_R)^c} \leq (C^*)^{1/2} \sqrt{\mathbb{P}((\mathcal{H}_S \cap \mathcal{H}_R)^c)} \leq \sqrt{\frac{C}{k^c}} + \delta.$$

□

To finish off the proof we shall show that the occupation measures are indeed close to the measure  $Q$  and the product Bernoulli measures respectively. To this end, we have the following lemma.

**Lemma 3.5.5.** *Fix  $L, R \in \mathbb{N}$  and  $A \subseteq \{0, \}^{L+1}$ . Recall that  $T_m := T_{\mathbf{0},(m,m)}$  in the reinforced model and  $Z_R := \frac{1}{R} \int_{T_{\mathbf{0},(n+k,n)}}^{T_{\mathbf{0},(n+k+R,n+R)}} \mathbf{1}(\eta_t[k, k+L] \in A) dt$ . Then for  $n \geq s \gg k^2$ ,  $k \geq R$ ,*

$$\left| \mathbb{E}(Z_R) - \mathbb{E} \left( \frac{1}{sR} \int_{T_n}^{T_{n+sR}} \mathbf{1}(\eta_t[k, k+L] \in A) dt \right) \right| \leq \frac{C'k}{sR} + C' e^{-cs^{1/4}} \leq \frac{C'}{kR} + C' e^{-ck^{1/2}},$$

where  $C', c$  are constants not depending on  $k, n, s, R$ .

*Proof.* Let  $T^{n,k,a} := T_{\mathbf{0},(n+k+a,n+a)}$ . Recall the definitions of  $D_R$  and  $A_R$  from Lemma 3.15. Using standard arguments, it is easy to see that, for  $n \geq s \gg k^2$ , the geodesics in  $A_R$  and the geodesics starting from  $(n-L-s, n-L-s)$  to the corresponding points in  $D_R$  meet the diagonal simultaneously with probability atleast  $1 - e^{-cs^{1/4}}$  by Corollary 3.2.8. This also holds true for  $n$  replaced by  $n+R, n+2R, \dots, n+(s-1)R$ .

Define the random variables

$$\begin{aligned} Y_i &:= \frac{1}{R} \int_{T^{n,k,(i-1)R}}^{T^{n,k,iR}} \mathbf{1}(\eta_t[k, k+L] \in A) dt, \quad i = 1, 2, \dots, s, \\ Z &:= \frac{1}{R} \int_{T^{s+L,k,0}}^{T^{s+L,k,R}} \mathbf{1}(\eta_t[k, k+L] \in A) dt. \end{aligned}$$

Then, using standard arguments, it follows that there exist random variables  $Z_i, i = 1, 2, \dots, s$ , such that  $Z_i \stackrel{d}{=} Z$  for each  $i$ , and  $\mathbb{P}(Y_i = Z_i \text{ for all } i) \geq 1 - se^{-cs^{1/4}}$ . Hence, using the  $\mathcal{L}^2$  boundedness of the random variables and Cauchy-Schwarz inequality, as in the previous proposition, we have,

$$|\mathbb{E}(Y_i) - \mathbb{E}(Y_1)| \leq |\mathbb{E}(Y_i) - \mathbb{E}(Z)| + |\mathbb{E}(Z) - \mathbb{E}(Y_1)| \leq C' e^{-\frac{c}{2}s^{1/4}}, \text{ for each } i = 1, 2, \dots, s,$$

where  $C'$  is some constant not depending on  $n, k, R, s$ . Hence,

$$\left| \mathbb{E} \left( \frac{1}{s} \sum_{i=1}^s Y_i \right) - \mathbb{E}(Y_1) \right| \leq C' e^{-\frac{c}{2}s^{1/4}}.$$

Now note that,

$$\frac{1}{s} \sum_{i=1}^s Y_i = \frac{1}{sR} \int_{T^{n,k,0}}^{T^{n,k,sR}} \mathbf{1}(\eta_t[k, k+L] \in A) dt,$$

and  $\mathbb{E} \left( \frac{T^{n,k,0} - T_n}{sR} \right)$  and  $\mathbb{E} \left( \frac{T^{n,k,sR} - T_{n+sR}}{sR} \right)$  are less than  $\frac{C'k}{sR}$  by again using Lemma 3.6.3.  $\square$

Finally putting all of these together we get the result.

**Theorem 3.5.6.** *Let  $\nu_\rho$  denote the product Bernoulli( $\rho$ ) measure. Then, for the given  $\delta > 0$  and any set  $A \subseteq \{0, 1\}^{L+1}$ ,*

$$\limsup_{k \rightarrow \infty} |Q_{[k, k+L]}(A) - \nu_\rho(A)| \leq (2\sqrt{\delta} + 4\delta)(4 + \varepsilon)^{-1}. \quad (3.17)$$

As this holds for all  $\delta > 0$  and all  $A \subseteq \{0, 1\}^{L+1}$ , this completes the proof of Theorem 4.

*Proof.* Fix  $A \subseteq \{0, 1\}^{L+1}$ . Observe that, following similar arguments as in Lemma 3.6.3, the random variables  $Z_S$  (which are bounded by  $\frac{T_{k'+R, k'+R}^S - T_{k', k'}^S}{R}$ ), are  $\mathcal{L}^2$  bounded, hence uniformly integrable. This, together with stationarity, would imply that, for the given  $\delta > 0$ , we can choose  $R = R(\delta)$  large enough not depending on  $k'$ , such that

$$\left| \mathbb{E} \left( \frac{1}{R} \int_{T_{k', k'}^S}^{T_{k'+R, k'+R}^S} \mathbf{1}(\eta_t^S[1, L+1] \in A) dt \right) - (4 + \varepsilon)\nu_\rho(A) \right| \leq \delta.$$

This is proved in Lemma 3.5.7 below. Fix such an  $R$ . Then, by Proposition 3.5.4 and Lemma 3.5.5, it follows that for  $n \geq s \gg k^2$ ,

$$\left| \mathbb{E} \left( \frac{1}{sR} \int_{T_n}^{T_{n+sR}} \mathbf{1}(\eta_t[k, k+L] \in A) dt \right) - (4 + \varepsilon)\nu_\rho(A) \right| \leq \delta + \sqrt{\left(\frac{C}{k^c} + \delta\right)} + \frac{C'}{kR} + C'e^{-ck^{1/2}}, \quad (3.18)$$

where  $C, C', c$  depend only on  $R, \delta, L$ . Choose  $k$  large enough so that the right side of (3.18) is less than  $2\sqrt{\delta} + 2\delta$ . Fix such a  $k$ .

Applying Theorem 3.3.1 and uniform integrability of the random variables, there exist constants  $\tilde{C}, \tilde{c}$  depending on  $k$ , such that for every  $n \in \mathbb{N}$

$$\left| \mathbb{E} \left( \frac{1}{sR} \int_{T_n}^{T_{n+sR}} \mathbf{1}(\eta_t[k, k+L] \in A) dt \right) - (4 + \varepsilon)Q_{[k, k+L]}(A) \right| \leq \delta + \tilde{C}e^{-\tilde{c}\delta s^{1/13}}.$$

Choose  $s$  large so that the right hand side of the above equation is at most  $2\delta$ .

Combining all this, we get, for any fixed  $A \subseteq \{0, 1\}^{L+1}$ , and for all large  $k$  (depending on  $\delta$ ),

$$|Q_{[k, k+L]}(A) - \nu_\rho(A)| \leq (2\sqrt{\delta} + 4\delta)(4 + \varepsilon)^{-1}.$$

□

**Lemma 3.5.7.** *In the setting of the proof of Theorem 3.5.6, for the given  $\delta > 0$ , there exists  $R = R(\delta)$  such that*

$$\sup_{k'} \left| \mathbb{E} \left( \frac{1}{R} \int_{T_{k', k'}^S}^{T_{k'+R, k'+R}^S} \mathbf{1}(\eta_t^S[1, L+1] \in A) dt \right) - (4 + \varepsilon)\nu_\rho(A) \right| \leq \delta.$$

*Proof.* The proof is by a standard size biasing argument. Let  $\kappa > 0$  be fixed sufficiently small depending on  $\delta$ . Let  $\mathcal{A}_{k',R}$  denote the event that

$$\left| \left( \frac{1}{R} \int_{T_{k',k'}^S}^{T_{k'+R,k'+R}^S} \mathbf{1}(\eta_t^S[1, L+1] \in A) dt \right) - (4 + \varepsilon)\nu_\rho(A) \right| \geq \kappa.$$

Clearly it suffices to show that  $\sup_{k'} \mathbb{P}(\mathcal{A}_{k',R}) \rightarrow 0$  as  $R \rightarrow \infty$ . Now, for the process in equilibrium let us denote the law by  $\tilde{\mathbb{P}}$  and the expectation by  $\tilde{\mathbb{E}}$  and let  $\Delta$  denote the time difference between two consecutive jumps at the origin. Clearly the distribution of the time difference between the jumps straddling time 0 is size biased distribution of  $\Delta$ , and Cauchy-Schwarz inequality then implies

$$\mathbb{P}(\mathcal{A}_{k',R}) \leq \frac{(\tilde{\mathbb{E}}\Delta^2)^{1/2} \tilde{\mathbb{P}}(\mathcal{A}_{k',R})^{1/2}}{\tilde{\mathbb{E}}\Delta}.$$

We know that  $\tilde{\mathbb{E}}\Delta = (4 + \varepsilon)$  (cf. Remark 3.1.2) and we have already shown (by Lemma 3.6.3) that  $\tilde{\mathbb{E}}\Delta^2 < \infty$ , hence it suffices to prove that  $\sup_{k'} \tilde{\mathbb{P}}(\mathcal{A}_{k',R}) \rightarrow 0$  as  $R \rightarrow \infty$ . Now observe that  $\tilde{\mathbb{P}}$  measure of  $\mathcal{A}_{k',R}$  is independent of  $k'$ , and hence it suffices to show that

$$\mathbb{P} \left( \left| \frac{1}{R} \int_0^{T_R} \mathbf{1}(\eta_t^{\tilde{\nu}}[1, L+1] \in A) dt - (4 + \varepsilon)\nu_\rho(A) \right| \geq \kappa \right) \rightarrow 0$$

as  $R \rightarrow \infty$  where  $\eta^{\tilde{\nu}}$  denotes the process started from the hitting distribution  $\tilde{\nu}$  of  $\mathcal{B}$  in the stationary chain where  $\mathcal{B}$  denotes the set of configurations immediately after a jump at the origin. The result now follows by observing that starting from  $\tilde{\nu}$ , TASEP converges to  $\nu_\rho$  weakly and the fact that  $\frac{T_R}{R} \rightarrow (4 + \varepsilon)$  almost surely.  $\square$

### 3.5.3 A sketch of proof of Theorem 3.1.4

We end with a sketch of the proof of Theorem 3.1.4. As will be clear shortly, the proof is quite similar to the proof of Theorem 4, so we shall omit the details. Fix  $p < \rho$ . We shall show that starting from  $\nu_p$  initial condition, TASEP with a slow bond converges to a stationary distribution  $\nu_p^*$ , moreover,  $\nu_p^*$  is asymptotically equivalent to  $\nu_p$  at  $\pm\infty$ . A similar argument applies for  $p > 1 - \rho$ .

Using the correspondence between TASEP starting from a stationary distribution and last passage percolation described in Subsection 3.5.1, it follows that jump times in TASEP started with product  $\text{Ber}(p)$  initial condition corresponds roughly to last passage times to the line  $\mathbb{L}$  with the equation

$$y = -\frac{p}{1-p}x.$$

Using coalescence of geodesics from points near  $(n, n)$  to the line  $\mathbb{L}$  it follows as before that average occupation measures over large intervals  $T_n$  to  $T_{n+k}$  converge to a measure  $\nu_p^*$  on the

space of all configurations. However, as the geodesics now will typically not remain pinned to the diagonal, instead of the strong coalescence results of Theorem 3.2.4 used earlier, here one has to use Theorem 3.1.5 for the coalescence of geodesics. To show that the process itself converges to the measure  $\nu_p^*$  (and hence  $\nu_p^*$  is stationary), one needs a smoothing argument as in Section 3.4. However as the vertices on the diagonal closer to the origin are no longer pivotal, a different argument would be needed. Consider TASEP with a slow bond. By coalescence, the geodesics from points near  $(2n, 2n)$  to  $\mathbb{L}$  are very unlikely to be affected by the first  $\frac{n}{4}$  many passage times on the diagonal, in particular one can replace these by i.i.d.  $\text{Exp}(1)$  variables and get a coupling between TASEP with a slow bond; and Stationary TASEP with density  $p$  run for time  $n$  followed by TASEP with a slow bond, such that the average occupation measure of the former in an interval around time  $T_{2n}$  is with high probability identical to that of a slow bond at time  $T_{2n} - \delta n$  for all  $\delta \in (0, 1)$  as running the stationary TASEP for time  $n$  does not change the marginal distribution. Since the occupation measures are close to one another in total variation distance (and each of them are close to  $\nu_p^*$ ) the process must converge to the limiting distribution  $\nu_p^*$ .

It remains to show that  $\nu_p^*$  is asymptotically equivalent to  $\nu_p$  at  $\pm\infty$ . We shall only sketch that  $\nu_p^*$  is asymptotically equivalent to  $\nu_p$  at  $\infty$ , the other part is easier. As in the proof of Theorem 4, the basic objects of study are the geodesics to the points  $(n+k, n)$  for  $n \gg k \gg 1$ . The important observation is the following. If  $p < \rho$ , then the geodesics from  $(n+k, n)$  to  $\mathbb{L}$  spends only  $O(1)$  time on the diagonal, in a deterministic interval of length  $O(k^{2/3})$ . This can be checked by doing a first order calculation as in Subsection 3.1.4, and a variant of Theorem 3. So the geodesic from  $(n+k, n)$  to  $\mathbb{L}$  is asymptotically a straight line that has the same slope (asymptotically for  $n \gg k \gg 1$ ) as the geodesic from  $(k, k)$  to  $\mathbb{L}$  in the unreinforced DLPP. Using this and coalescence one can again couple the occupation measure of stationary TASEP of density  $p$  near the origin, to be close in total variation distance to the occupation measure of TASEP with a slow bond at some large time and at sites near the point  $k$  for some large  $k$ . The proof of Theorem 3.1.4 is then completed as in the proof of Theorem 4. We omit the details.

## 3.6 Proofs of a few technical results

### 3.6.1 Lemmas used in Section 3.4

**Lemma 3.6.1.** *In the set up of Lemma 3.4.3, there exist two absolute positive constants  $a, b$ , such that  $\tau^2 \in [a, b]$ .*

The non trivial part in this lemma is to prove the lower bound for  $\tau^2$ . For this, we follow the proof of Lemma 3.2.6 and even construct the same events to ensure that  $\Gamma_1$  touches  $(a_1, a_1)$  on some event whose probability is bounded away from 0 (not depending on  $n$ ). There is one additional technicality here which is taken care of by the monotonicity of the events.

*Proof.* Let  $X_1$  be the weight of the geodesic from  $(m_1, m_1)$  to  $(a_1, a_1)$  (excluding the weight of  $\xi_{(a_1, a_1)}$ ), and  $X_2$  be the weight of the geodesic from  $(a_1, a_1)$  to  $(n_1, n_1)$  (excluding the weight of  $\xi_{(a_1, a_1)}$ ), and let  $X$  be the weight of the geodesic from  $(m_1, m_1)$  to  $(n_1, n_1)$  avoiding the point  $(a_1, a_1)$ . Then clearly,

$$T_1(C_1) = \max\{X_1 + X_2 + \xi_{(a_1, a_1)}, X\} \leq \max\{X_1 + X_2, X\} + \xi_{(a_1, a_1)}.$$

Hence,

$$\max\{X_1 + X_2, X\} \leq \mathbb{E}(T_1(C_1)|\mathcal{F}_n) \leq \max\{X_1 + X_2, X\} + \mathbb{E}(\xi_{(a_1, a_1)}).$$

Hence,

$$|T_1(C_1) - \mathbb{E}(T_1(C_1)|\mathcal{F}_n)| \leq \xi_{(a_1, a_1)} + \mathbb{E}(\xi_{(a_1, a_1)}).$$

Thus, the upper bound of  $\tau^2$  is immediate. Hence we only need to prove the lower bound of  $\tau^2$ .

Let  $C$  be an absolute positive constant to be chosen appropriately later and consider the two boxes of size  $C$  whose top left or bottom right vertex is  $(a_1, a_1)$ . Let  $D_1$  denotes the event that the sum of all  $C^2 - 1$  many exponential random variables excluding  $\xi_{(a_1, a_1)}$  inside each of these boxes is less than  $2C^2$ . Let  $D_2$  denote the event that  $\Gamma_1$  is within a vertical and horizontal distance of  $C$  from  $(a_1, a_1)$ . It is not hard to see that the same argument as in Lemma 3.2.1 and Proposition 3.2.2 works even when one point on the diagonal is conditioned to have 0 weight, and it follows that  $\mathbb{P}(D_2|\xi_{(a_1, a_1)} = 0) \geq 1 - e^{-cC^{1/2}}$ , where  $c$  is some absolute positive constant. Let  $\Pi_B$  denote the configuration restricted to the set  $B$ . Define the event  $D_0 \subseteq \Pi_{\mathbb{Z}^2 \setminus (a_1, a_1)}$  as,

$$D_0 = \{\omega \in \Pi_{\mathbb{Z}^2 \setminus (a_1, a_1)} : \omega \cap \{\xi_{a_1, a_1} = 0\} \in D_2\}.$$

Note that  $D_0$  is independent of  $\xi_{a_1, a_1}$ . Also as  $D_2$  is an increasing event in  $\xi_{a_1, a_1}$ , hence  $D_0 \cap \{\xi_{a_1, a_1} \geq x\} \subseteq D_2$  for all  $x \geq 0$ . Also, since  $D_0 \cap \{\xi_{a_1, a_1} = 0\} = D_2 \cap \{\xi_{a_1, a_1} = 0\}$ , hence,

$$\mathbb{P}(D_0) = \mathbb{P}(D_2|\xi_{a_1, a_1} = 0) \geq 1 - e^{-cC^{1/2}}.$$

Since  $D_1$  is also a high probability event, one can choose  $C$  a large constant (not depending on  $n$ ) such that  $\mathbb{P}(D_0 \cap D_1) \geq \frac{1}{2}$ . Observe that  $D_1 \cap D_0 \cap \{\xi_{(a_1, a_1)} > 2C^2\} \subseteq D_1 \cap D_2 \cap \{\xi_{(a_1, a_1)} > 2C^2\}$ . Also it follows from the proof of Lemma 3.2.6, that on  $D_1 \cap D_2 \cap \{\xi_{(a_1, a_1)} > 2C^2\}$ ,  $\Gamma_1$  touches the point  $(a_1, a_1)$ . As  $D_1 \cap D_0 \in \mathcal{F}_n$ ,

$$\begin{aligned} (T_1(C_1) - \mathbb{E}(T_1(C_1)|\mathcal{F}_n))^2 &\geq (T_1(C_1) - \mathbb{E}(T_1(C_1)|\mathcal{F}_n))^2 \mathbf{1}_{D_1 \cap D_0} \\ &= (T_1(C_1) \mathbf{1}_{D_1 \cap D_0} - \mathbb{E}(T_1(C_1) \mathbf{1}_{D_1 \cap D_0} | \mathcal{F}_n))^2. \end{aligned}$$

Also,

$$\begin{aligned} &T_1(C_1) \mathbf{1}_{D_1 \cap D_0} - \mathbb{E}(T_1(C_1) \mathbf{1}_{D_1 \cap D_0} | \mathcal{F}_n) \\ &= T_1(C_1) \mathbf{1}_{D_1 \cap D_0 \cap \{\xi_{(a_1, a_1)} > 2C^2\}} - \mathbb{E}(T_1(C_1) \mathbf{1}_{D_1 \cap D_0 \cap \{\xi_{(a_1, a_1)} > 2C^2\}} | \mathcal{F}_n) \\ &\quad + (T_1(C_1) \mathbf{1}_{D_1 \cap D_0 \cap \{\xi_{(a_1, a_1)} \leq 2C^2\}} - \mathbb{E}(T_1(C_1) \mathbf{1}_{D_1 \cap D_0 \cap \{\xi_{(a_1, a_1)} \leq 2C^2\}} | \mathcal{F}_n)). \end{aligned}$$

Clearly the second summand is at most  $2C^2$  in absolute value (by a similar calculation as done for the upper bound of  $\tau$ ). Hence enough to show that the first summand is bigger than  $4C^2$  with a positive probability that does not depend on  $n$ .

On  $D_1 \cap D_0 \cap \{\xi_{(a_1, a_1)} > 2C^2\}$ ,  $\Gamma_1$  passes through  $(a_1, a_1)$ , hence,

$$T_1(C_1)\mathbf{1}_{D_1 \cap D_0 \cap \{\xi_{(a_1, a_1)} > 2C^2\}} = (X_1 + \xi_{(a_1, a_1)} + X_2)\mathbf{1}_{D_1 \cap D_0 \cap \{\xi_{(a_1, a_1)} > 2C^2\}},$$

and,

$$\mathbb{E}(T_1(C_1)\mathbf{1}_{D_1 \cap D_0 \cap \{\xi_{(a_1, a_1)} > 2C^2\}} | \mathcal{F}_n) = ((X_1 + X_2)\mathbb{E}(\mathbf{1}_{\xi_{(a_1, a_1)} > 2C^2}) + \mathbb{E}(\xi_{(a_1, a_1)}\mathbf{1}_{\xi_{(a_1, a_1)} > 2C^2}))\mathbf{1}_{D_1 \cap D_0}$$

Let  $D_3 := \{\xi_{(a_1, a_1)}\mathbf{1}_{\{\xi_{(a_1, a_1)} > 2C^2\}} \geq \mathbb{E}(\xi_{(a_1, a_1)}\mathbf{1}_{\{\xi_{(a_1, a_1)} > 2C^2\}}) + 4C^2\}$ . Note that

$$\{\mathbf{1}_{\{\xi_{(a_1, a_1)} > 2C^2\}} \geq \mathbb{E}(\mathbf{1}_{\{\xi_{(a_1, a_1)} > 2C^2\}})\} = \{\xi_{(a_1, a_1)} > 2C^2\} \supset D_3.$$

Let  $\mathbb{P}(D_3) = p$ . As  $\mathbb{P}(D_0 \cap D_1) \geq \frac{1}{2}$  and  $D_0 \cap D_1$  is independent of  $D_3$ , hence  $\mathbb{P}(D_0 \cap D_1 \cap D_3) \geq \frac{p}{2}$ . Hence with probability atleast  $\frac{p}{2}$ ,  $T_1(C_1)\mathbf{1}_{D_1 \cap D_0} - \mathbb{E}(T_1(C_1)\mathbf{1}_{D_1 \cap D_0} | \mathcal{F}_n) \geq 2C^2$ . As  $C, p$  are constants not depending on  $n$ , this proves the claim.  $\square$

**Lemma 3.6.2.** Fix  $b \in \mathbb{N}, I = [-b, b]$  and  $A \subseteq \{0, 1\}^I$ . For  $s \geq 0$ , define  $\rho(s) := \mathbb{P}(\eta_s(I) \in A)$ . Then  $\rho(s)$  is uniformly continuous in  $s$ .

*Proof.* To see this, note that, for any  $\delta > 0$ ,

$$\rho(s + \delta) = \mathbb{P}(\eta_{s+\delta}(I) \in A | \xi_{(0,0)} > \delta)\mathbb{P}(\xi_{(0,0)} > \delta) + \mathbb{P}(\eta_{s+\delta}(I) \in A | \xi_{(0,0)} \leq \delta)\mathbb{P}(\xi_{(0,0)} \leq \delta).$$

If  $\xi_{(0,0)} > \delta$ , then the exponential clock at site 0 of the TASEP has not yet ticked, and as the TASEP starts from step initial conditions, so  $\mathbb{P}(\eta_{s+\delta}(I) \in A | \xi_{(0,0)} > \delta) = \mathbb{P}(\eta_s(I) \in A)$ . Also as  $\mathbb{P}(\xi_{(0,0)} > \delta) = e^{-r\delta}$  Hence,

$$|\rho(s + \delta) - \rho(s)| \leq 2(1 - e^{-\delta}),$$

which shows the uniform continuity of  $\rho(s)$ .  $\square$

### 3.6.2 Regularity estimate and uniform integrability used in Section 3.5

In this subsection, we prove the following lemma that is used to get the uniform integrability conditions of random variables used earlier. This is a direct consequence of Corollary 2.2.4(a) from the previous chapter.

**Lemma 3.6.3.** Let  $k, R \in \mathbb{N}$  and  $T_m^0$  denotes the length of the geodesic from  $(0, 0)$  to  $(m, m)$  in the Exponential DLPP. Then

$$\sup_{k, R} \mathbb{E} \left( \frac{T_{k+R}^0 - T_k^0}{R} \right)^2 < C^* < \infty.$$

*Proof.* Let

$$X = \min\{x \mid (x, k) \in \Gamma_{k+R} \text{ or } (k, x) \in \Gamma_{k+R}\}.$$

If  $X = k$ , then  $\Gamma_{k+R}$  passes through  $(k, k)$ , hence,  $T_{k+R}^0 - T_k^0 = T_{(k,k),(k+R),k+R}^0 \stackrel{d}{=} T_R^0$ . Similarly, if  $X = k - \ell$ , then  $T_{k+R}^0 - T_k^0 \prec T_{R+\ell}^0$ . We need to bound  $\mathbb{P}\left(\frac{T_{k+R}^0 - T_k^0}{R} > \sqrt{m}\right)$  for each  $m$  by a term not depending on  $k, R$ , such that the bound is summable. Hence for  $\ell$  small (i.e.,  $X$  large), we bound the probability by  $\mathbb{P}\left(\frac{T_{R+\ell}^0}{R} > \sqrt{m}\right)$ , and for  $\ell$  large, we bound the probability by Corollary 2.2.4(a). Let  $\Gamma'$  be the geodesic  $\Gamma_{R+k}$  viewed from  $(R+k, R+k)$  to  $(0, 0)$ , i.e.,  $\Gamma'(\ell) = \Gamma_{k+R}(k+R-\ell)$ . We shall apply Corollary 2.2.4(a) to the geodesic  $\Gamma'$ . Then, for  $k \geq R$ , we get for any  $m \geq 1$ ,

$$\begin{aligned} & \mathbb{P}\left(\frac{T_{k+R}^0 - T_k^0}{R} > \sqrt{m}\right) \\ & \leq \mathbb{P}\left(\frac{T_{k+R}^0 - T_k^0}{R} > \sqrt{m}, X \geq (k - (R + m^{1/3}))_+\right) + \\ & \quad \sum_{\ell=R+m^{1/3}}^{\infty} \mathbb{P}\left(\frac{T_{k+R}^0 - T_k^0}{R} > \sqrt{m}, k - X = \ell\right) \\ & \leq \mathbb{P}\left(\frac{T_{2R+m^{1/3}}^0}{R} \geq \sqrt{m}\right) + \sum_{\ell=R+m^{1/3}}^{\infty} \mathbb{P}\left(\frac{T_{k+R}^0 - T_k^0}{R} > \sqrt{m}, k - X = \ell\right) \\ & \leq \mathbb{P}\left(\frac{T_{2R+m^{1/3}}^0}{2R+m^{1/3}} \geq \frac{m^{1/6}}{2}\right) + \sum_{\ell=R+m^{1/3}}^{\infty} \mathbb{P}\left(|\Gamma'(R+\ell) - (R+\ell)| \geq \ell\right) \\ & \leq 2^{2R+m^{1/3}} C e^{-\frac{1}{4}(2R+m^{1/3})m^{1/6}} + \sum_{s=m^{1/3}}^{\infty} \mathbb{P}\left(|\Gamma'_s - s| \geq \frac{s}{2}\right) \\ & \leq C e^{-cm^{1/3}} + C e^{-cm^{1/10}} \leq C e^{-cm^{1/10}}. \end{aligned}$$

The result follows immediately.  $\square$

### 3.7 A Central Limit Theorem for the slow bond model

As remarked before, here we provide a proof of a central limit theorem for the last passage time in the slow bond model. Recall that this is a consequence of the path getting pinned to the diagonal at a constant rate; and using Theorem 3.2.4 one can argue that  $T_n$  can be approximated by partial sums of stationary processes. This argument was outlined in [16]; we provide a complete proof here for the sake of completeness. Also observe that we do not really need this central limit theorem for the other results in this chapter, however we believe it is an interesting result in its own right, hence the proof.

**Theorem 3.7.1.** *For any  $r < 1$ ; we have*

$$\frac{T_n^{(r)} - \mathbb{E}T_n^{(r)}}{\sqrt{\text{Var } T_n^{(r)}}} \Rightarrow N(0, 1).$$

Furthermore, there exists  $\sigma = \sigma(r) \in (0, \infty)$  such that  $\lim_{n \rightarrow \infty} \frac{\text{Var } T_n^{(r)}}{n} = \sigma^2$ .

*Proof.* We suppress the dependence on  $r$ , and write  $T_n^{(r)}$  simply as  $T_n$ . Note that  $\frac{T_n - \mathbb{E}(T_n)}{\sqrt{n}} = \frac{T_n - T_{n^{1/3}} - \mathbb{E}(T_n - T_{n^{1/3}})}{\sqrt{n}} + R$ , where  $R := \frac{T_{n^{1/3}} - \mathbb{E}(T_{n^{1/3}})}{\sqrt{n}} \xrightarrow{\mathbb{P}} 0$ . Also  $T_n - T_{n^{1/3}} = \sum_{i=n^{1/3}}^{n-1} (T_{i+1} - T_i)$ . Define

$$X_i := T_{n^{1/3+i}} - T_{n^{1/3+i-1}} \text{ for } i = 1, 2, \dots, t,$$

where  $t = n - n^{1/3}$ . Then enough to show  $\frac{\sum_{i=1}^t (X_i - \mathbb{E}(X_i))}{\sqrt{t}} \Rightarrow N(0, \sigma^2)$ . As stated earlier, we would apply central limit theorem for stationary processes.

To this end, we first show that  $X_1, X_2, \dots, X_t$  is equal to a stationary sequence with high probability. Fix  $1 \leq k \leq t$ . Then fix  $\ell \geq 0$  such that  $k + \ell \leq t$ , let  $\Gamma_\ell$  be the geodesics from  $(0, 0)$  to  $(n^{1/3} + k + \ell, n^{1/3} + k + \ell)$ , and  $\Gamma_\ell^k$  be the geodesics from  $(k, k)$  to  $(n^{1/3} + k + \ell, n^{1/3} + k + \ell)$ . Let  $E_k$  denote the event that there exists some  $u \in \llbracket k, n^{1/3} + k \rrbracket$  such that  $(u, u) \in \bigcap_{\ell=0}^{t-k} (\Gamma_\ell \cap \Gamma_\ell^k)$ . That is,  $E_k$  denotes the event that all these paths meet together on the diagonal. Then by Corollary 3.2.7,  $\mathbb{P}(E_k) \geq 1 - e^{-cn^{1/12}}$ . Let  $Y_i^k := T_{(k,k),(n^{1/3+k+i}, n^{1/3+k+i})} - T_{(k,k),(n^{1/3+k+i-1}, n^{1/3+k+i-1})}$ . Clearly  $(Y_1^k, Y_2^k, \dots, Y_{t-k}^k) \stackrel{d}{=} (X_1, X_2, \dots, X_{t-k})$ . Note that on  $E_k$ , the differences in the lengths of geodesics starting from  $(0, 0)$  coincide with those starting from  $(k, k)$ . Hence, on  $E_k$ ,

$$(X_{k+1}, X_{k+2}, \dots, X_t) = (Y_1^k, Y_2^k, \dots, Y_{t-k}^k).$$

Let  $E = \bigcap_{k=1}^t E_k$ . Then  $\mathbb{P}(E) \geq 1 - ne^{-cn^{1/12}}$ . And, for all  $1 \leq k \leq t$ , there exist random variables  $Y_1^k, Y_2^k, \dots, Y_{t-k}^k$  such that  $(Y_1^k, Y_2^k, \dots, Y_{t-k}^k) \stackrel{d}{=} (X_1, X_2, \dots, X_{t-k})$ ; and on  $E$ ,  $(X_{k+1}, X_{k+2}, \dots, X_t) = (Y_1^k, Y_2^k, \dots, Y_{t-k}^k)$ .

Next we show that the sequence is  $\alpha$ -mixing. For this, we consider two sets  $(X_1, X_2, \dots, X_\ell)$  and  $(X_{\ell+m+1}, X_{\ell+m+2}, \dots)$  such that the indices are separated by a distance of  $m$ . For any  $\ell, s \geq 1$  and  $m \geq n^{1/3}$  such that  $n^{1/3} + \ell + m + s \leq n$ , let  $F$  denote the event that all the geodesics from  $(0, 0)$  to  $(n^{1/3} + \ell + m + s, n^{1/3} + \ell + m + s)$ , and all geodesics from  $(n^{1/3} + \ell + 1, n^{1/3} + \ell + 1)$  to  $(n^{1/3} + \ell + m + s, n^{1/3} + \ell + m + s)$  meet the diagonal simultaneously in the interval  $\llbracket n^{1/3} + \ell + 1, n^{1/3} + \ell + m \rrbracket$ . Then as in previous paragraph, using Corollary 3.2.7 and union bound (and the fact that  $m \geq n^{1/3}$ ), it follows that  $\mathbb{P}(F) \geq 1 - e^{-cm^{1/4}}$  for some absolute positive constant  $c$ . For  $j \geq \ell + 2$ , define  $Z_s = T_{(n^{1/3+\ell+1}, n^{1/3+\ell+1}), (n^{1/3+j}, n^{1/3+j})} - T_{(n^{1/3+\ell+1}, n^{1/3+\ell+1}), (n^{1/3+j-1}, n^{1/3+j-1})}$  to be the difference in the lengths of the corresponding geodesics starting from  $(n^{1/3} + \ell + 1, n^{1/3} + \ell + 1)$  instead of  $(0, 0)$ . Then, as before, on  $F$ ,

$$(X_{\ell+m}, X_{\ell+m+1}, \dots) = (Z_{\ell+m}, Z_{\ell+m+1}, \dots).$$

Now for  $A = f(X_1, X_2, \dots, X_\ell)$ ,  $B = g(X_{\ell+m+1}, X_{\ell+m+2}, \dots)$ , and  $B' := g(Z_{\ell+m+1}, Z_{\ell+m+2}, \dots)$ ,

$$\begin{aligned} & |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \\ & \leq |\mathbb{P}(A \cap B \cap F) - \mathbb{P}(A)\mathbb{P}(B \cap F)| + 2\mathbb{P}(F^c) \\ & = |\mathbb{P}(A \cap B' \cap F) - \mathbb{P}(A)\mathbb{P}(B' \cap F)| + 2\mathbb{P}(F^c) \\ & \leq |\mathbb{P}(A \cap B') - \mathbb{P}(A)\mathbb{P}(B')| + 4\mathbb{P}(F^c) \\ & = 4\mathbb{P}(F^c) \leq 4e^{-cm^{1/4}}, \end{aligned}$$

where we have used the fact that  $A$  and  $B'$  are independent.

It is easy to see using Proposition 3.2.2 and Theorem 3.2.4 that the geodesics  $\Gamma_n$  and  $\Gamma_{n-1}$  meet the diagonal simultaneously in the interval  $[[n-h, n]]$  with probability at least  $1 - e^{-ch^{1/4}}$ . From this it is not too hard to see that  $\sup_n \mathbb{E}(T_n - T_{n-1})^{12} < \infty$ . Hence, following the proof of Central Limit Theorem for stationary processes, (see e.g. Theorem 27.4 in [19]), with obvious modifications, the theorem follows. That  $\sigma > 0$  follows from the following Proposition 3.7.2. This completes the proof.  $\square$

The following proposition shows that  $\sigma > 0$  in Theorem 3.7.1.

**Proposition 3.7.2.** *Let  $T_n$  denote the last passage time from  $(0, 0)$  to  $(n, n)$  in the slow bond model. There exists  $C > 0$  such that  $\text{Var } T_n \geq Cn$  for all  $n$ .*

Recall that the individual passage time of vertex  $v$  is denoted by  $\xi_v$ . We shall decompose  $\text{Var}(T_n)$  by revealing vertex weights in  $[0, n]^2$  in some order. First fix a bijection  $\pi : [n^2] \rightarrow [0, n] \times [0, n]$ . Let  $\mathcal{F}_i$  denote the  $\sigma$ -field generated by  $\{\xi_{\pi(1)}, \xi_{\pi(2)}, \dots, \xi_{\pi(i)}\}$ . Considering the Doob martingale  $M_i := \mathbb{E}[T | \mathcal{F}_i]$ , it follows that we have

$$\text{Var}(T) = \mathbb{E} \left[ \sum_{i=1}^{n^2} \text{Var}(M_i | \mathcal{F}_{i-1}) \right] \quad (3.19)$$

Also let  $D \subseteq [n^2]$  denote the set such that  $\pi(D)$  is the set of all vertices on the diagonal. Clearly

$$\text{Var}(T) \geq \mathbb{E} \left[ \sum_{i \in D} \text{Var}(M_i | \mathcal{F}_{i-1}) \right]. \quad (3.20)$$

The proposition will follow from the next lemma which provides a lower bound on the individual terms in the above sum.

**Lemma 3.7.3.** *Let  $M_i$  be as above and let  $\gamma$  denote the geodesic from  $(0, 0)$  to  $(n, n)$ . Then for each  $i \in D$*

$$\text{Var}(M_i | \mathcal{F}_{i-1}) \geq h \left( \mathbb{P}[\pi(i) \in \gamma | \mathcal{F}_{i-1}] \right)$$

where  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a function such that  $h(t)$  is bounded away from 0 (uniformly in  $n$ ) as soon as  $t$  is bounded away from 0.

We postpone the proof of this lemma for the moment and first show how this implies Proposition 3.7.2.

*Proof of Proposition 3.7.2.* First observe that since the expected number of vertices on the diagonal that  $\gamma$  intersects in linear it follows by Cauchy-Schwarz inequality that

$$\mathbb{E} \left[ \sum_{i \in D} \mathbb{P}[\pi(i) \in \gamma \mid \mathcal{F}_{i-1}]^2 \right] \geq \delta_1 n \quad (3.21)$$

for all  $n$  sufficiently large for some  $\delta_1 > 0$ . Notice further that for any  $\delta_2 > 0$ , we have

$$\begin{aligned} \mathbb{E} \left[ \sum_{i \in D} \mathbb{P}[\pi(i) \in \gamma \mid \mathcal{F}_{i-1}]^2 \right] &\leq \delta_2 \mathbb{E} \left[ \sum_{i \in D} \mathbb{P}[\pi(i) \in \gamma \mid \mathcal{F}_{i-1}] \right] \\ &+ \mathbb{E} \left[ \sum_{i \in D} \mathbb{P}[\pi(i) \in \gamma \mid \mathcal{F}_{i-1}]^2 I(\mathbb{P}[\pi(i) \in \gamma \mid \mathcal{F}_{i-1}] > \delta_2) \right] \\ &\leq 2\delta_2 n + \mathbb{E} \left[ \sum_{i \in D} I(\mathbb{P}[\pi(i) \in \gamma \mid \mathcal{F}_{i-1}] > \delta_2) \right]. \end{aligned}$$

By choosing  $\delta_2$  sufficiently small compared to  $\delta_1$  we get

$$\mathbb{E} \left[ \sum_{i \in D} I(\mathbb{P}[\pi(i) \in \gamma \mid \mathcal{F}_{i-1}] > \delta_2) \right] \geq \frac{\delta_1 n}{2}$$

which implies the desired linear lower bound on  $V_n$  using Lemma 3.7.3.  $\square$

*Proof of Lemma 3.7.3.* Fix  $i \in D$  and condition on  $\mathcal{F}_{i-1}$ . Let  $T^*$  be the last passage time in the environment where  $\xi_{\pi(i)}$  is resampled by an independent copy  $Z_{\pi(i)}$ . Observe that

$$M_i - M_{i-1} = \mathbb{E}[T - T^* \mid \mathcal{F}_i].$$

Let the optimizing paths in the two environments be denote by  $\gamma_1$  and  $\gamma_2$  respectively. Let  $x_0 > 0$  and set  $w_0 := W(x_0)$  where  $W(x) := \mathbb{P}[\pi(i) \in \gamma \mid \xi_{\pi(i)} = x, \mathcal{F}_{i-1}]$  is an increasing function of  $x$ . Define the events

$$A_1 := \{\pi(i) \in \gamma_1 \text{ if } \xi_{\pi(i)} \geq x_0\}; \quad A_2 := \{\pi(i) \in \gamma_2 \text{ if } Z_{\pi(i)} \geq x_0\}.$$

Clearly,  $A_1 = A_2 := A$  as the environments differ only in the weight of vertex  $\pi(i)$  and also notice that  $A$  is independent of  $\xi_{\pi(i)}, Z_{\pi(i)}$ . Further observe that  $\mathbb{P}[A \mid \mathcal{F}_{i-1}] \geq w_0$ . Indeed,  $A \cap \{\xi_{\pi(i)} \geq x_0\} = \{\pi(i) \in \gamma_1, \xi_{\pi(i)} \geq x_0\}$  and hence  $\mathbb{P}[A \cap \{\xi_{\pi(i)} \geq x_0\}] \geq w_0 \mathbb{P}[\xi_{\pi(i)} \geq x_0]$ .

The desired inequality follows from observing that  $A$  and  $\{\xi_{\pi(i)} \geq x_0\}$  are conditionally independent given  $\mathcal{F}_{i-1}$ . Observe that on  $\{\xi_{\pi(i)} > (\ell + 2)x_0\}$ , we have

$$T - T^* \geq \ell x_0 \mathbf{1}_{\{Z_{\pi(i)} \in [x_0, 2x_0], A\}} + ((\ell + 2)x_0 - Z_{\pi(i)}) \mathbf{1}_{\{Z_{\pi(i)} > (\ell + 2)x_0\}}$$

and hence

$$\mathbb{E}[T - T^* \mid \mathcal{F}_i] \geq \ell x_0 w_0 \mathbb{P}[Z_{\pi(i)} \in [x_0, 2x_0]] - q(x_0, \ell)$$

where  $q(x_0, \ell) := \mathbb{E}(Z_{\pi(i)} - (\ell + 2)x_0) \mathbf{1}_{\{Z_{\pi(i)} > (\ell + 2)x_0\}}$  decreases to 0 as  $\ell$  increases, hence by choosing  $\ell = \ell(x_0)$  sufficiently large, on  $\{\xi_{\pi(i)} > (\ell + 2)x_0\}$ , we have

$$M_i - M_{i-1} \geq x_0 w_0.$$

It follows that

$$\text{Var}(M_i \mid \mathcal{F}_{i-1}) = \mathbb{E}[(M_i - M_{i-1})^2 \mid \mathcal{F}_{i-1}] \geq x_0^2 w_0^2 \mathbb{P}[\xi_{\pi(i)} \geq (\ell + 2)x_0].$$

The proof of the lemma is completed by observing that exponential distribution has unbounded support and hence if  $\mathbf{p} := \mathbb{P}[\pi(i) \in \gamma \mid \mathcal{F}_{i-1}]$  is bounded away from 0, then one can choose  $x_0 = x_0(\mathbf{p})$  and  $w_0 = w_0(\mathbf{p})$  to be bounded away from 0 as well.  $\square$

# Chapter 4

## Modulus of continuity for polymer fluctuations and weight profiles

### 4.1 Introduction and main results

Recall that Poissonian last passage percolation specifies a growth process whose height at a given moment is the maximum number of points (or the *energy*) obtainable in a directed path through a planar Poisson point process. These are exactly solvable models, for which certain exact distributional formulas are available, and the derivations of these formulas typically employ deep machinery from algebraic combinatorics or random matrix theory. It is interesting to study geometric properties of universal KPZ objects by approaches that, while they are reliant on certain integrable inputs, are probabilistic in flavour: for example, [16],[14] and [15] are recent results and applications concerning geometric properties of last passage percolation paths.

It is rigorously understood, then, that last passage percolation paths experience fluctuation in their energy and transversal fluctuation governed by scaling exponents of one-third and two-thirds. It is very natural to view such paths via the lens of scaled coordinates, in which transversal fluctuation and path energy have unit order. We will be more precise very shortly, when suitable notation has been introduced, but for now we mention that our aim in this article is to refine rigorous understanding of the magnitude and geometry of fluctuation in last passage percolation paths. We shall call the scaled geodesics *polymers*, and refer to the scaled energy as *weight*. We will see that polylogarithmic corrections to the scaled laws implied by the exponents of one-third and two-thirds arise when we consider natural geometric problems concerning the weights and the maximum fluctuation among polymers in a unit order region. The techniques for verifying our claims will employ geometric and probabilistic tools rather than principally integrable ones, since problems involving maxima as both endpoints of a last passage percolation path are varied are not usually amenable to integrable techniques. We will draw on the integrable approach in a way that, while essential, is limited to a simple aspect of this theory, namely by applying bounds on the upper

and lower tails of the fluctuation of point-to-point polymer weights; the needed results will be recalled in Section 4.2.

### 4.1.1 Model definition and main results

Let  $\Pi$  be a homogeneous rate one Poisson point process (PPP) on  $\mathbb{R}^2$ . We introduce a partial order on  $\mathbb{R}^2$ :  $(x_1, y_1) \preceq (x_2, y_2)$  if and only if  $x_1 \leq x_2$  and  $y_1 \leq y_2$ . For  $u \preceq v$ ,  $u, v \in \mathbb{R}^2$ , an increasing path  $\gamma$  from  $u$  to  $v$  is a piecewise affine path, viewed as a subset of  $\mathbb{R}^2$ , that joins points  $u = u_0 \preceq u_1 \preceq u_2 \preceq \dots \preceq u_k = v$  such that  $u_i \in \Pi$  for  $i \in \llbracket 1, k-1 \rrbracket$ . Here and later,  $\llbracket a, b \rrbracket$  for  $a, b \in \mathbb{Z}$  with  $a \leq b$  denotes the integer interval  $\{a, \dots, b\}$ . Also let  $|\gamma|$  denote the *energy* of  $\gamma$ , namely the number of points in  $\Pi \setminus \{v\}$  that lie on  $\gamma$ ; (the last vertex is excluded from the definition of energy so that the sum of the energies of two paths equals the energy of the concatenated path, as we will see in Section 4.3.1). Then we define the last passage time from  $u$  to  $v$ , denoted by  $X_u^v$ , to be the maximum of  $|\gamma|$  as  $\gamma$  varies over all increasing paths from  $u$  to  $v$ . Any such maximizing path is called a geodesic. There may be several such, but if  $\Gamma_u^v$  denotes any one of them, we have

$$X_u^v = |\Gamma_u^v|. \tag{4.1}$$

Note that, in this notation, the starting and ending points of the geodesic,  $u$  and  $v$ , are assigned subscript and superscript placements. We will often use this convention, including in the case of the scaled coordinates that we will introduce momentarily.

When  $u \preceq v$ , any geodesic from  $u$  to  $v$  may be viewed as a function of its horizontal coordinate, since it contains a vertical line segment with probability zero. The operations of maximum and minimum may be applied to any pair of such geodesics, and the results are also geodesics. For this reason, we may speak unambiguously of  $\Gamma_u^{\leftarrow;v}$ , the uppermost geodesic between  $u$  and  $v$ , and of  $\Gamma_u^{\rightarrow;v}$ , the lowermost geodesic between  $u$  and  $v$ . (The notation  $\leftarrow$  and  $\rightarrow$  is compatible with these two paths being equally well described as the leftmost and rightmost geodesics. This choice of notation also anticipates the form of these paths when viewed in the scaled coordinates that we are about to introduce.) When the endpoints are  $(0, 0)$  and  $(n, n)$ , we will call these geodesics  $\Gamma_n^{\leftarrow}$  and  $\Gamma_n^{\rightarrow}$ .

### Introducing scaled coordinates

We rotate the plane about the origin counterclockwise by 45 degrees, squeeze the vertical coordinate by a factor  $2^{1/2}n$  and the horizontal one by  $2^{1/2}n^{2/3}$ , thus setting

$$T_n : (x, y) \mapsto (2^{-1}n^{-2/3}(x - y), 2^{-1}n^{-1}(x + y)). \tag{4.2}$$

The horizontal line at vertical coordinate  $t$  is the image under  $T_n$  of the anti-diagonal line through  $(nt, nt)$ . It is easy to see that, for  $(x, t) \in \mathbb{R}^2$ ,  $T_n^{-1}(x, t) = (nt + xn^{2/3}, nt - xn^{2/3})$ .

Paths that are the image of geodesics under  $T_n$  will be called *polymers*; we might say  $n$ -polymers, but the suppressed parameter will always be  $n$ . Geodesics from  $(0, 0)$  to  $(n, n)$

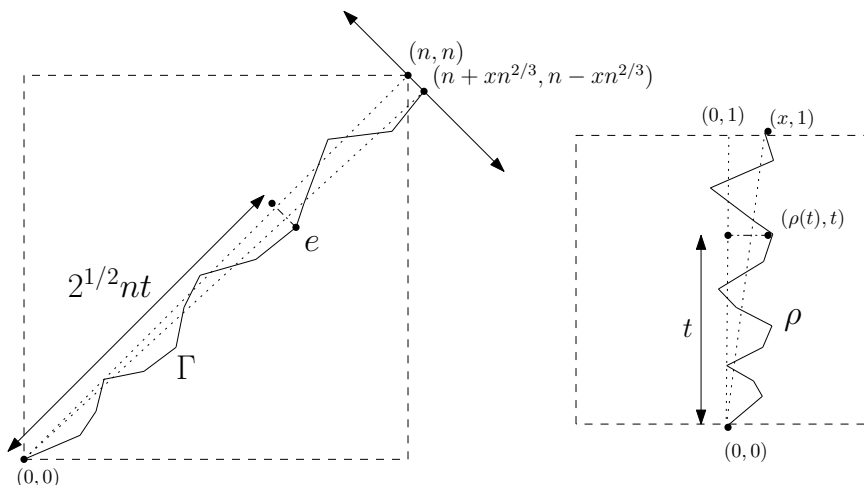


Figure 4.1: The scaling map  $T_n$  applied to the left figure produces the figure on the right. The point  $e$  in the geodesic  $\Gamma$  is the preimage of the point  $(\rho(t), t)$  in the polymer  $\rho$ .

transform to polymers  $(0, 0)$  to  $(0, 1)$ . Figure 4.1 depicts a geodesic  $\Gamma$  and its image polymer  $\rho$ . The polymer between planar points  $u$  and  $v$  that is the image of the uppermost geodesic given the preimage endpoints will be denoted by  $\rho_{n;u}^{\leftarrow;v}$ , and, naturally enough, called the leftmost polymer from  $u$  to  $v$ . The rightmost polymer from  $u$  to  $v$  is the image of the corresponding lowermost geodesic and will be denoted by  $\rho_{n;u}^{\rightarrow;v}$ . The simpler notation  $\rho_n^{\leftarrow}$  and  $\rho_n^{\rightarrow}$  will be adopted when  $u = (0, 0)$  and  $v = (0, 1)$ . When  $u = (x_1, t_1), v = (x_2, t_2)$ , with  $x_1, x_2, t_1, t_2 \in \mathbb{R}$ ,  $t_1 < t_2$ , such that  $T_n^{-1}(x_1, t_1) \preceq T_n^{-1}(x_2, t_2)$ , we will, when it is convenient, regard any polymer  $\rho$  from  $u$  to  $v$  as a function of its vertical coordinate: that is, for  $t \in [t_1, t_2]$ ,  $\rho(t)$  will denote the unique point such that  $(\rho(t), t) \in \rho$ . (This definition makes sense since an increasing path can intersect any anti-diagonal at most once.) We regard the vertical coordinate as time, as the  $t$ -notation suggests, and will sometimes refer to the interval  $[t_1, t_2]$  as the *lifetime* of the polymer. In particular, when  $t_1 = 0$  and  $t_2 = 1$ , writing  $C[0, 1]$  for the space of continuous real-valued functions on  $[0, 1]$  (equipped for later purposes with the topology of uniform convergence), we may thus view  $\rho = \{\rho(t)\}_{t \in [0, 1]}$  as an element of  $C[0, 1]$ .

### Condition for existence of polymers

For  $u = (x_1, t_1), v = (x_2, t_2)$  with  $x_1, x_2, t_1, t_2 \in \mathbb{R}$ ,  $t_1 < t_2$ , we have that  $T_n^{-1}(u) = (nt_1 + x_1n^{2/3}, nt_1 - x_1n^{2/3})$  and  $T_n^{-1}(v) = (nt_2 + x_2n^{2/3}, nt_2 - x_2n^{2/3})$ . Thus  $T_n^{-1}(u) \preceq T_n^{-1}(v)$  if and only if  $|x_1 - x_2| < n^{1/3}(t_2 - t_1)$ . Indeed, we will write  $u \preceq_n v$  to mean that  $|x_1 - x_2| < n^{1/3}(t_2 - t_1)$ ; this condition ensures that polymers exist between the endpoints  $u$  and  $v$ .

The first of our three main results shows that polymers, viewed as functions of the vertical coordinate, enjoy modulus of continuity of order  $t^{2/3}(\log t^{-1})^{1/3}$ .

**Theorem 4.1.1.** (a) The sequence  $\{\rho_n^{\leftarrow}\}_{n \in \mathbb{N}}$  is tight in  $(C[0, 1], \|\cdot\|_\infty)$ .

(b) There exists a constant  $C > 0$  such that, for the weak limit  $\rho_*^{\leftarrow}$  of any weakly converging subsequence of  $\{\rho_n^{\leftarrow}\}_{n \in \mathbb{N}}$ , almost surely,

$$\limsup_{t \searrow 0} \sup_{0 \leq z \leq 1-t} t^{-2/3} (\log t^{-1})^{-1/3} |\rho_*^{\leftarrow}(z+t) - \rho_*^{\leftarrow}(z)| \leq C. \quad (4.3)$$

The same result holds for the rightmost polymer.

Note that the constant  $C$  does not depend on the choice of the weakly converging subsequence.

The exponent pair  $(2/3, 1/3)$  for power law and polylogarithmic correction is thus demonstrated to hold in an upper bound on polymer fluctuation. We believe that a lower bound holds as well, in the sense that the limit infimum counterpart to (4.3) is positive. A polymer is an object specified by a global constraint, and it by no means clearly enjoys independence properties as it traverses disjoint regions, even though the underlying Poisson randomness does. In order to demonstrate the polymer fluctuation lower bound, this subtlety would have to be addressed. We choose instead to demonstrate that the exponent pair  $(2/3, 1/3)$  describes polymer fluctuation by proving a lower bound of this form for the maximum fluctuation witnessed among a natural class of short polymers in a unit region. This alternative formulation offers a greater supply of independent randomness.

Indeed, we now specify a notion of *maximum transversal fluctuation* over a collection of short polymers. Fix any two points  $u = (x_1, t_1), v = (x_2, t_2)$  such that  $t_2 > t_1$ . Let  $\Phi_{n;u}^v$  denote the set of all polymers  $\rho$  from  $u$  to  $v$ . Let  $\ell_u^v$  denote the planar line segment that joins  $u$  and  $v$ ; extending an abuse of notation that we have already made, we write  $\ell_u^v(t)$  for the unique point such that  $(\ell_u^v(t), t) \in \ell_u^v$ , where  $t \in [t_1, t_2]$ . Then, for any polymer  $\rho$ , the transversal fluctuation  $\text{TF}(\rho)$  of  $\rho$  is specified to be

$$\text{TF}(\rho) := \sup_{t \in [t_1, t_2]} |\rho(t) - \ell_u^v(t)|, \quad (4.4)$$

and the transversal fluctuation between the points  $u$  and  $v$  to be

$$\text{TF}_{n;u}^v := \max_{\rho \in \Phi_{n;u}^v} \text{TF}(\rho) = \max \{ \text{TF}(\rho_{n;u}^{\leftarrow;v}), \text{TF}(\rho_{n;u}^{\rightarrow;v}) \}. \quad (4.5)$$

Also, let

$$\text{InvSlope}_{(x_1, t_1)}^{(x_2, t_2)} = \frac{x_2 - x_1}{t_2 - t_1}$$

denote the reciprocal of the slope of the interpolating line. Since  $t_2 > t_1$ ,  $\text{InvSlope}_{(x_1, t_1)}^{(x_2, t_2)} \in \mathbb{R}$ .

Now fix some large constant  $\psi > 0$ . Then, for any fixed parameter  $t \in (0, 1]$  and any  $n \in \mathbb{N}, n > \psi^3$ , we define the set of *admissible endpoint pairs*

$$\text{AdEndPair}_{n,\psi}(t) := \left\{ ((x_1, t_1), (x_2, t_2)) : t_2 - t_1 \in (0, t], \left| \text{InvSlope}_{(x_1, t_1)}^{(x_2, t_2)} \right| \leq \psi, \right. \\ \left. x_1, x_2 \in [-1, 1], t_1, t_2 \in [0, 1] \right\}. \quad (4.6)$$

Since  $n > \psi^3$ ,

$$|x_2 - x_1|n^{2/3} \leq \psi(t_2 - t_1)n^{2/3} < (t_2 - t_1)n.$$

Recalling the notation at the start of Subsection 4.1.1, we thus have  $(x_1, t_1) \stackrel{n}{\preceq} (x_2, t_2)$ , so that polymers do exist between such endpoint pairs.

We then define

$$\text{MTF}_n(t) = \text{MTF}_{n,\psi}(t) := \sup \left\{ \text{TF}_{n;u}^v : (u, v) \in \text{AdEndPair}_{n,\psi}(t) \right\}, \quad (4.7)$$

so that  $\text{MTF}_n(t)$  is the *maximum transversal fluctuation* over polymers between all endpoint pairs at vertical distance at most  $t$  such that the slope of the interpolating line segment is bounded away from being horizontal; (we suppress the parameter  $\psi$  in the notation). Our second theorem demonstrates that the exponent pair  $(2/3, 1/3)$  governs this maximum traversal fluctuation.

**Theorem 4.1.2.** *There exist  $\psi$ -determined constants  $0 < c < C < \infty$  such that*

$$\liminf_n \mathbb{P} \left( t^{-2/3} (\log t^{-1})^{-1/3} \text{MTF}_n(t) \in [c, C] \right) \rightarrow 1 \quad \text{as } t \searrow 0.$$

### Scaled energies are called weights

It is natural to scale the energy of a geodesic when we view the geodesic as a polymer after scaling. Scaled energy will be called *weight* and specified so that it is of unit order for polymers that cross unit-order distances. For  $t_1 < t_2$ , let  $t_{1,2}$  denote  $t_2 - t_1$ ; (this is a notation that we will often use). Let  $(x, t_1), (y, t_2) \in \mathbb{R}^2$  be such that  $|x - y| < t_{1,2}n^{1/3}$ . (This condition ensures that  $(x, t_1) \stackrel{n}{\preceq} (y, t_2)$ , so that polymers exist between this pair of points.) Since  $T_n^{-1}((x, t_1)) = (nt_1 + xn^{2/3}, nt_1 - xn^{2/3})$  and  $T_n^{-1}((y, t_2)) = (nt_2 + yn^{2/3}, nt_2 - yn^{2/3})$ , it is natural to define the scaled energies, which we call *weights*, in the following way. Define

$$\mathbf{W}_{n;(x,t_1)}^{(y,t_2)} = n^{-1/3} \left( X_{(nt_1+n^{2/3}x, nt_1-n^{2/3}x)}^{(nt_2+n^{2/3}y, nt_2-n^{2/3}y)} - 2nt_{1,2} \right). \quad (4.8)$$

Because of translation invariance of the underlying Poisson point process,  $t_{1,2}$  is a far more relevant parameter than  $t_1$  or  $t_2$ . The notation on the left-hand side of (4.8) is characteristic of our presentation in this article: a scaled object is being denoted, with planar points  $(\cdot, \cdot)$  in the subscript and superscript indicating starting and ending points.

### A continuous modification of the weight function

For the statement of our third theorem, we prefer to make an adjustment to the polymer weight to cope with a minor problem concerning discontinuity of geodesic energy under endpoint perturbation. For  $n \in \mathbb{N}$ , define  $X_n : [1, 2] \mapsto [0, \infty)$ ,

$$X_n(t) := X_{(0,0)}^{(nt,nt)}.$$

Observe that  $X_n(t)$  is integer-valued, non-decreasing, right continuous and has almost surely a finite number of jump discontinuities. Let  $d_0 = 1$  and  $d_m = 2$ . Record in increasing order the points of discontinuity of  $X_n$  as a list  $(d_1, d_2, \dots, d_{m-1})$ . We specify a *modified* and continuous form of the function  $X_n$  by linearly interpolating it between these points of discontinuity, setting

$$X_n^{\text{mod}}(t) := X_n(d_i) + (t - d_i)(d_{i+1} - d_i)^{-1}(X_n(d_{i+1}) - X_n(d_i)), \text{ for } t \in [d_i, d_{i+1}],$$

for  $i = 1, 2, \dots, m-1$ . Because almost surely no two points in a planar Poisson point process share either their horizontal or vertical coordinate,  $X_n(d_{i+1}) - X_n(d_i) = 1$  for all  $i$ . Thus, for all  $t \in [1, 2]$ ,

$$X_n(t) \leq X_n^{\text{mod}}(t) \leq X_n(t) + 1. \quad (4.9)$$

Now define the *modified weight* function  $\text{Wgt}_n : [1, 2] \mapsto \mathbb{R}$  for polymers from  $(0, 1)$  to  $(\cdot, 1)$ :

$$\text{Wgt}_n(t) := n^{-1/3} (X_n^{\text{mod}}(t) - 2nt). \quad (4.10)$$

Because of (4.9),

$$\left| \text{Wgt}_n(t) - \text{W}_{n;(0,0)}^{(0,t)} \right| \leq n^{-1/3}. \quad (4.11)$$

By construction,  $\text{Wgt}_n$  sending  $t \in [1, 2]$  to  $\text{Wgt}_n(t)$  is an element of  $C[1, 2]$ , the space of continuous functions on  $[1, 2]$ ; (similarly to before, this space will be equipped with the topology of uniform convergence).

Our third main result demonstrates that the exponent pair  $(1/3, 2/3)$  offers a description of the modulus of continuity of polymer weight when one endpoint is varied vertically.

**Theorem 4.1.3.** *The sequence  $\{\text{Wgt}_n\}_{n \in \mathbb{N}}$  is tight in  $(C[1, 2], \|\cdot\|_\infty)$ . There exist constants  $0 < c < C < \infty$  such that, for the weak limit  $\text{Wgt}_*$  of any weakly converging subsequence of  $\{\text{Wgt}_n\}_{n \in \mathbb{N}}$ , almost surely*

$$\begin{aligned} c &\leq \liminf_{t \searrow 0} \sup_{1 \leq z \leq 2-t} t^{-1/3} (\log t^{-1})^{-2/3} \left| \text{Wgt}_*(z+t) - \text{Wgt}_*(z) \right| \\ &\leq \limsup_{t \searrow 0} \sup_{1 \leq z \leq 2-t} t^{-1/3} (\log t^{-1})^{-2/3} \left| \text{Wgt}_*(z+t) - \text{Wgt}_*(z) \right| \leq C. \end{aligned} \quad (4.12)$$

Note that, as in Theorem 4.1.1, the constants  $c$  and  $C$  do not depend on the choice of weak limit point or converging subsequence.

Beyond these three theorems, we present a proposition, which is needed for the proof of Theorem 4.1.2 and which may have an independent interest. That the maximum fluctuation of any geodesic joining  $(0, 0)$  and  $(n, n)$  around the interpolating line is of the order  $n^{2/3}$  was first shown in [51]. We first state Johansson's result using scaled coordinates. Observe from (4.4) that, for any polymer  $\rho$  between  $(0, 0)$  and  $(0, 1)$ ,  $\text{TF}(\rho) = \sup_{y \in [0, 1]} |\rho(y)|$ . Recall that  $\Phi_{n;(0,t_1)}^{(0,t_2)}$  is the set of all polymers from  $(0, t_1)$  to  $(0, t_2)$ , and define

$$\xi := \inf \left\{ \theta > 0 : \lim_n \mathbb{P} \left( \max \left\{ \text{TF}(\rho) : \rho \in \Phi_{n;(0,0)}^{(0,1)} \right\} \geq n^{\theta-2/3} \right) = 0 \right\}.$$

Johansson [51] proved that  $\xi = 2/3$ . This value appears in the scaled coordinates in (4.2). His result is an upper bound on the maximum fluctuation from the diagonal of the geodesic joining  $(0, 0)$  and  $(n, n)$  whose order is  $n^{2/3+o(1)}$ . That this fluctuation has probability at most  $e^{-ck}$  of exceeding  $kn^{2/3}$  has been obtained in [16, Theorem 11.1 and Corollary 11.7]; the concerned proof may be straightforwardly varied to obtain an upper bound of the form  $e^{-ck^3}$ , and later we will state and prove the result in such a form: see Theorem 4.2.6. Our next proposition is the matching lower bound, stated using scaled coordinates. We adopt such coordinates throughout because they offer a coherent notation for the central aims of this chapter, but in the present case it is worth noting the simple expression of the result in unscaled terms: it is with probability at least  $e^{-ck^3}$  that the maximum fluctuation from the diagonal of the geodesics joining  $(0, 0)$  and  $(n, n)$  exceeds  $kn^{2/3}$ .

**Proposition 4.1.4.** *There exist positive constants  $c^*$ ,  $n_0$ ,  $s_0$  and  $\alpha_0$  such that, for all  $t_1, t_2$  with  $t_{1,2} = t_2 - t_1 > 0$  and all  $nt_{1,2} \geq n_0$  and  $s \in [s_0, \alpha_0(nt_{1,2})^{1/3}]$ ,*

$$\mathbb{P} \left( \min \left\{ \text{TF}(\rho) : \rho \in \Phi_{n;(0,t_1)}^{(0,t_2)} \right\} \geq st_{1,2}^{2/3} \right) \geq \exp \{ -c^* s^3 \}.$$

## 4.1.2 A few words about the proofs

The main ingredients in the proofs of Theorem 4.1.1 and Theorem 4.1.2 are tail estimates on polymer weight arising from integrable probability (and certain ramifications thereof) assembled in Section 4.2, and a *polymer ordering* property elaborated in Lemma 4.3.2 that propagates control on polymer fluctuation among polymers whose endpoints lie in a discrete mesh to all polymers in the region of this mesh. The basic tools in the proof of the upper bound in Theorem 4.1.3 and that of Proposition 4.1.4 are surgical techniques and comparisons of the weights of polymers, and are reminiscent of the techniques developed and extensively used in [16] and [14].

The proofs in this chapter depend on the moderate deviation estimates of the point-to-point energies proved in [64, Theorem 1.3] and [65, Theorem 1.2], which are recalled here in Theorem 4.2.2 and Theorem 4.2.3. As we record shortly in Section 4.2, further inputs that we use from [16] and [14], namely [16, Propositions 10.1 and 10.5], [16, Theorem 11.1] and [14, Theorem 3], are results that are themselves derived from the same integrable input point-to-point estimates of Theorem 4.2.2. It is thus plausible that our results concerning modulus of continuity may be proved for other models in the KPZ universality class that enjoy the same point-to-point estimates, for example, the exponential directed last passage percolation.

## 4.1.3 Phase separation and KPZ

Certain random models manifest the scaling exponents of KPZ universality and some of its qualitative features, without exhibiting the richness of behaviour of models in this class. For example, the least convex majorant of the stochastic process  $\mathbb{R} \rightarrow \mathbb{R} : x \rightarrow B(x) - t^{-1}x^2$  is

comprised of planar line segments, or facets, the largest of which in a compact region has length of order  $t^{2/3+o(1)}$  when  $t > 0$  is high; and the typical deviation of the process from its majorant scales as  $t^{1/3+o(1)}$ .

Some such models form a testing ground for KPZ conjectures. Phase separation concerns the form of the boundary of a droplet of one substance suspended in another. When supercritical bond percolation on  $\mathbb{Z}^2$  is conditioned on the cluster (or droplet) containing the origin being finite and large, namely of finite size at least  $n^2$ , with  $n$  high, the interface at the boundary of this cluster is expected to exhibit KPZ scaling characteristics, with the scaling parameter  $n$  playing a comparable role to  $t$  in the preceding example. Indeed, the papers [42, 43, 44], which develop the study made in [3, 76], a surrogate of this interface, expressed in terms of the random cluster model, was investigated. The maximum length of the facets that comprise the boundary of the interface's convex hull was proved to typically have the order  $n^{2/3}(\log n)^{1/3}$ , while the maximum local roughness, namely the maximum distance from a point on the interface to the convex hull boundary, was shown to be of the order of  $n^{1/3}(\log n)^{2/3}$ .

Viewed in this light, the present article validates for the KPZ universality class the implied predictions: that exponent pairs of  $(1/3, 2, 3)$  and  $(2/3, 1/3)$  for power-law and logarithmic-power govern maximal polymer weight change under vertical endpoint displacement and maximal transversal polymer fluctuation.

In a natural sense, these two exponent pairs are accompanied by a third, namely  $(1/2, 1/2)$ , for interface regularity. In the example of parabolically curved Brownian motion,  $x \rightarrow B(x) - x^2t^{-1}$ , the modulus of continuity of the process on  $[-1, 1]$  is easily seen to have the form  $s^{1/2}(\log s^{-1})^{1/2}$ , up to a random constant, and uniformly in  $t \geq 1$ . In KPZ, this assertion finds a counterpart when it is made for the  $\text{Airy}_2$  process, which offers a limiting description in scaled coordinates of the weight of polymers of given lifetime with first endpoint fixed. This assertion has been proved in [39, Theorem 1.11(1)]. Recently, for a very broad class of initial data, the polymer weight profile was shown in [40, Theorem 1.3] to have a modulus of continuity of the order of  $s^{1/2}(\log s^{-1})^{2/3}$ , uniformly in the scaling parameter and the initial condition. Also see [29, Theorem 1.4] for a modulus of continuity statement that is uniform in all the lines of the Airy line ensemble. The present article and [40] derive different modulus of continuity results for polymer weight profiles. In [40], the weight profiles that are considered may be called ‘spatial’, in the sense that the variation of the polymer endpoint is *horizontal*. In contrast, Theorem 4.1.3 addresses ‘temporal’ weight profiles, where the variation in polymer endpoint is instead *vertical*. The two articles share a perspective of employing probabilistic and geometric techniques that harness limited integrable inputs, but those techniques are rather different: in [40], the tools concern resampling associated to the *Brownian Gibbs* property enjoyed by the Airy line ensemble, while, for Theorem 4.1.3, the key tools are surgeries on polymers allied with prelimiting moderate deviation estimates.

### 4.1.4 Organization

We continue with two sections that offer basic general tools. The first, Section 4.2, provides useful estimates including the basic integrable input Theorem 4.2.2 concerning the tail of polymer weights. Then, in Section 4.3, we state and prove the polymer ordering lemmas and some other basic results, which are essential tools in the proofs of the main theorems.

The remaining four sections, 4.4 – 4.7, contain the main proofs. Consecutively, these sections are devoted to proving:

- the polymer Hölder continuity upper bound Theorem 4.1.1;
- the modulus of continuity for maximum transversal fluctuation over short polymers, Theorem 4.1.2, subject to assuming Proposition 4.1.4;
- Hölder continuity for the polymer weight profile, Theorem 4.1.3;
- and the lower bound on transversal polymer fluctuation, Proposition 4.1.4.

We will stick to scaled coordinates in the results' statements and, except in Section 4.2, in their proofs. A bridge between scaled coordinates and the original ones is offered in this next section, in whose proofs we use the scaling map  $T_n$  from (4.2) and weight function  $W$  from (4.8) to transfer unscaled results to their scaled counterparts.

## 4.2 Scalings and estimates: input results and their adaptations

In this section, we assemble the results that we will quote in our arguments. Most of these results were derived in terms of unscaled coordinates in [16] and [14]. Point-to-point estimates of last passage percolation energies were used crucially in [16] to resolve the *slow bond* conjecture, and in [14] to show the coalescence of nearby geodesics, and those estimates will be employed in this chapter as well. The concerned results will either be recalled from [16] and [14] or proved in this section: in each case, the underlying integrable input is the pair of tail estimates concerning point-to-point polymer weights stated here in Theorem 4.2.2 and 4.2.3.

We state results in scaled coordinates – valuable we believe for grasping the putatively KPZ universal behaviour at stake – and the proofs explain how to obtain these statements from their unscaled and largely already available counterparts. The transformation from unscaled to scaled uses the definitions of the scaling map in (4.2) and the weight in (4.8).

First we observe some simple relations enjoyed by polymers and weights.

**The scaling principle.** Because of translation invariance and the definition (4.2), it is easy to see that for any  $x, y, t_1, t_2 \in \mathbb{R}$  with  $t_{1,2} = t_2 - t_1 > 0$  and  $(x, t_1) \stackrel{n}{\preceq} (y, t_2)$  (see Subsection

4.1.1), for any  $\theta \in [0, 1]$ ,

$$\rho_{n;(x,t_1)}^{\leftarrow;(y,t_2)}(t_1 + \theta t_{1,2}) \stackrel{d}{=} t_{1,2}^{2/3} \rho_{n;(xt_{1,2}^{-2/3},0)}^{\leftarrow;(yt_{1,2}^{-2/3},1)}(\theta) \stackrel{d}{=} t_{1,2}^{2/3} \rho_{n;(0,0)}^{\leftarrow;((y-x)t_{1,2}^{-2/3},1)}(\theta). \quad (4.13)$$

The same statement holds for the rightmost polymers as well. Here and throughout  $\stackrel{d}{=}$  denotes that the two random variables on either side have the same distribution. We will sometimes call the displayed assertion the *scaling principle*.

Also by translation invariance and the definition of weight in (4.8), it follows that

$$t_{1,2}^{-1/3} \mathbb{W}_{n;(x,t_1)}^{(y,t_2)} \stackrel{d}{=} \mathbb{W}_{nt_{1,2};(xt_{1,2}^{-2/3},0)}^{(yt_{1,2}^{-2/3},1)} \stackrel{d}{=} \mathbb{W}_{nt_{1,2};(0,0)}^{((y-x)t_{1,2}^{-2/3},1)}. \quad (4.14)$$

**Boldface notation for applying results.** In our proofs, we will naturally often be applying tools such as those stated in this section. Sometimes the notation of the tool and of the context of the application will be in conflict. To alleviate this conflict, we will use boldface notation when we specify the values of the parameters of a given tool in terms of quantities in the context of the application. We will first use this notational device shortly, in one of the upcoming proofs.

The next theorem, which was proved in [6], indicates a basic aspect of the role of scaled coordinates, though in fact we will never use the result.

**Theorem 4.2.1.** *As  $n \rightarrow \infty$ ,*

$$\mathbb{W}_{n;(0,0)}^{(0,1)} \Rightarrow F_{TW},$$

*where the convergence is in distribution and  $F_{TW}$  denotes the GUE Tracy-Widom distribution.*

For a definition of the GUE Tracy-Widom distribution, also called the  $F_2$  distribution, see [6].

The next two results, concerning moderate deviations for the polymer weight, are the only inputs from integrable probability used in this chapter. The first follows immediately from [64, Theorem 1.3], [65, Theorem 1.2] and (4.14); the second from [64, Theorem 1.3] and the same identity (4.14).

**Theorem 4.2.2.** *There exist positive constants  $c, s_0$  and  $n_0$  such that, for all  $t_1 < t_2$  with  $nt_{1,2} > n_0$  and  $s > s_0$ ,*

$$\mathbb{P}\left(t_{1,2}^{-1/3} \mathbb{W}_{n;(0,t_1)}^{(0,t_2)} \geq s\right) \leq e^{-cs^{3/2}},$$

and

$$\mathbb{P}\left(t_{1,2}^{-1/3} \mathbb{W}_{n;(0,t_1)}^{(0,t_2)} \leq -s\right) \leq e^{-cs^{3/2}}.$$

**Theorem 4.2.3.** *There exist constants  $c_2, s_0, n_0 > 0$  such that, for all  $t_1 < t_2$  with  $nt_{1,2} > n_0$  and  $s > s_0$ ,*

$$\mathbb{P}\left(t_{1,2}^{-1/3} \mathbf{W}_{n;(0,t_1)}^{(0,t_2)} \geq s\right) \geq e^{-c_2 s^{3/2}}.$$

We shall need not just tail bounds for weights of point-to-point polymers, but uniform tail bounds on polymer weights whose endpoints vary over fixed unit order intervals. The unscaled version of this theorem follows from [16, Propositions 10.1 and 10.5], results that themselves make essential use of Theorem 4.2.2.

**Theorem 4.2.4.** *There exist  $C, c \in (0, \infty), C_0 \in (1, \infty)$  and  $n_0 \in \mathbb{N}$  such that, for all  $t_1 < t_2$  with  $nt_{1,2} \geq n_0$ ,  $s \in [0, 10(nt_{1,2})^{2/3}]$ ,  $A = C_0^{-1} s^{1/4} n^{1/6} t_{1,2}^{5/6}$  and  $I$  and  $J$  intervals of length at most  $t_{1,2}^{2/3}$  that are contained in  $[-A, A]$ ,*

$$\mathbb{P}\left(\sup_{x \in I, y \in J} \left| t_{1,2}^{-1/3} \mathbf{W}_{n;(x,t_1)}^{(y,t_2)} + t_{1,2}^{-4/3} (x-y)^2 \right| > s\right) \leq C \exp\{-cs^{3/2}\}.$$

**Proof.** First we prove the theorem when  $t_1 = 0$  and  $t_2 = 1$  by invoking the unscaled version of this theorem from [16]. At the end we prove Theorem 4.2.4 for general  $t_1 < t_2$ . Observe that  $|x-y| < 2C_0^{-1} s^{1/4} n^{1/6} t_{1,2}^{5/6} < 2^{-1} n t_{1,2}^{1/3}$  for  $C_0 > 2 \cdot 10^{1/4}$  since  $s \leq 10(nt_{1,2})^{2/3}$ . This ensures that  $\mathbf{W}_{n;(x,t_1)}^{(y,t_2)}$  is well defined.

Let  $u = T_n^{-1}(x, 0) = (xn^{2/3}, -xn^{2/3})$  and  $v = T_n^{-1}(y, 1) = (n + yn^{2/3}, n - yn^{2/3})$ . If  $S_{u,v}$  denotes the slope of the line segment joining  $u$  and  $v$ , then  $|x-y| < 2^{-1}n$  ensures that  $3^{-1} < S_{u,v} < 3$ . Then, using the first order estimates (see [16, Corollary 9.1]) and a simple binomial expansion giving  $|(1-x)^{1/2} - (1-2^{-1}x)| \leq C_1 x^2$  for  $x \in (-1, 1)$ , we get that

$$|\mathbb{E}[X_u^v] - (2n - (x-y)^2 n^{1/3})| \leq C_2 n^{-1/3} (x-y)^4 + C_2 n^{1/3},$$

for some constants  $C_1, C_2 > 0$ , where  $X_u^v$  is defined in (4.1). Since  $|x-y| \leq 2C_0^{-1} s^{1/4} n^{1/6}$ ,

$$C_2 n^{-2/3} (x-y)^4 \leq 2^4 C_0^{-4} C_2 s < 2^{-1} s$$

for  $C_0 > 2^{5/4} C_2^{1/4}$ . Hence, using the definition of the weight function in (4.8), for all  $s \geq 6C_2$ ,

$$\begin{aligned} \left\{ \left| \mathbf{W}_{n;(x,0)}^{(y,1)} + (x-y)^2 \right| > s \right\} &\subseteq \left\{ n^{-1/3} |X_u^v - \mathbb{E}X_u^v| > s - C_2 n^{-2/3} (x-y)^4 - C_2 \right\} \\ &\subseteq \left\{ n^{-1/3} |X_u^v - \mathbb{E}X_u^v| > 3^{-1} s \right\}. \end{aligned}$$

Let  $U = T_n^{-1}(I \times \{0\})$  and  $V = T_n^{-1}(J \times \{1\})$ . For  $u \in U, v \in V$ , since  $3^{-1} < S_{u,v} < 3$ , we can invoke the proofs of [16, Propositions 10.1 and 10.5]. Observe that, for Poissonian last passage percolation, [16, Corollary 9.1] strengthens to

$$\mathbb{P}(|X_u^{u'} - \mathbb{E}X_u^{u'}| > \theta r^{1/3}) \leq e^{-C_1 \theta^{3/2}}. \quad (4.15)$$

Following the proofs of Proposition 10.1 and 10.5 of [16] verbatim, and using the above bound in (4.15) in place of Corollary 9.1 of [16], one thus has for all  $n, s$  large enough,

$$\mathbb{P}\left(\sup_{u \in U, v \in V} n^{-1/3} |X_u^v - \mathbb{E}X_u^v| > 2^{-1}s\right) \leq e^{-cs^{3/2}}.$$

Thus, for  $n$  large enough, and  $I$  and  $J$  intervals of at most unit length contained in the interval of length  $2C_0^{-1}s^{1/4}n^{1/6}$  centred at the origin,

$$\mathbb{P}\left(\sup_{x \in I, y \in J} \left| \mathbf{W}_{n;(x,0)}^{(y,1)} + (x-y)^2 \right| > s\right) \leq C \exp\{-cs^{3/2}\}. \quad (4.16)$$

We now make a first use of the boldface notation for applying results specified at the beginning of Section 4.2. For general  $t_1 < t_2$ , set  $\mathbf{n} = nt_{1,2}$ ,  $\mathbf{x} = xt_{1,2}^{-2/3}$ ,  $\mathbf{y} = yt_{1,2}^{-2/3}$ ,  $\mathbf{I} = t_{1,2}^{-2/3}I$ ,  $\mathbf{J} = t_{1,2}^{-2/3}J$  and  $\mathbf{s} = s$  in (4.16). Recall that the boldface variables are those of Theorem 4.2.4 and that these are written in terms of non-boldface parameters specified by the present context.

From the hypothesis of Theorem 4.2.4,  $\mathbf{I}$  and  $\mathbf{J}$  are intervals of at most unit length contained in  $[-\mathbf{n}^{1/6}, \mathbf{n}^{1/6}]$ . Thus, applying (4.16) and using the scaling principle (4.14), we get Theorem 4.2.4.  $\square$

Moving to unscaled coordinates, the transversal fluctuations for paths between  $(0, 0)$  and  $(n, n)$  around the diagonal were shown to be of the order  $n^{2/3+o(1)}$  with high probability in [51]. More precise estimates were established in [16]. However, the fluctuation of the geodesic at the point  $(r, r)$  for any  $r \leq n$  is only of the order  $r^{2/3}$ . This is the content of the next theorem which in essence is the scaled version of [14, Theorem 3] (Theorem 3 of Chapter 2 here) adapted for Poissonian LPP. In the proof of [14, Theorem 3], Theorem 4.2.2 is again essential, applied at several scales alongside a union bound. Recall that, for  $u, v \in \mathbb{R}^2$ ,  $\Phi_{n;u}^v$  is the set of all polymers from  $u$  to  $v$ , and  $\ell_u^v$  is the straight line joining  $u$  and  $v$ .

**Theorem 4.2.5.** *There exist positive constants  $n_0, s_1, c$  such that for all  $x, y, t_1, t_2 \in \mathbb{R}$  with  $t_{1,2} = t_2 - t_1 > 0$  and  $|x - y| \leq 2^{-1}n^{1/3}t_{1,2}$  and for all  $nt_{1,2} \geq n_0, s \geq s_1$  and  $t \in [t_1, t_2]$ ,*

$$\mathbb{P}\left(\max\left\{\left|\rho(t) - \ell_{(x,t_1)}^{(y,t_2)}(t)\right| : \rho \in \Phi_{n;(x,t_1)}^{(y,t_2)}\right\} \geq s\left((t-t_1) \wedge (t_2-t)\right)^{2/3}\right) \leq 2e^{-cs^3}. \quad (4.17)$$

Here  $a \wedge b$  denotes  $\min\{a, b\}$ .

**Proof of Theorem 4.2.5.** First we prove the theorem when  $t_1 = 0, t_2 = 1$ , and  $x = 0$ . Observe that in this case it is enough to bound the probabilities of the events

$$\left\{\left|\rho_{n;(0,0)}^{\leftarrow;(y,1)}(t) - \ell_{(0,0)}^{(y,1)}(t)\right| \geq s\left(t \wedge (1-t)\right)^{2/3}\right\} \text{ and}$$

$$\left\{\left|\rho_{n;(0,0)}^{\rightarrow;(y,1)}(t) - \ell_{(0,0)}^{(y,1)}(t)\right| \geq s\left(t \wedge (1-t)\right)^{2/3}\right\},$$

and use a union bound to obtain (4.17).

We first prove an upper bound for the probability of the first of these two events. Also, first assume that  $t \in [0, 2^{-1}]$ . To prove the bound in this case, we move to unscaled coordinates, and use [14, Theorem 3], which has been stated as Theorem 3 in Chapter 2 here.

To this end, let  $\Gamma := \Gamma_{(0,0)}^{\leftarrow; (n+yn^{2/3}, n-yn^{2/3})}$  be the leftmost geodesic, and  $\mathcal{S}$  the straight line from  $(0, 0)$  to  $(n + yn^{2/3}, n - yn^{2/3})$ . For  $r \in [0, n + yn^{2/3}]$ , let  $\Gamma(r)$  and  $\mathcal{S}(r)$  be such that  $(r, \Gamma(r)) \in \Gamma$  and  $(r, \mathcal{S}(r)) \in \mathcal{S}$ . Now, for  $r = nt$ ,

$$\begin{aligned} & \left\{ \left| \rho_{n;(0,0)}^{\leftarrow; (y, t_{1,2})}(t) - \ell_{(0,0)}^{(y, t_{1,2})}(t) \right| \geq st^{2/3} \right\} \\ &= \left\{ \left| n^{2/3} \rho_{n;(0,0)}^{\leftarrow; (y, 1)}(rn^{-1}) - n^{2/3} \ell_{(0,0)}^{(y, 1)}(rn^{-1}) \right| \geq sr^{2/3} \right\} \\ &\subseteq \left\{ |\Gamma(r') - \mathcal{S}(r')| \geq sr^{2/3} \right\} =: \mathbf{B}, \end{aligned} \quad (4.18)$$

where  $r'$  is such that the anti-diagonal line passing through  $(r, r)$  intersects  $\mathcal{S}$  at  $(r', \mathcal{S}(r'))$ . The last inclusion follows from the definition of the scaling map  $T_n$  in (4.2). Since  $|y| \leq 2^{-1}n^{1/3}$ ,  $2^{-1}r \leq r' \leq 2r$ . Thus,

$$\mathbf{B} \subseteq \left\{ |\Gamma(r') - \mathcal{S}(r')| \geq 2^{-1}s(r')^{2/3} \right\} =: \mathbf{C}. \quad (4.19)$$

Thus it is enough to bound the probability of the event  $\mathbf{C}$ . This local fluctuation estimate for the leftmost geodesic in (4.20) was proved for exponential directed last passage percolation in Theorem 3 in Chapter 2. The proof goes through verbatim for the leftmost (and also the rightmost) geodesic in Poissonian last passage percolation. Moreover, the refined bounds of Theorem 4.2.4 give corresponding improvements for Poissonian LPP: see Remark 2.1.3. This gives that, for some positive constants  $n_0, r_0, s_0$ , and for  $n \geq n_0, r' \geq r'_0$  and  $s \geq s_0$ ,

$$\mathbb{P}(\mathbf{C}) \leq e^{-cs^3}. \quad (4.20)$$

However, observe that (4.20) holds only when  $r' \geq r'_0$ . Now assume  $r' \leq r'_0$ , so that  $r \leq r_0$ , where  $r_0 = 2r'_0$ . Let the anti-diagonal passing through  $(r, r)$  intersect the geodesic  $\Gamma$  at  $v$  and the line  $\mathcal{S}$  at  $w$ . Clearly  $\|v - (r, r)\|_2 \leq 2^{1/2}r$ . Also, since  $|y| \leq 2^{-1}n^{1/3}$ ,

$$\|w - (r, r)\|_2 = 2^{1/2}|y|rn^{-1/3} \leq r.$$

Thus, with  $r = nt \leq r_0$ ,

$$\left| \rho_{n;(0,0)}^{\leftarrow; (y, 1)}(t) - \ell_{(0,0)}^{(y, 1)}(t) \right| = 2^{-1/2}n^{-2/3}\|v - w\|_2 \leq 2^{-1}(2^{1/2} + 1)n^{-2/3}r \leq 2r_0^{1/3}t^{2/3}.$$

Define  $s_1 = \max\{s_0, 2r_0^{1/3}\}$ . Then for  $n \geq n_0, s \geq s_1$  and  $t \in [0, 2^{-1}]$ ,

$$\mathbb{P} \left( \left| \rho_{n;(0,0)}^{\leftarrow; (y, 1)}(t) - \ell_{(0,0)}^{(y, 1)}(t) \right| \geq st^{2/3} \right) \leq e^{-cs^3}.$$

For  $t \in [2^{-1}, 1]$ , we consider the reversed polymer and translate it by  $-y$  so that its starting point is  $(0, 0)$ , that is,  $\rho'(v) = \rho_{n;(0,0)}^{\leftarrow;(y,1)}(1-v) - y$  for  $v \in [0, 1]$ . Now we follow the same arguments as above to get the bound for the probability of the event

$$\left\{ \left| \rho_{n;(0,0)}^{\leftarrow;(y,1)}(t) - \ell_{(0,0)}^{(y,1)}(t) \right| \geq s \left( t \wedge (1-t) \right)^{2/3} \right\}.$$

Since the same arguments work for the rightmost polymer  $\rho_{n;(0,0)}^{\rightarrow;(y,1)}$ , we get for  $n \geq n_0, s \geq s_1$  and all  $t \in [0, 1]$ ,

$$\mathbb{P} \left( \max \left\{ \left| \rho(t) - \ell_{(0,0)}^{(y,1)}(t) \right| : \rho \in \Phi_{n;(0,0)}^{(y,1)} \right\} \geq s \left( t \wedge (1-t) \right)^{2/3} \right) \leq 2e^{-cs^3}. \quad (4.21)$$

Now for general  $t_1 < t_2$ , set  $\mathbf{n} = nt_{1,2}$ ,  $\mathbf{y} = (y-x)t_{1,2}^{-2/3}$ ,  $\mathbf{s} = s$  and  $\mathbf{t} = t_{1,2}^{-1}(t-t_1)$ . Then from the hypothesis of Theorem 4.2.5,  $|\mathbf{y}| \leq 2^{-1}\mathbf{n}^{1/3}$  since  $|y-x| \leq 2^{-1}n^{1/3}t_{1,2}$ . Thus applying (4.21) and using the scaling principle (4.13), we obtain the theorem.  $\square$

The following theorem bounds the transversal fluctuation of polymers; (recall the definitions in (4.4) and (4.5)). The theorem essentially follows from [16, Theorem 11.1]; however, we replace the exponent in the upper bound with its optimal value. Again, [16, Theorem 11.1] uses the point-to-point estimate Theorem 4.2.2 at several scales.

**Theorem 4.2.6.** *There exist positive constants  $c, n_0$  and  $k_0$  such that, for  $t \in (0, 1]$ ,  $k \geq k_0$  and  $n \geq n_0 t^{-1}$ ,*

$$\mathbb{P} \left( \text{TF}_{n;(0,0)}^{(0,t)} \geq kt^{2/3} \right) \leq 2e^{-ck^3}.$$

**Proof.** Because of (4.5), it is enough to bound the probabilities of the events

$\left\{ \text{TF}(\rho_{n;(0,0)}^{\leftarrow;(0,t)}) \geq kt^{2/3} \right\}$  and  $\left\{ \text{TF}(\rho_{n;(0,0)}^{\rightarrow;(0,t)}) \geq kt^{2/3} \right\}$  and use a union bound. We bound only the first event, the arguments for the second event being the same. Then, as in the proof of Theorem 4.2.5, going to the unscaled coordinates, and defining  $\Gamma = \Gamma_{(0,0)}^{\leftarrow;(nt,nt)}$ , it is enough to show that

$$\mathbb{P} \left( \sup_{r \in [0, nt]} |\Gamma(r) - r| \geq k(nt)^{2/3} \right) \leq e^{-ck^3}. \quad (4.22)$$

From Theorem 4.2.5, it is easy to see that there exist constants  $c > 0$  and  $n_0, k_0 > 0$  such that, for all  $k > k_0$  and  $nt \geq n_0$ ,

$$\mathbb{P} \left( \left| \Gamma(2^{-1}nt) - 2^{-1}nt \right| > k(nt)^{2/3} \right) \leq e^{-ck^3}.$$

Using the above bound in place of [16, Lemma 11.3], and following the rest of the proof of [16, Theorem 11.1] verbatim, we get (4.22).  $\square$

## 4.3 Basic tools

Fundamental facts about ordering and concatenation of polymers will be used repeatedly in the proofs of the main theorems.

### 4.3.1 Polymer concatenation and superadditivity of weights

Let  $n \in \mathbb{N}$  and  $(x, t_1), (y, t_2) \in \mathbb{R}^2$  with  $t_1 < t_2$  and  $|x - y| < n^{1/3}(t_2 - t_1)$ . (This condition ensures that  $(x, t_1) \stackrel{n}{\preceq} (y, t_2)$ , see Subsection 4.1.1.) Let  $u = T_n^{-1}(x, t_1)$  and  $v = T_n^{-1}(y, t_2)$  and let  $\zeta$  be an increasing path from  $u$  to  $v$ . Let  $\gamma = T_n(\zeta)$ . We call  $\gamma$  an  $n$ -path. We shall often consider  $\gamma$  as a subset of  $\mathbb{R}^2$ , and call  $(x, t_1)$  its *starting point* and  $(y, t_2)$  its *ending point*. Moreover, similarly to the definition of the weight of a polymer in (4.8), we define the weight of an  $n$ -path as

$$n^{-1/3} (|\zeta| - 2nt_{1,2}), \quad (4.23)$$

where  $|\zeta|$  denotes the energy of  $\zeta$ , that is, the number of points in  $\Pi \setminus \{v\}$  that lie on  $\zeta$ .

Now, let  $(x, t_1), (y, t_2), (z, t_3) \in \mathbb{R}^2$  be such that  $t_1 < t_2 < t_3$ ,  $|x - y| < n^{1/3}(t_2 - t_1)$  and  $|y - z| < n^{1/3}(t_3 - t_2)$ , so that there exist polymers from  $(x, t_1)$  to  $(y, t_2)$ ; and from  $(y, t_2)$  to  $(z, t_3)$ . Let  $\rho_1$  be any polymer from  $(x, t_1)$  to  $(y, t_2)$ , and  $\rho_2$  any polymer from  $(y, t_2)$  to  $(z, t_3)$ . The union of these two subsets of  $\mathbb{R}^2$  is an  $n$ -path from  $(x, t_1)$  to  $(z, t_3)$ . We call this  $n$ -path the concatenation of  $\rho_1$  and  $\rho_2$  and denote it by  $\rho_1 \circ \rho_2$ . The weight of  $\rho_1 \circ \rho_2$  is  $W_{n;(x,t_1)}^{(y,t_2)} + W_{n;(y,t_2)}^{(z,t_3)}$ . This additivity is the reason that the endpoint  $v$  was excluded from the definition of path energy in Section 4.1.1.

Again, let  $n \in \mathbb{N}$  and  $(x, t_1), (y, t_2), (z, t_3) \in \mathbb{R}^2$  be such that  $t_1 < t_2 < t_3$  and  $|x - y| < n^{1/3}(t_2 - t_1)$  and  $|y - z| < n^{1/3}(t_3 - t_2)$ . Then

$$W_{n;(x,t_1)}^{(z,t_3)} \geq W_{n;(x,t_1)}^{(y,t_2)} + W_{n;(y,t_2)}^{(z,t_3)}. \quad (4.24)$$

Indeed, taking a polymer  $\rho_1$  from  $(x, t_1)$  to  $(y, t_2)$  and a polymer  $\rho_2$  from  $(y, t_2)$  to  $(z, t_3)$ , the weight of  $\rho_1 \circ \rho_2$  is a lower bound on  $W_{n;(x,t_1)}^{(z,t_3)}$ .

### 4.3.2 Polymer ordering lemmas

The first lemma roughly says that if two polymers intersect at two points during their lifetimes, then they are identical between these points.

**Lemma 4.3.1.** *Let  $n \in \mathbb{N}$  and  $(x_1, t_1), (x_2, t_2), (y_1, s_1), (y_2, s_2) \in \mathbb{R}^2$  and  $t, s \in \mathbb{R}$  be such that  $t_1 < t < s < s_1$ ,  $t_2 < t < s < s_2$ ,  $|x_1 - y_1| < n^{1/3}(s_1 - t_1)$  and  $|x_2 - y_2| < n^{1/3}(s_2 - t_2)$ . Suppose that  $\rho_{n;(x_1,t_1)}^{\leftarrow,(y_1,s_1)}$  and  $\rho_{n;(x_2,t_2)}^{\leftarrow,(y_2,s_2)}$  intersect at two points  $z_1 = (x, t)$  and  $z_2 = (y, s)$ . Then  $\rho_{n;(x_1,t_1)}^{\leftarrow,(y_1,s_1)}$  and  $\rho_{n;(x_2,t_2)}^{\leftarrow,(y_2,s_2)}$  are identical between  $t$  and  $s$ . The same statement holds for the rightmost polymers.*

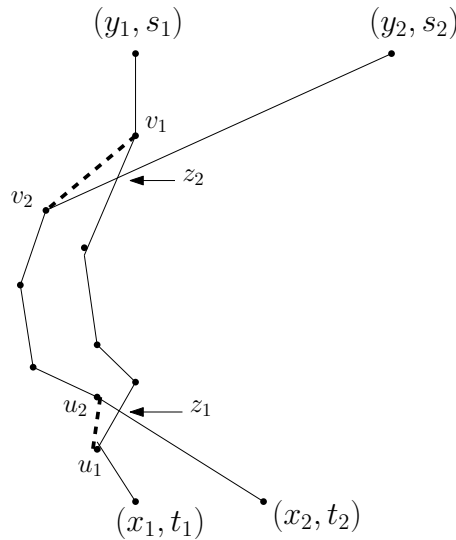


Figure 4.2: This illustrates Lemma 4.3.2. The points of the underlying Poisson process lying on a polymer are marked by dots, and the polymer is obtained by linearly interpolating between the points. The figure shows that both the paths cannot be leftmost polymers between their respective endpoints, since by joining the dashed lines, one obtains an alternative increasing path where the Poisson points between the intersecting points  $z_1$  and  $z_2$  in the two polymers are interchanged.

To simplify notation in the proof, we write  $\rho_1 = \rho_{n; (x_1, t_1) \leftarrow (y_1, s_1)}$  and  $\rho_2 = \rho_{n; (x_2, t_2) \leftarrow (y_2, s_2)}$ .

**Proof of Lemma 4.3.1.** First, for any polymer  $\rho$ , call a point  $u \in \rho$  a *Poisson* point of  $\rho$  if  $T_n^{-1}(u) \in \Pi \cap \Gamma$ , where  $\Gamma$  is the geodesic  $T_n^{-1}(\rho)$  and  $\Pi$  is the underlying unit rate Poisson point process. Also, for  $r_1, r_2 \in \rho$ , let  $\rho[r_1, r_2]$  denote the part of the polymer between the points  $r_1$  and  $r_2$ , and let  $\#\rho[r_1, r_2]$  denote the number of Poisson points that lie in  $\rho[r_1, r_2]$ . We first claim that  $\#\rho_1[z_1, z_2] = \#\rho_2[z_1, z_2]$  where  $z_1$  and  $z_2$  appear in the lemma's statement. For, if not, without loss of generality assume that  $\#\rho_1[z_1, z_2] < \#\rho_2[z_1, z_2]$  and let  $u_1$  and  $v_1$  be the Poisson points of  $\rho_1$  immediately before  $z_1$  and immediately after  $z_2$ ; and let  $u_2$  and  $v_2$  be the Poisson points of  $\rho_2$  immediately after  $z_1$  and immediately before  $z_2$ : see Figure 4.2. Then joining  $u_1$  to  $u_2$  and  $v_1$  to  $v_2$  (shown in the figure by dashed lines), one gets an alternative path  $\rho'$  between  $(x_1, t_1)$  and  $(y_1, s_1)$  that has more Poisson points than  $\rho_1$ , thereby contradicting that  $\rho_1$  is a polymer between  $(x_1, t_1)$  and  $(y_1, s_1)$ . Thus,  $\#\rho_1[z_1, z_2] = \#\rho_2[z_1, z_2]$ . Since both  $\rho_1$  and  $\rho_2$  are leftmost polymers between their respective endpoints, we see that  $\rho_1[z_1, z_2] = \rho_2[z_1, z_2]$ . This proves the lemma.  $\square$

The next result roughly says that two polymers that begin and end at the same heights, with the endpoints of one to the right of the other's, cannot cross during their shared lifetime.

**Lemma 4.3.2 (Polymer Ordering).** *Fix  $n \in \mathbb{N}$ . Consider the points  $(x_1, t_1), (x_2, t_1), (y_1, t_2), (y_2, t_2) \in \mathbb{R}^2$  such that  $t_1 < t_2$ ,  $x_1 \leq x_2$ ,  $y_1 \leq y_2$ ,  $|x_1 - y_1| < n^{1/3}(t_2 - t_1)$  and*

$|x_2 - y_2| < n^{1/3}(t_2 - t_1)$ . Then  $\rho_{n;(x_1,t_1)}^{\leftarrow;(y_1,t_2)}(t) \leq \rho_{n;(x_2,t_1)}^{\leftarrow;(y_2,t_2)}(t)$  and  $\rho_{n;(x_1,t_1)}^{\rightarrow;(y_1,t_2)}(t) \leq \rho_{n;(x_2,t_1)}^{\rightarrow;(y_2,t_2)}(t)$  for all  $t \in [t_1, t_2]$ .

Let  $\rho_1 = \rho_{n;(x_1,t_1)}^{\leftarrow;(y_1,t_2)}$  and  $\rho_2 = \rho_{n;(x_1,t_1)}^{\rightarrow;(y_1,t_2)}$ .

**Proof of Lemma 4.3.2.** Supposing otherwise, there exists  $z = (x, y) \in \rho_2$  such that  $x < \rho_1(y)$ . But then there exist  $z_1, z_2 \in \rho_1 \cap \rho_2$  straddling the point  $z$ . By Lemma 4.3.1,  $\rho_1[z_1, z_2] = \rho_2[z_1, z_2]$ , and hence  $z \in \rho_1 \cap \rho_2$ , a contradiction.  $\square$

By ordering, a polymer whose endpoints are straddled between those of a pair of polymers becomes sandwiched between those polymers.

**Corollary 4.3.3.** Fix  $n \in \mathbb{N}$ . Consider points  $(x_1, t_1), (x_2, t_1), (x_3, t_1), (y_1, t_2), (y_2, t_2), (y_3, t_2) \in \mathbb{R}^2$  such that  $t_1 < t_2$ ,  $x_1 \leq x_2 \leq x_3$ ,  $y_1 \leq y_2 \leq y_3$  and  $|x_i - y_i| < n^{1/3}(t_2 - t_1)$  for  $i = 1, 2, 3$ . Let  $t \in (t_1, t_2)$ . Let  $\rho_i = \rho_{n;(x_i,t_1)}^{\leftarrow;(y_i,t_2)}$  for  $i = 1, 2, 3$ . Then

$$|\rho_2(t) - \rho_2(t_1)| \leq \max_{i \in \{1,3\}} |\rho_i(t) - \rho_i(t_1)| + \max_{i \in \{1,3\}} |x_i - x_2|.$$

The same result holds for rightmost polymers.

**Proof.** By Lemma 4.3.2,

$$\rho_1(t) \leq \rho_2(t) \leq \rho_3(t).$$

The result now follows immediately.  $\square$

## 4.4 Exponent pair $(2/3, 1/3)$ for a single polymer: Proof of Theorem 4.1.1

In this section, we show that the sequence  $\{\rho_n^{\leftarrow} : n \in \mathbb{N}\}$  of leftmost  $n$ -polymers from  $(0, 0)$  to  $(0, 1)$  is tight, and any weak limit is Hölder  $2/3$ -continuous with a polylogarithmic correction of order  $1/3$ . The main two ingredients in this proof are the local regularity estimate Theorem 4.2.5 and the polymer ordering Lemma 4.3.2. First, we bound the fluctuation of the polymer near any given point  $z \in [0, 1]$ .

**Proposition 4.4.1.** There exist positive constants  $n_0, s_1$  and  $c$  such that, for all  $n \geq n_0, s \geq s_1, z \in [0, 1]$  and  $0 \leq t \leq 1 - z$ ,

$$\mathbb{P}(|\rho_n^{\leftarrow}(z+t) - \rho_n^{\leftarrow}(z)| \geq st^{2/3}) \leq 10t^{-2/3}e^{-cs^3}. \quad (4.25)$$

The same statement holds for  $\rho_n^{\rightarrow}$ .

As we now explain, the proposition will be proved by reducing to the case that  $z = 0$ , when the result follows from Theorem 4.2.5. For any fixed  $z \in (0, 1)$ , Theorem 4.2.5 again guarantees that the polymer  $\rho_n^{\leftarrow}$  is at distance at most  $s$  from the point  $(0, z)$  with probability at least  $1 - e^{-cs^3}$ . We break the horizontal line segment of length  $2s$  centred at  $(0, z)$  into



Fix  $s \leq (nt)^{1/3}$  and  $z \in [0, 1]$ . For  $t \geq 8^{-3}$ ,

$$\{|\rho_n^\leftarrow(z+t) - \rho_n^\leftarrow(z)| \geq st^{2/3}\} \subseteq \{|\rho_n^\leftarrow(z+t) - \rho_n^\leftarrow(z)| \geq 8^{-2}s\} \subseteq \left\{ \text{TF}_{n;(0,0)}^{(0,1)} \geq 2^{-1}8^{-2}s \right\},$$

where  $\text{TF}_{n;(0,0)}^{(0,1)}$  is defined in (4.5). Hence, applying Theorem 4.2.6 with the parameter specifications  $\mathbf{t} = 1$  and  $\mathbf{k} = 2^{-1}8^{-2}s$ , we get that (4.25) holds for all  $n, s$  large enough. Hence we assume that  $t \leq 8^{-3}$ . Also, let us assume for now that  $z \in [0, 2^{-1}]$ .

Let  $L$  be the line segment  $[-s, s] \times \{z\}$ . Let  $\mathbf{E}$  be the event that  $\rho_n^\leftarrow$  passes through  $L$ . By Theorem 4.2.5 with  $\mathbf{n} = n, \mathbf{t} = z, \mathbf{x} = 0, \mathbf{y} = 0, \mathbf{s} = s, \mathbf{t}_1 = 0$  and  $\mathbf{t}_2 = 1$ , we have that, for  $n \geq n_1$  and  $s \geq s_1$ ,

$$\mathbb{P}(\mathbf{E}) \geq 1 - 2e^{-cs^3}.$$

Now, we divide  $L$  into  $\lceil 4t^{-2/3} \rceil$ -many adjacent intervals of length at most  $2^{-1}st^{2/3}$ , and let  $(x_i, z), i = 0, 1, 2, \dots, \lceil 4t^{-2/3} \rceil$  be the endpoints of these intervals, i.e.,

$$x_i = -s + 2^{-1}ist^{2/3} \quad \text{for } i = 0, 1, 2, \dots, \lceil 4t^{-2/3} \rceil.$$

Let  $\rho_n^{(i)} := \rho_{n;(x_i, z)}^{\leftarrow; (0,1)}$  be the leftmost polymer from  $(x_i, z)$  to  $(0, 1)$ .

By Corollary 4.3.3, on  $\mathbf{E}$ ,

$$|\rho_n^\leftarrow(z+t) - \rho_n^\leftarrow(z)| \leq \max_{i \in \llbracket 0, \lceil 4t^{-2/3} \rceil \rrbracket} |\rho_n^{(i)}(z+t) - \rho_n^{(i)}(z)| + 2^{-1}st^{2/3}. \quad (4.26)$$

Also, for any fixed  $i \in \llbracket 0, \lceil 4t^{-2/3} \rceil \rrbracket$ , let  $\ell^{(i)} = \ell_{(x_i, z)}^{(0,1)}$  be the straight line segment joining  $(x_i, z)$  and  $(0, 1)$ . Then, since  $z \in [0, 2^{-1}]$  and  $t \leq 8^{-3}$ , for any  $i \in 0, 1, 2, \dots, \lceil 4t^{-2/3} \rceil$ ,

$$|\ell^{(i)}(z) - \ell^{(i)}(z+t)| \leq \frac{st}{1-z} \leq 2st \leq 4^{-1}st^{2/3}.$$

Since  $\rho_n^{(i)}(z) = \ell^{(i)}(z) = x_i$ ,

$$\begin{aligned} |\rho_n^{(i)}(z+t) - \rho_n^{(i)}(z)| &\leq |\rho_n^{(i)}(z+t) - \ell^{(i)}(z+t)| + |\ell^{(i)}(z+t) - \ell^{(i)}(z)| \\ &\leq |\rho_n^{(i)}(z+t) - \ell^{(i)}(z+t)| + 4^{-1}st^{2/3}. \end{aligned}$$

Thus, on the event  $\mathbf{E}$ , by (4.26),

$$|\rho_n^\leftarrow(z+t) - \rho_n^\leftarrow(z)| \leq \max_{i \in \llbracket 0, \lceil 4t^{-2/3} \rceil \rrbracket} |\rho_n^{(i)}(z+t) - \ell^{(i)}(z+t)| + \frac{3}{4}st^{2/3}.$$

From here, it follows by taking a union bound that

$$\begin{aligned} &\mathbb{P}(|\rho_n^\leftarrow(z+t) - \rho_n^\leftarrow(z)| \geq st^{2/3}) \\ &\leq \mathbb{P}(\mathbf{E}^c) + \sum_{i=0}^{\lceil 4t^{-2/3} \rceil} \mathbb{P}(|\rho_n^{(i)}(z+t) - \ell^{(i)}(z+t)| \geq 4^{-1}st^{2/3}) \\ &\leq 10t^{-2/3}e^{-cs^3}, \end{aligned}$$

for some absolute positive constant  $c$  and all  $n \geq 2n_0$ . Here the last inequality follows by applying Theorem 4.2.5 to each of the polymers  $\rho^{(i)}$ . For given  $i$ , set the parameters  $\mathbf{n} = n, \mathbf{t}_1 = z, \mathbf{t}_2 = 1, \mathbf{t} = t + z, \mathbf{x} = -s + 2^{-1}ist^{2/3}, \mathbf{y} = 0$  and  $\mathbf{s} = 4^{-1}s$ . Since  $z \in [0, 2^{-1}]$  and  $s \leq (nt)^{1/3}$ , we have that  $|\mathbf{x} - \mathbf{y}| \leq s \leq n^{1/3}t^{1/3} \leq 8^{-1}n^{1/3} \leq 4^{-1}\mathbf{n}^{1/3}\mathbf{t}_{1,2}$ . Thus one can apply Theorem 4.2.5 to get the above inequality for all  $\mathbf{n}\mathbf{t}_{1,2} \geq 2^{-1}n \geq n_0$ .

For  $z \in [2^{-1}, 1]$ , define the reversed polymer  $\widehat{\rho}_n^{\leftarrow}$  by  $\widehat{\rho}_n^{\leftarrow}(a) = \rho_n^{\leftarrow}(1 - a)$  for  $a \in [0, 1]$ , and follow the above argument.  $\square$

Next we show the tightness of the members of the sequence  $\{\rho_n^{\leftarrow}\}_{n \in \mathbb{N}}$  as elements in the space  $(C[0, 1], \|\cdot\|_\infty)$ . We prove that Proposition 4.4.1 guarantees that Kolmogorov-Chentsov's tightness criterion is satisfied.

**Proof of Theorem 4.1.1(a).** Fix  $n \geq n_0$  and any  $\lambda > 0$ . Fix  $t \in (0, 1]$  small enough that  $\lambda t^{-2/3} \geq s_1$ , where  $n_0$  and  $s_1$  are as in Proposition 4.4.1. Also fix some  $M \in \mathbb{N}$  large enough that  $2M - 2/3 > 1$ . Then it follows from Proposition 4.4.1 that for any  $z, z' \in [0, 1]$  with  $|z - z'| = t$ ,

$$\mathbb{P}(|\rho_n^{\leftarrow}(z') - \rho_n^{\leftarrow}(z)| \geq \lambda) \leq 10t^{-2/3}e^{-c(\lambda^3 t^{-2})} \leq K_M \lambda^{-3M} t^{2M-2/3} = K_M \lambda^{-3M} |z' - z|^{2M-2/3}, \quad (4.27)$$

where  $K_M := \sup_{x \geq 0} x^M e^{-cx} < \infty$ . Since  $2M - 2/3 > 1$ , by Kolmogorov-Chentsov's tightness criterion (see for example [31, Theorem 8.1.3]), it follows that the sequence  $\{\rho_n^{\leftarrow}\}_{n \in \mathbb{N}}$  is tight in  $(C[0, 1], \|\cdot\|_\infty)$ .  $\square$

#### 4.4.1 Modulus of continuity

Here we prove Theorem 4.1.1(b), thus finding the modulus of continuity for any weak limit of a weakly converging subsequence of  $\{\rho_n^{\leftarrow}\}_{n \in \mathbb{N}}$ . We will follow the arguments used to derive the Kolmogorov continuity criterion, where one infers Hölder continuity of a stochastic process from moment bounds on the difference of the process between pairs of times. Thus we introduce the set of dyadic rationals

$$D = \bigcup_{i=0}^{\infty} 2^{-i}\mathbb{Z}.$$

Next is the first step towards proving the modulus of continuity.

**Lemma 4.4.2.** *Let  $\rho_*^{\leftarrow}$  be the weak limit of a weakly converging subsequence of  $\{\rho_n^{\leftarrow}\}_{n \in \mathbb{N}}$ . Then there exists a universal positive constant  $C$  (not depending on the particular weak limit  $\rho_*^{\leftarrow}$ ) such that, almost surely, for some random  $m_0(\omega) \in \mathbb{N}$  and for all  $s, t \in D \cap [0, 1]$  with  $|t - s| \leq 2^{-m_0(\omega)}$ ,*

$$|\rho_*^{\leftarrow}(t) - \rho_*^{\leftarrow}(s)| \leq C(t - s)^{2/3} (\log(t - s)^{-1})^{1/3}.$$

**Proof.** For  $m \in \mathbb{N}$ , let  $S_m$  be the set of all intervals of the form  $[j2^{-m}, (j+1)2^{-m}]$ , for  $j \in \{0, 1, 2, \dots, 2^m - 1\}$ . Fix  $c_0 > (\frac{5}{3c})^{1/3}$ , where  $c$  is the constant in Proposition 4.4.1.

Writing  $\Rightarrow$  for convergence in distribution, let  $\{\rho_{n_k}^\leftarrow\}_{k \in \mathbb{N}}$  be a subsequence of  $\{\rho_n^\leftarrow\}_{n \in \mathbb{N}}$  such that  $\rho_{n_k}^\leftarrow \Rightarrow \rho_*^\leftarrow$  as random variables in  $(C[0, 1], \|\cdot\|_\infty)$ . Since for  $a, b \in [0, 1]$ , the map  $\tau_{a,b}$  defined by  $(C[0, 1], \|\cdot\|_\infty) \mapsto (\mathbb{R}, |\cdot|) : f \mapsto |f(a) - f(b)|$  is continuous,

$$U := \bigcup \left\{ \tau_{(j+1)2^{-m}, j2^{-m}}^{-1} \left( c_0 2^{-\frac{2m}{3}} (\log 2^m)^{1/3}, \infty \right) : j = 0, 1, \dots, 2^m - 1 \right\}$$

is an open set. Thus, by the Portmanteau theorem,

$$\begin{aligned} & \mathbb{P} \left( \sup_{j \in \{0, 1, \dots, 2^m - 1\}} |\rho_*^\leftarrow((j+1)2^{-m}) - \rho_*^\leftarrow(j2^{-m})| > c_0 2^{-\frac{2m}{3}} (\log 2^m)^{1/3} \right) \quad (4.28) \\ & \leq \liminf_k \mathbb{P} \left( \sup_{j \in \{0, 1, \dots, 2^m - 1\}} |\rho_{n_k}^\leftarrow((j+1)2^{-m}) - \rho_{n_k}^\leftarrow(j2^{-m})| > c_0 2^{-\frac{2m}{3}} (\log 2^m)^{1/3} \right) \\ & \leq \limsup_n \mathbb{P} \left( \sup_{j \in \{0, 1, \dots, 2^m - 1\}} |\rho_n^\leftarrow((j+1)2^{-m}) - \rho_n^\leftarrow(j2^{-m})| > c_0 2^{-\frac{2m}{3}} (\log 2^m)^{1/3} \right). \end{aligned}$$

Now, for all  $m$  large enough that  $(\log 2^m)^{1/3} \geq s_1$ , where  $s_1$  is as in Proposition 4.4.1, and all  $n \geq n_0$ , applying Proposition 4.4.1 and a union bound,

$$\begin{aligned} & \mathbb{P} \left( \sup_{j \in \{0, 1, \dots, 2^m - 1\}} |\rho_n^\leftarrow((j+1)2^{-m}) - \rho_n^\leftarrow(j2^{-m})| > c_0 2^{-\frac{2m}{3}} (\log 2^m)^{1/3} \right) \\ & \leq 10 \cdot 2^m \left( \frac{1}{2^m} \right)^{c_0^3 c - 2/3} \leq 10 \left( \frac{1}{2^m} \right)^{c_0^3 c - 5/3}. \end{aligned}$$

Hence, from (4.28),

$$\mathbb{P} \left( \sup_{j \in \{0, 1, \dots, 2^m - 1\}} |\rho_*^\leftarrow((j+1)2^{-m}) - \rho_*^\leftarrow(j2^{-m})| > c_0 2^{-\frac{2m}{3}} (\log 2^m)^{1/3} \right) \leq 10 \cdot 2^{-m(c_0^3 c - 5/3)}.$$

As the right hand side is summable in  $m$  (by the choice of  $c_0$  made at the beginning of the proof), the Borel-Cantelli lemma implies that there exists a null set  $N_0$ , such that, for each  $\omega \notin N_0$ , there is some  $m_0(\omega)$  for which  $m \geq m_0(\omega)$  entails that

$$|\rho_*^\leftarrow(t) - \rho_*^\leftarrow(s)| \leq c_0 (t - s)^{2/3} (\log(t - s)^{-1})^{1/3} \quad \text{for all } [s, t] \in S_m. \quad (4.29)$$

Now, let  $\omega \notin N_0$  and  $s, t \in D \cap [0, 1]$  be such that  $|s - t| \leq 2^{-m_0(\omega)}$ . Let  $m = m(s, t)$  be the greatest integer such that  $|s - t| \leq 2^{-m}$ ; then clearly,  $m \geq m_0(\omega)$ . Also, consider the binary expansions of  $s$  and  $t$ :

$$s = s_0 + \sum_{j>m} \sigma_j 2^{-j}, \quad t = t_0 + \sum_{j>m} \tau_j 2^{-j},$$

where  $\sigma_j, \tau_j \in \{0, 1\}$ , and each of the sequences is eventually zero. Either  $s_0 = t_0$  or  $[s_0, t_0] \in S_m$ . Moreover, for  $n \geq 1$ , let

$$s_n = s_0 + \sum_{m < j \leq m+n} \sigma_j 2^{-j}.$$

Then, for  $n \geq 1$ , either  $s_n = s_{n-1}$  or  $[s_{n-1}, s_n] \in S_{m+n}$ . Since  $m \geq m_0(\omega)$ , by (4.29),

$$|\rho_*^\leftarrow(t_0)(\omega) - \rho_*^\leftarrow(s_0)(\omega)| \leq c_0 2^{-\frac{2m}{3}} (\log 2^m)^{1/3}.$$

Also,

$$\begin{aligned} |\rho_*^\leftarrow(s)(\omega) - \rho_*^\leftarrow(s_0)(\omega)| &\leq \sum_{n=1}^{\infty} |\rho_*^\leftarrow(s_n)(\omega) - \rho_*^\leftarrow(s_{n-1})(\omega)| \leq \sum_{n=1}^{\infty} c_0 2^{-\frac{2(m+n)}{3}} (\log 2^{m+n})^{1/3} \\ &\leq C_1 2^{-\frac{2(m+1)}{3}} (\log 2^{m+1})^{1/3}, \end{aligned}$$

and similarly

$$|\rho_*^\leftarrow(t)(\omega) - \rho_*^\leftarrow(t_0)(\omega)| \leq C_2 2^{-\frac{2(m+1)}{3}} (\log 2^{m+1})^{1/3},$$

for some absolute constants  $C_1$  and  $C_2$ . Hence,

$$|\rho_*^\leftarrow(t) - \rho_*^\leftarrow(s)| \leq |\rho_*^\leftarrow(t) - \rho_*^\leftarrow(t_0)| + |\rho_*^\leftarrow(t_0) - \rho_*^\leftarrow(s_0)| + |\rho_*^\leftarrow(s) - \rho_*^\leftarrow(s_0)| \leq C 2^{-\frac{2m}{3}} (\log 2^m)^{1/3}.$$

Since by definition  $2^{-m-1} \leq |s - t| \leq 2^{-m}$ , the result follows.  $\square$

**Proof of Theorem 4.1.1(b).** For any  $s, t \in [0, 1]$  satisfying  $s < t$  and  $|s - t| \leq 2^{-m_0(\omega)}$ , choose  $s_k, t_k \in D \cap [s, t]$  such that  $s_k \searrow s$  and  $t_k \nearrow t$ . Then, since  $|s_k - t_k| \leq |s - t| \leq 2^{-m_0(\omega)}$ , by Lemma 4.4.2,

$$|\rho_*^\leftarrow(t_k) - \rho_*^\leftarrow(s_k)| \leq C(t_k - s_k)^{2/3} (\log(t_k - s_k)^{-1})^{1/3}.$$

Since  $\rho_*^\leftarrow(t_k)(\omega) \rightarrow \rho_*^\leftarrow(t)(\omega)$  and  $\rho_*^\leftarrow(s_k)(\omega) \rightarrow \rho_*^\leftarrow(s)(\omega)$ , the theorem follows by taking the limit as  $k \rightarrow \infty$ . The same argument applies without any change for the rightmost polymers as well.  $\square$

## 4.5 Exponent pair $(2/3, 1/3)$ for maximum fluctuation over short polymers: Proof of Theorem 4.1.2

In this section, we shall prove Theorem 4.1.2. It is the upper bound that is the more subtle. Recall the notation of transversal fluctuations from (4.4) and (4.5),  $\text{AdEndPair}_n(t)$  from (4.6) and  $\text{MTF}_n(t)$  from (4.7).

Here is the idea behind the proof. Proposition 4.1.4 offers a lower bound on the transversal fluctuation of a polymer between two given points. By considering order- $t^{-1}$  endpoint

pairs with disjoint intervening lifetimes of length  $t$ , we obtain a collection of independent opportunities for the fluctuation lower bound to occur. By tuning the probability of the individual event to have order  $t$ , at least one among the constituent events typically does occur, and the lower bound in Theorem 4.1.2 follows.

On the other hand, suppose that a big swing in the unit order region happens between a certain endpoint pair, with an intervening duration, or height difference, of order  $t$ . Members of the endpoint pair may be exceptional locations when viewed as functions of the underlying Poisson point field, both in horizontal and vertical coordinate. Thus, the upper bound in Theorem 4.1.2 does not follow directly from a union bound of a given endpoint estimate over elements in a discrete mesh, since such a mesh may not capture the exceptional endpoints. However, polymer ordering forces exceptional behaviour to become typical and to occur between an endpoint pair in a discrete mesh. To see this, assume that the original polymer between exceptional endpoints makes a big left swing. (Figure 4.4 illustrates the argument.) We take a discrete mesh endpoint pair whose lifetime includes that of the original polymer but has the same order  $t$ , and whose lower and upper points lie to the left of the original endpoint locations, about halfway between these and the leftmost coordinate visited by the original polymer. Then we consider the leftmost mesh polymer at the beginning and ending times of the original polymer. If the mesh polymer is to the right of the original polymer at any of these endpoints, then the mesh polymer has already made a big rightward swing at one of these endpoints. If, on the other hand, the mesh polymer is to the left of the original polymer at both the endpoints of the original polymer, then by polymer ordering Lemma 4.3.2, the mesh polymer cannot cross the original polymer during the latter's lifetime. Hence the big left swing of the original polymer forces a significant left swing for the mesh polymer as well.

**Proof of Theorem 4.1.2.** The lower bound follows in a straightforward way from Proposition 4.1.4. For any  $t \in (0, 1)$  and  $i \in \{0, 1, 2, \dots, [t^{-1}] - 1\}$ , define

$$F_{i,t,n} = \left\{ \text{TF}_{n;(0,it)}^{(0,(i+1)t)} \geq ct^{2/3} (\log t^{-1})^{1/3} \right\}.$$

For given such  $(t, i)$ , we apply Proposition 4.1.4 with parameter settings  $\mathbf{n} = n, \mathbf{t}_1 = it, \mathbf{t}_2 = (i + 1)t$  and  $\mathbf{s} = c(\log t^{-1})^{1/3}$ , to find that, when  $c(\log t^{-1})^{1/3} \geq s_0$  and  $n \geq \max\{\alpha_0^{-3} c^3 t^{-1} \log t^{-1}, n_0 t^{-1}\}$ ,

$$\mathbb{P}(F_{i,t,n}) \geq e^{-c^* c^3 \log t^{-1}} = t^{c^* c^3},$$

where the proposition specifies the quantities  $\alpha_0, n_0$  and  $s_0$ .

Thus, for all  $t \leq e^{-(c^{-1} s_0)^3}$  and  $i \in \{0, 1, 2, \dots, [t^{-1}] - 1\}$ ,

$$\liminf_n t^{-1} \mathbb{P}(F_{i,t,n}) = \liminf_n t^{-1} \mathbb{P}(F_{0,t,n}) \geq t^{c^* c^3 - 1}.$$

By choosing  $c > 0$  small enough that  $c^* c^3 < 1$ , one has  $\liminf_n t^{-1} \mathbb{P}(F_{0,t,n}) \rightarrow \infty$  as  $t \searrow 0$ . For such  $c > 0$ , using the definition (4.7) of  $\text{MTF}_n(t)$  and independence of the events

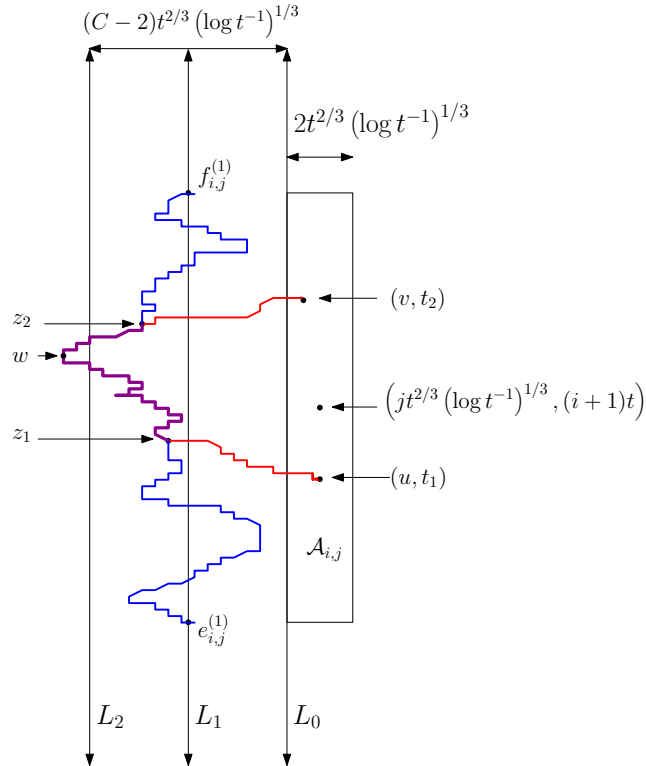


Figure 4.4: The figure illustrates the proof of the upper bound in Theorem 4.1.2. If the leftmost polymer between  $(u, t_1)$  and  $(v, t_2)$  (shown in red) makes a huge leftward fluctuation and the leftmost polymer between points  $e_{i,j}^{(1)}$  and  $f_{i,j}^{(1)}$  (shown in blue) is to the left of  $u$  and  $v$  at  $t_1$  and  $t_2$  respectively, then the blue polymer stays to the left of the red polymer between times  $t_1$  and  $t_2$  by polymer ordering. Thus the big left fluctuation transmits from the red to the blue polymer. If, however, the blue polymer reaches to the right of either  $u$  or  $v$ , then it creates a big right fluctuation for the blue polymer. Thus by bounding the fluctuations of a small number of polymers between deterministic endpoints, one can bound the fluctuation between all admissible endpoint pairs.

$F_{i,t,n}$  for  $i \in \{0, 1, 2, \dots, [t^{-1}] - 1\}$ ,

$$\mathbb{P} \left( \text{MTF}_n(t) t^{-2/3} (\log t^{-1})^{-1/3} < c \right) \leq \mathbb{P} \left( \bigcap_{i=0}^{[t^{-1}]-1} F_{i,t,n}^c \right) = \prod_{i=0}^{[t^{-1}]-1} \mathbb{P} (F_{i,t,n}^c) .$$

Thus,

$$\begin{aligned} & \limsup_n \mathbb{P} \left( \text{MTF}_n(t) t^{-2/3} (\log t^{-1})^{-1/3} < c \right) \\ & \leq \limsup_n \left( 1 - \mathbb{P}(F_{0,t,n}) \right)^{[t^{-1}]} \leq \limsup_n \exp \left\{ - [t^{-1}] \mathbb{P}(F_{0,t,n}) \right\} \rightarrow 0 , \end{aligned}$$

the latter convergence as  $t \searrow 0$ .

Now we show the upper bound. Fix  $t \in (0, 1]$  small enough that  $\psi t \leq t^{2/3}$ , where the parameter  $\psi$  appears in the definition (4.6) of  $\text{AdEndPair}_n(t)$ .

For any  $i = 0, 1, 2, \dots, \lceil t^{-1} \rceil$  and  $j \in \left[ \left[ -\lceil t^{-2/3}(\log t^{-1})^{-1/3} \rceil, \lceil t^{-2/3}(\log t^{-1})^{-1/3} \rceil \right] \right]$ , define the rectangle  $\mathcal{A}_{i,j}$  with lower-left corner  $\left( (j-1)t^{2/3}(\log t^{-1})^{1/3}, it \right)$ , width  $2t^{2/3}(\log t^{-1})^{1/3}$  and height  $2t$ . Figure 4.4 illustrates this rectangle and the arguments that follow.

Let  $C > 0$  be an even integer whose value will later be specified. For such  $i, j$  as above, define planar points

$$\begin{aligned} e_{i,j}^{(1)} &:= \left( (j - 2^{-1}C)t^{2/3}(\log t^{-1})^{1/3}, it \right), & f_{i,j}^{(1)} &:= \left( (j - 2^{-1}C)t^{2/3}(\log t^{-1})^{1/3}, (i+2)t \right), \\ e_{i,j}^{(2)} &:= \left( (j + 2^{-1}C)t^{2/3}(\log t^{-1})^{1/3}, it \right), & f_{i,j}^{(2)} &:= \left( (j + 2^{-1}C)t^{2/3}(\log t^{-1})^{1/3}, (i+2)t \right). \end{aligned}$$

Then we claim that, whatever the value of  $C > 0$ ,

$$\mathbf{B}_{i,j} := \left\{ \sup \left\{ \text{TF}_{n;(x_1, y_1)}^{(x_2, y_2)} : (x_1, y_1), (x_2, y_2) \in \mathcal{A}_{i,j}, y_2 > y_1 \right\} > Ct^{2/3}(\log t^{-1})^{1/3} \right\} \subseteq \mathbf{D}_{i,j}^{(1)} \cup \mathbf{D}_{i,j}^{(2)}, \quad (4.30)$$

where

$$\mathbf{D}_{i,j}^{(1)} := \left\{ \text{TF}_{n;e_{i,j}^{(1)}}^{f_{i,j}^{(1)}} \geq (2^{-1}C - 1)t^{2/3}(\log t^{-1})^{1/3} \right\}$$

and

$$\mathbf{D}_{i,j}^{(2)} := \left\{ \text{TF}_{n;e_{i,j}^{(2)}}^{f_{i,j}^{(2)}} \geq (2^{-1}C - 1)t^{2/3}(\log t^{-1})^{1/3} \right\}.$$

To see (4.30), define the vertical lines:

$$L_2 = \{x = (j - C + 1)t^{2/3}(\log t^{-1})^{1/3}\} \quad \text{and} \quad L'_2 = \{x = (j + C - 1)t^{2/3}(\log t^{-1})^{1/3}\}.$$

Then, on the event  $\mathbf{B}_{i,j}$ , there exists a pair of points  $(u, t_1), (v, t_2) \in \mathcal{A}_{i,j}$  such that either  $\rho_{n;(u, t_1)}^{\leftarrow; (v, t_2)}$  intersects  $L_2$  or  $\rho_{n;(u, t_1)}^{\rightarrow; (v, t_2)}$  intersects  $L'_2$ . We now show that, when  $\rho_{n;(u, t_1)}^{\leftarrow; (v, t_2)}$  intersects  $L_2$ , the event  $\mathbf{D}_{i,j}^{(1)}$  occurs. Let

$$\rho := \rho_{n;e_{i,j}^{(1)}}^{\leftarrow; f_{i,j}^{(1)}}.$$

Let  $\ell_{i,j}^{(1)}$  be the line segment joining  $e_{i,j}^{(1)}$  and  $f_{i,j}^{(1)}$ . If  $\rho(t_1) > u$ , then

$$\rho(t_1) - \ell_{i,j}^{(1)}(t_1) \geq (j-1)t^{2/3}(\log t^{-1})^{1/3} - (j-2^{-1}C)t^{2/3}(\log t^{-1})^{1/3} \geq (2^{-1}C-1)t^{2/3}(\log t^{-1})^{1/3},$$

and thus  $\mathbf{D}_{i,j}^{(1)}$  holds. Similarly, if  $\rho(t_2) > v$ , then  $\mathbf{D}_{i,j}^{(1)}$  holds. Now assume that  $\rho(t_1) < u$  and  $\rho(t_2) < v$ . Polymer ordering Lemma 4.3.2 then implies that  $\rho(t) \leq \rho_{n;(u, t_1)}^{\leftarrow; (v, t_2)}(t)$  for all  $t \in [t_1, t_2]$ . Thus  $\rho$  intersects  $L_2$  as well, and hence  $\mathbf{D}_{i,j}^{(1)}$  occurs.

By similar reasoning, we see that, when  $\rho_{n;(u,t_1)}^{\rightarrow;(v,t_2)}$  intersects  $L'_2$ , the event  $D_{i,j}^{(2)}$  occurs. We have proved (4.30).

For any compatible pair of points  $(u, v) \in \text{AdEndPair}_n(t)$ , there exists a pair  $(i, j)$  for which  $u, v \in \mathcal{A}_{i,j}$ ; here we use  $\psi t \leq t^{2/3}$ . Hence,

$$\begin{aligned} & \left\{ t^{-2/3} (\log t^{-1})^{-1/3} \text{MTF}_n(t) > C \right\} \\ & \subseteq \bigcup \left\{ \mathcal{B}_{i,j} : i \in \llbracket 0, \lceil t^{-1} \rceil \rrbracket, j \in \llbracket -\lceil t^{-2/3} (\log t^{-1})^{-1/3} \rceil, \lceil t^{-2/3} (\log t^{-1})^{-1/3} \rceil \rrbracket \right\} \\ & \subseteq \bigcup \left\{ \mathcal{D}_{i,j}^{(1)} \cup \mathcal{D}_{i,j}^{(2)} : i \in \llbracket 0, \lceil t^{-1} \rceil \rrbracket, j \in \llbracket -\lceil t^{-2/3} (\log t^{-1})^{-1/3} \rceil, \lceil t^{-2/3} (\log t^{-1})^{-1/3} \rceil \rrbracket \right\}, \end{aligned}$$

where (4.30) was used in the latter inclusion.

Thus, with  $c, k_0, n_0$  as in the statement of Theorem 4.2.6, for any fixed  $t$  small enough that  $\log t^{-1} \geq 2^2 k_0^3$ , and all  $n \geq n_0(2t)^{-1}$ , we have by a union bound and the translation invariance of the environment,

$$\begin{aligned} & \mathbb{P} \left( t^{-2/3} (\log t^{-1})^{-1/3} \text{MTF}_n(t) > C \right) \\ & \leq (2t^{-2/3} (\log t^{-1})^{-1/3} + 2)(t^{-1} + 2) \mathbb{P} \left( \text{TF}_{n;(0,0)}^{(0,2t)} > (2^{-1}C - 1)t^{2/3} (\log t^{-1})^{1/3} \right) \\ & \leq 2(t^{-2/3} + 1)(t^{-1} + 2) \exp \left\{ -c(2^{-1}C - 1)^3 \log t^{-1} \right\} \leq 8 \cdot t^{c(C/2-1)^3-5/3}. \end{aligned}$$

Here the second inequality follows from Theorem 4.2.6 with  $\mathbf{t} = 2t, \mathbf{k} = 2^{-2/3}(2^{-1}C - 1)(\log t^{-1})^{1/3}$  and  $\mathbf{n} = n$  being the parameter settings. The assumptions  $\log t^{-1} \geq 2^2 k_0^3$ , and  $n \geq n_0(2t)^{-1}$  ensure that  $\mathbf{n} \geq n_0 \mathbf{t}^{-1}$  and  $\mathbf{k} \geq k_0$  for any  $C \geq 2$ .

Finally, choosing  $C$  large enough that  $c(C/2 - 1)^3 > 5/3$ , we learn that

$$\mathbb{P} \left( t^{-2/3} (\log t^{-1})^{-1/3} \text{MTF}_n(t) > C \right) \rightarrow 0 \quad \text{as } t \searrow 0,$$

whenever  $n = n(t)$  verifies  $n \geq n_0(2t)^{-1}$ .

This completes the proof of Theorem 4.1.2.  $\square$

## 4.6 Exponent pair $(1/3, 2/3)$ for polymer weight: Proof of Theorem 4.1.3

A lemma and two propositions will lead to the proof of Theorem 4.1.3 on the Hölder continuity of  $[1, 2] \mapsto \mathbb{R} : t \mapsto \text{Wgt}_n(t)$ , the polymer weight profile under vertical displacement.

**Lemma 4.6.1.** *There exist positive constants  $n_0, r_0, s_0, c_0$  such that, for all  $n \geq n_0, z \in [1, 2], t \in [r_0 n^{-1}, 2 - z]$  and  $s \in [s_0, 10(nt)^{2/3}]$ ,*

$$\mathbb{P} \left( |\text{Wgt}_n(z+t) - \text{Wgt}_n(z)| \geq st^{1/3} \right) \leq 5e^{-c_0 s^{3/2}}.$$

We postpone the proof to Section 4.6.1 and first see how the lemma implies the upper bound in Theorem 4.1.3. This bound follows from Lemma 4.6.1 similarly to how Theorem 4.1.1 is derived from Proposition 4.4.1.

**Proposition 4.6.2.** *The sequence  $\{\text{Wgt}_n\}_{n \in \mathbb{N}}$  is tight in  $(C[1, 2], \|\cdot\|_\infty)$ . Moreover, if  $\text{Wgt}_*$  is the weak limit of a weakly converging subsequence of  $\{\text{Wgt}_n\}_{n \in \mathbb{N}}$ , then there exists a positive constant  $C$  not depending on the particular weak limit  $\text{Wgt}_*$  such that, almost surely,*

$$\limsup_{t \searrow 0} \sup_{1 \leq z \leq 2-t} |\text{Wgt}_*(z+t) - \text{Wgt}_*(z)| t^{-1/3} (\log t^{-1})^{-2/3} \leq C. \quad (4.31)$$

Lemma 4.6.1 holds only for  $t \in [\max\{r_0 n^{-1}, 10^{-3/2} s^{3/2} n^{-1}\}, 2-z]$  for some fixed constant  $r_0 > 0$ , and not for all  $t \in [0, 1-z]$ , as was the case in Proposition 4.4.1. Hence, we directly show tightness in the following proof instead of applying Kolmogorov-Chentsov's tightness criterion.

**Proof of Proposition 4.6.2.** To show the first statement, concerning tightness, we follow the proof of the tightness criterion used to derive [18, Theorem 12.3]. To this end, it is enough to show that, for given  $\varepsilon, \eta > 0$ , there exist  $\delta \in [0, 1]$ , which we may harmlessly suppose to verify  $\delta^{-1} \in \mathbb{N}$ , and  $N_0 \in \mathbb{N}$  such that, for all  $n \geq N_0$ ,

$$\sum_{j < \delta^{-1}} \mathbb{P} \left( \sup_{j\delta \leq u \leq (j+1)\delta} |\text{Wgt}_n(1+u) - \text{Wgt}_n(1+j\delta)| \geq \varepsilon \right) < \eta. \quad (4.32)$$

Assume then that  $\varepsilon, \eta > 0$  are given small constants. For the time being, fix some  $\delta > 0$  small to be chosen later (depending on  $\varepsilon$  and  $\eta$ ).

Now fix any  $M > 1$ . For any  $z_1, z_2 \in [1, 2]$  such that  $|z_1 - z_2| = 10^{-1} \varepsilon n^{-2/3}$ , set  $t = |z_1 - z_2|$ . For all  $\lambda \in [0, \varepsilon]$ , clearly  $\lambda t^{-1/3} \leq 10(nt)^{2/3}$ . Hence, choosing  $s = \lambda t^{-1/3}$  in Lemma 4.6.1, one gets, for all  $n$  large enough,

$$\mathbb{P} \left( |\text{Wgt}_n(z_1) - \text{Wgt}_n(z_2)| \geq \lambda \right) \leq K_M \lambda^{-3M} |z_1 - z_2|^M, \quad (4.33)$$

for some constant  $K_M$  depending only on  $M$ .

To establish tightness, the general strategy is to bound the distribution of the maximum of certain fluctuations. To achieve this, we crucially use the bound in (4.33) together with the inequality in [18, Theorem 12.2] that bounds the maximum of partial sums. To this end, fix  $j < \delta^{-1}$ , and break the interval  $[j\delta, (j+1)\delta]$  into  $[\delta\beta^{-1}]$ -many subintervals of length  $\beta := 10^{-1} \varepsilon n^{-2/3}$  each, and follow the proof of the inequality in [18, Theorem 12.2] to obtain

$$\mathbb{P} \left( \max_{0 \leq i \leq [\delta\beta^{-1}]} |\text{Wgt}_n(1+j\delta+i\beta) - \text{Wgt}_n(1+j\delta)| \geq \frac{\varepsilon}{2} \right) \leq K'_M \varepsilon^{-3M} \delta^M, \quad (4.34)$$

for some appropriate constant  $K'_M$  depending only on  $M$ . Note that by [18, Theorem 12.2] it directly follows that if (4.33) holds for all  $\lambda > 0$ , then (4.34) holds for all  $\varepsilon > 0$ . However,

in our case (4.33) holds for all  $\lambda \in [0, \varepsilon]$ , instead of all  $\lambda > 0$ . Hence, we resort to the proof of [18, Theorem 12.2] which shows that if for some fixed  $\varepsilon > 0$ , (4.33) holds for all  $\lambda \in [0, \varepsilon]$ , then (4.34) holds for that particular  $\varepsilon$ .

Now, fix any  $i \in \llbracket 0, \lceil \delta\beta^{-1} \rceil - 1 \rrbracket$ . For any  $u \in [j\delta + i\beta, j\delta + (i+1)\beta]$ , it clearly follows from the definition (4.8),

$$\mathbf{W}_{n;(0,1+u)}^{(0,1+j\delta+(i+1)\beta)} \geq -2n^{2/3}(1 + j\delta + (i+1)\beta - (1+u)) \geq -2n^{2/3}\beta, \text{ and}$$

$$\mathbf{W}_{n;(0,1+j\delta+i\beta)}^{(0,1+u)} \geq -2n^{2/3}(1+u - (1+j\delta+i\beta)) \geq -2n^{2/3}\beta.$$

Thus, for any  $u \in [j\delta + i\beta, j\delta + (i+1)\beta]$ , by superadditivity of polymer weights described in (4.24),

$$\mathbf{W}_{n;(0,0)}^{(0,1+j\delta+i\beta)} - 2n^{2/3}\beta \leq \mathbf{W}_{n;(0,0)}^{(0,1+j\delta+i\beta)} + \mathbf{W}_{n;(0,1+j\delta+i\beta)}^{(0,1+u)} \leq \mathbf{W}_{n;(0,0)}^{(0,1+u)} \text{ and}$$

$$\mathbf{W}_{n;(0,0)}^{(0,1+u)} \leq \mathbf{W}_{n;(0,0)}^{(0,1+j\delta+(i+1)\beta)} - \mathbf{W}_{n;(0,1+u)}^{(0,1+j\delta+(i+1)\beta)} \leq \mathbf{W}_{n;(0,0)}^{(0,1+j\delta+(i+1)\beta)} + 2n^{2/3}\beta.$$

This, together with (4.11), imply that for any  $i \in \llbracket 0, \lceil \delta\beta^{-1} \rceil - 1 \rrbracket$  and  $u \in [j\delta + i\beta, j\delta + (i+1)\beta]$ ,

$$\begin{aligned} & n^{1/3} |\mathbf{Wgt}_n(1+u) - \mathbf{Wgt}_n(1+j\delta)| \leq 2n\beta + 2 + \\ & n^{1/3} \max \left\{ |\mathbf{Wgt}_n(1+j\delta+i\beta) - \mathbf{Wgt}_n(1+j\delta)|, |\mathbf{Wgt}_n(1+j\delta+(i+1)\beta) - \mathbf{Wgt}_n(1+j\delta)| \right\}. \end{aligned} \quad (4.35)$$

Since  $2n\beta = 5^{-1}\varepsilon n^{1/3}$ , for all  $n$  large enough that  $2n^{-1/3} \leq \varepsilon/5$ , (4.34) and (4.35) imply

$$\begin{aligned} & \mathbb{P} \left( \sup_{j\delta \leq u \leq (j+1)\delta} |\mathbf{Wgt}_n(1+u) - \mathbf{Wgt}_n(1+j\delta)| \geq \varepsilon \right) \\ & \leq \mathbb{P} \left( \max_{0 \leq i \leq \lceil \delta\beta^{-1} \rceil} |\mathbf{Wgt}_n(1+j\delta+i\beta) - \mathbf{Wgt}_n(1+j\delta)| \geq \frac{\varepsilon}{2} \right) \leq K'_M \varepsilon^{-3M} \delta^M. \end{aligned}$$

Thus, by choosing  $\delta$  small enough that  $K'_M \varepsilon^{-3M} \delta^{M-1} < \eta$ , we obtain (4.32), and hence tightness.

To show (4.31), we follow the proof of Theorem 4.1.1(b). Let  $n_0, r_0, s_0$  and  $c_0$  be as in Lemma 4.6.1. For any fixed  $m \in \mathbb{N}$  such that  $c_1 (\log 2^m)^{2/3} \geq s_0$ , and any  $j \in \{0, 1, 2, \dots, 2^m - 1\}$ , and all  $n \geq \max\{r_0 2^m, 10^{-3/2} c_1^{3/2} 2^m \log 2^m\}$ , by applying Lemma 4.6.1 with the parameters  $\mathbf{n} = n, \mathbf{t} = 2^{-m}$  and  $\mathbf{s} = c_1 (\log 2^m)^{2/3}$ , it follows that

$$\mathbb{P} \left( \left| \mathbf{Wgt}_n(1 + (j+1)2^{-m}) - \mathbf{Wgt}_n(1 + j2^{-m}) \right| > c_1 2^{-\frac{m}{3}} (\log 2^m)^{2/3} \right) \leq 5 \cdot 2^{-m(c_0 c_1^{3/2})}.$$

Now, observe that (4.28) in the proof of Lemma 4.4.2 carries over verbatim to the present case. By choosing  $c_1$  high enough that  $c_0 c_1^{3/2} > 1$ , and exactly imitating the rest of the proof of Lemma 4.4.2 followed by the proof of Theorem 4.1.1(b), we complete the proof of Proposition 4.6.2.  $\square$

Turning to prove the lower bound in (4.12), we restate it now.

**Proposition 4.6.3.** *There exists a constant  $c > 0$  such that, almost surely,*

$$\liminf_{t \searrow 0} \sup_{1 \leq z \leq 2-t} t^{-1/3} (\log t^{-1})^{-2/3} |\text{Wgt}_*(z+t) - \text{Wgt}_*(z)| \geq c.$$

This result will follow directly from weight superadditivity, i.e.  $\text{W}_{n;(0,0)}^{(0,1+z+t)} - \text{W}_{n;(0,0)}^{(0,1+z)} \geq \text{W}_{n;(0,1+z)}^{(0,1+z+t)}$  for  $z, t > 0$ , control on weight with given endpoints via Theorem 4.2.3, independence in disjoint strips, and the weight  $\text{W}_{n;(0,1+z)}^{(0,1+z+t)}$  depending on the configuration in the strip delimited by the lines  $y = 1 + z$  and  $y = 1 + z + t$ . The proof is reminiscent of an argument for a similar statement made for Brownian motion: see the proof on page 362 of Exercise 1.7 in the book [66].

**Proof of Proposition 4.6.3.** We need to show that, for some constant  $c > 0$ , almost surely, there exists  $\varepsilon > 0$  such that, for all  $0 < t < \varepsilon$  and some  $z \in [1, 2 - t]$ ,

$$|\text{Wgt}_*(z+t) - \text{Wgt}_*(z)| \geq ct^{1/3} (\log t^{-1})^{2/3}.$$

Let  $c > 0$  satisfy  $2^{3/2} c_2 c^{3/2} < 1$ , where  $c_2$  arises from Theorem 4.2.3. For integers  $n, m \geq 1$  and  $k \in \{0, 1, 2, \dots, m-1\}$ , we define the events

$$\mathbf{A}_{k,m,n} = \left\{ \text{Wgt}_n(1 + (k+1)m^{-1}) - \text{Wgt}_n(1 + km^{-1}) \geq cm^{-1/3} (\log m)^{2/3} \right\}$$

and

$$\mathbf{A}_{k,m} = \left\{ \text{Wgt}_*(1 + (k+1)m^{-1}) - \text{Wgt}_*(1 + km^{-1}) \geq cm^{-1/3} (\log m)^{2/3} \right\}.$$

Also let

$$\mathbf{B}_{k,m,n} = \left\{ \text{W}_{n;(0,1+km^{-1})}^{(0,1+(k+1)m^{-1})} \geq cm^{-1/3} (\log m)^{2/3} + 2n^{-1/3} \right\}.$$

Let  $n_0, s_0$  and  $c_2$  be as in Theorem 4.2.3, and let  $m_0$  be large enough that  $2c(\log m_0)^{2/3} \geq \max\{s_0, 4n_0^{-1/3}\}$ . Then from Theorem 4.2.3 with parameter settings  $\mathbf{t}_1 = 1 + km^{-1}$ ,  $\mathbf{t}_2 = 1 + (k+1)m^{-1}$ ,  $\mathbf{t}_{1,2} = m^{-1}$ ,  $\mathbf{n} = n$  and  $\mathbf{s} = 2c(\log m)^{2/3}$ , for all  $m \geq m_0$  and  $n \geq n_0 m$ ,

$$\mathbb{P}(\mathbf{B}_{0,m,n}) \geq \mathbb{P} \left( \text{W}_{n;(0,1+km^{-1})}^{(0,1+(k+1)m^{-1})} \geq 2cm^{-1/3} (\log m)^{2/3} \right) \geq e^{-2^{3/2} c_2 c^{3/2} \log m} = m^{-2^{3/2} c_2 c^{3/2}}. \quad (4.36)$$

Here the first inequality follows because

$$cm^{-1/3} (\log m)^{2/3} \geq cm^{-1/3} (\log m_0)^{2/3} \geq 2n_0^{-1/3} m^{-1/3} \geq 2n^{-1/3}$$

for  $m \geq m_0$ ,  $2c(\log m_0)^{2/3} \geq 4n_0^{-1/3}$  and  $n \geq n_0 m$ .

Now  $\mathbf{B}_{k,m,n}$  are i.i.d. random variables for  $k \in \{0, 1, 2, \dots, m-1\}$  as the weights of polymers over disjoint regions are independent. Also using  $\text{W}_{n;(0,0)}^{(0,1+(k+1)m^{-1})} - \text{W}_{n;(0,0)}^{(0,1+km^{-1})} \geq$

$W_{n;(0,1+km^{-1})}^{(0,1+(k+1)m^{-1})}$  by superadditivity of polymer weights, together with (4.11), we get that  $B_{k,m,m} \subseteq A_{k,m,n}$ . Thus, using (4.36), for all  $m \geq m_0$  and  $n \geq n_0m$ ,

$$\begin{aligned} \mathbb{P} \left( \bigcap_{k=0}^{m-1} A_{k,m,n}^c \right) &\leq \mathbb{P} \left( \bigcap_{k=0}^{m-1} B_{k,m,n}^c \right) = (1 - \mathbb{P}(B_{0,m,n}))^m \\ &\leq \exp \{ -m\mathbb{P}(B_{0,m,n}) \} \leq \exp \left\{ -m^{1-2^{3/2}c_2c^{3/2}} \right\}, \end{aligned} \quad (4.37)$$

where we use that  $1 - x \leq e^{-x}$  for all  $x \geq 0$ .

Next, similarly to the first part of the proof of Lemma 4.4.2, let  $\{\text{Wgt}_{n_r}\}_r$  be a subsequence of  $\{\text{Wgt}_n\}_n$  such that  $\text{Wgt}_{n_r} \Rightarrow \text{Wgt}_*$  as random variables in  $(C[1, 2], \|\cdot\|_\infty)$  (where  $\Rightarrow$  denotes convergence in distribution). Since for  $a, b \in [1, 2]$ , the map  $T_{a,b}$  defined by  $(C[1, 2], \|\cdot\|_\infty) \mapsto (\mathbb{R}, |\cdot|) : f \mapsto f(a) - f(b)$  is continuous, the set

$$U := \bigcap \left\{ T_{1+(k+1)m^{-1}, 1+km^{-1}}^{-1} \left( -\infty, cm^{-1/3}(\log m)^{2/3} \right) : k = 0, 1, \dots, m-1 \right\}$$

is open. Thus, by the Portmanteau theorem,

$$\mathbb{P} \left( \bigcap_{k=0}^{m-1} A_{k,m}^c \right) \leq \liminf_r \mathbb{P} \left( \bigcap_{k=0}^{m-1} A_{k,m,n_r}^c \right) \leq \limsup_n \mathbb{P} \left( \bigcap_{k=0}^{m-1} A_{k,m,n}^c \right).$$

From here, using (4.37) and that our given choice of the constant  $c$  ensures  $2^{3/2}c_2c^{3/2} < 1$ , we get

$$\sum_{m=m_0}^{\infty} \mathbb{P} \left( \bigcap_{k=0}^{m-1} A_{k,m}^c \right) \leq \sum_{m=m_0}^{\infty} \limsup_n \mathbb{P} \left( \bigcap_{k=0}^{m-1} A_{k,m,n}^c \right) \leq \sum_{m=m_0}^{\infty} \exp \left\{ -m^{1-2^{3/2}c_2c^{3/2}} \right\} < \infty.$$

Hence, using the Borel-Cantelli lemma, almost surely there exists  $M_0 \in \mathbb{N}$  such that for all  $m \geq M_0$ , one has some  $k_m \leq m-1$  with  $z = 1 + k_m m^{-1}$  satisfying

$$|\text{Wgt}_*(z + m^{-1}) - \text{Wgt}_*(z)| \geq cm^{-1/3}(\log m)^{2/3}.$$

Let  $\varepsilon = M_0^{-1}$ . Also let  $M_0^{-1}$  be small enough in the sense of Proposition 4.6.2: namely, almost surely for all  $t \in [0, M_0^{-1}]$ ,  $\sup_{1 \leq z \leq 2-t} |\text{Wgt}_*(z+t) - \text{Wgt}_*(z)| t^{-1/3} (\log t^{-1})^{-2/3} \leq 2C$ . Then, for any given  $t \in [0, \varepsilon]$ , let  $m$  be such that  $(m+1)^{-1} < t \leq m^{-1}$ . Then for  $z = 1 + k_m m^{-1}$ ,

$$\begin{aligned} &|\text{Wgt}_*(z+t) - \text{Wgt}_*(z)| \\ &\geq |\text{Wgt}_*(z+m^{-1}) - \text{Wgt}_*(z)| - |\text{Wgt}_*(z+t) - \text{Wgt}_*(z+m^{-1})| \\ &\geq cm^{-1/3}(\log m)^{2/3} - 2C(m^{-1} - (m+1)^{-1})^{1/3} \left( \log(m^{-1} - (m+1)^{-1})^{-1} \right)^{2/3}. \end{aligned}$$

As the second term decays much faster than the first, choosing  $M_0$  large enough so that the second term is smaller than  $2^{-1}cm^{-1/3}(\log m)^{2/3}$  gives the result.  $\square$

**Proof of Theorem 4.1.3.** This result follows from Proposition 4.6.2 and Proposition 4.6.3.  $\square$

### 4.6.1 Upper bound on polymer weight fluctuation: Proof of Lemma 4.6.1

In this subsection, we complete the proof of Theorem 4.1.3. The remaining element, Lemma 4.6.1, will be derived from Lemmas 4.6.4 and 4.6.5.

**Lemma 4.6.4.** *There exist positive constants  $s_0, r_0$  and  $c_0$  such that for  $s \geq s_0$ ,  $z \in [1, 2]$  and  $t \in [r_0 n^{-1}, 2 - z]$ ,*

$$\mathbb{P} \left( \mathbb{W}_{n;(0,0)}^{(0,z)} \geq \mathbb{W}_{n;(0,0)}^{(0,z+t)} + st^{1/3} \right) \leq e^{-c_0 s^{3/2}}.$$

**Proof.** Using  $\mathbb{W}_{n;(0,0)}^{(0,z+t)} \geq \mathbb{W}_{n;(0,0)}^{(0,z)} + \mathbb{W}_{n;(0,z)}^{(0,z+t)}$ , we see that, for  $nt \geq r_0$  and  $s \geq s_0$ ,

$$\mathbb{P} \left( \mathbb{W}_{n;(0,0)}^{(0,z)} \geq \mathbb{W}_{n;(0,0)}^{(0,z+t)} + st^{1/3} \right) \leq \mathbb{P} \left( \mathbb{W}_{n;(0,z)}^{(0,z+t)} \leq -st^{1/3} \right) \leq e^{-cs^{3/2}},$$

where the latter inequality follows from the moderate deviation estimate Theorem 4.2.2, with  $\mathbf{t}_1 = z$ ,  $\mathbf{t}_2 = z + t$ ,  $\mathbf{n} = n$  and  $\mathbf{s} = s$ , and setting  $r_0$  and  $s_0$  to equal  $n_0$  and  $s_0$  respectively from the statement of Theorem 4.2.2.  $\square$

Next is the more subtle of the two constituents of Lemma 4.6.1.

**Lemma 4.6.5.** *There exist positive constants  $n_0, s_2, r_1$  and  $c_0$  such that, for  $n \geq n_0$ ,  $t \in [r_1 n^{-1}, 2 - z]$ ,  $s \in [s_2, 10(nt)^{2/3}]$  and  $z \in [1, 2]$ ,*

$$\mathbb{P} \left( \mathbb{W}_{n;(0,0)}^{(0,z+t)} \geq \mathbb{W}_{n;(0,0)}^{(0,z)} + st^{1/3} \right) \leq 4 e^{-c_0 s^{3/2}}. \quad (4.38)$$

This proof is reminiscent of arguments used in [16] and [14]. We first explain the basic idea, which is illustrated in Figure 4.5. A path may be formed from  $(0, 0)$  to  $(0, z)$  by following the route of a polymer from  $(0, 0)$  to  $(0, z + t)$  until its location,  $(U, z - t)$  say, at height  $z - t$ ; and then following a polymer from  $(U, z - t)$  to  $(0, z)$ . The discrepancy in weight between the original polymer, from  $(0, 0)$  to  $(0, z + t)$ , and the newly formed path, from  $(0, 0)$  to  $(0, z)$ , is equal to the difference in weights between the polymer from  $(U, z - t)$  to  $(0, z + t)$  and that from  $(U, z - t)$  to  $(0, z)$ . The latter two polymers have duration of order  $t$ ; Theorem 4.2.4 may then show that their weights have order  $t^{1/3}$ . Thus, the weight difference  $\mathbb{W}_{n;(0,0)}^{(0,z+t)} - \mathbb{W}_{n;(0,0)}^{(0,z)}$ , which is at most the discrepancy we are considering, is seen to be unlikely to exceed order  $t^{1/3}$ .

**Proof of Lemma 4.6.5.** To implement this idea, we will consider, for definiteness, the leftmost polymer from  $(0, 0)$  to  $(z + t, 0)$ , namely  $\rho_{n;(0,0)}^{\leftarrow; (z+t, 0)}$ . In accordance with the notation in the plan, we will set  $U = \rho_{n;(0,0)}^{\leftarrow; (z+t, 0)}(z - t)$ .

The height- $(z - t)$  polymer location  $U$  typically has order  $t^{2/3}$ . The plan will run into trouble if  $U$  is atypically high, because then the two short polymers running to  $(0, z + t)$  and  $(0, z)$  from  $(U, z - t)$  will have large negative weights dictated by parabolic curvature.

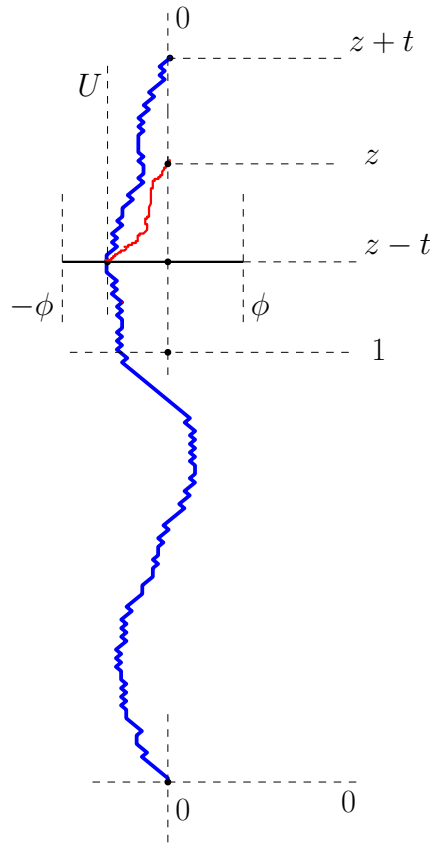


Figure 4.5: When the thick blue polymer  $\rho_{n;(0,0)}^{\leftarrow;(z+t,0)}$  crosses height  $z-t$  without immoderately high fluctuation, it may be diverted via the red polymer to form a path of comparable weight from  $(0,0)$  to  $(z,0)$ .

To cope with this difficulty, we introduce a *good* event  $\mathbf{G}$ ,

$$\mathbf{G} = \{|U| \leq \phi\},$$

specified in terms of a parameter  $\phi$  that is set equal to  $D^{-1}s^{1/2}(2t)^{2/3}$ . Here, the constant  $D$  is chosen to be  $2^{2/3}10^{1/2}C_0$ , with  $C_0$  given by Theorem 4.2.4. In view of Theorem 4.2.5, this choice of  $\phi$  ensures that the event  $\mathbf{G}$  fails to occur with probability of order  $\exp\{-\Theta(1)s^{3/2}\}$ . (The appearance of the factor of  $D^{-1}$  in  $\phi$  is a detail concerning values of  $s$  in Lemma 4.6.5 close to the maximum value  $10(nt)^{2/3}$ .)

Indeed, applying Theorem 4.2.5 with  $\mathbf{n} = n, \mathbf{t}_1 = 0, \mathbf{t}_2 = z+t, \mathbf{t} = z-t, \mathbf{x} = 0, \mathbf{y} = 0$  and  $\mathbf{s} = D^{-1}s^{1/2}$ , we find that, when  $n \geq n_0$  (a bound which ensures that the hypothesis that  $n\mathbf{t}_{1,2} \geq n_0$  is met) and  $s \geq s_1$ ,

$$\mathbb{P}(\mathbf{G}^c) \leq 2 \exp\{-cD^{-3}s^{3/2}\}, \tag{4.39}$$

where the positive constants  $c$  and  $s_1$  are provided by the theorem being applied.

When  $\mathbf{G}$  occurs,

$$|U| \leq D^{-1}s^{1/2}(2t)^{2/3} \leq D^{-1}2^{2/3}10^{1/2}tn^{1/3} < tn^{1/3},$$

because  $s \leq 10(nt)^{2/3}$ ,  $D = 2^{2/3}10^{1/2}C_0$  and  $C_0 > 1$ . As we saw in Subsection 4.1.1, it is this bound on  $|U|$  that ensures the existence of polymers between  $(U, z-t)$  and  $(0, z)$ . By superadditivity of polymer weights, we thus have

$$\mathbb{W}_{n;(0,0)}^{(0,z)} \geq \mathbb{W}_{n;(0,0)}^{(U,z-t)} + \mathbb{W}_{n;(U,z-t)}^{(0,z)}.$$

Thus, when  $\mathbf{G}$  occurs,

$$\begin{aligned} \mathbb{W}_{n;(0,0)}^{(0,z+t)} - \mathbb{W}_{n;(0,0)}^{(0,z)} &\leq \mathbb{W}_{n;(0,0)}^{(0,z+t)} - \mathbb{W}_{n;(0,0)}^{(U,z-t)} - \mathbb{W}_{n;(U,z-t)}^{(0,z)} \\ &= \mathbb{W}_{n;(U,z-t)}^{(0,z+t)} - \mathbb{W}_{n;(U,z-t)}^{(0,z)} \leq \sup_{x \in [-\phi, \phi]} \left( \mathbb{W}_{n;(x,z-t)}^{(0,z+t)} - \mathbb{W}_{n;(x,z-t)}^{(0,z)} \right), \end{aligned}$$

where the equality is dependent on the definition of  $U$  and the final inequality on the occurrence of  $\mathbf{G}$ . We see then that

$$\begin{aligned} &\mathbb{P} \left( \mathbf{G} \cap \left\{ \mathbb{W}_{n;(0,0)}^{(0,z+t)} \geq \mathbb{W}_{n;(0,0)}^{(0,z)} + st^{1/3} \right\} \right) \\ &\leq \mathbb{P} \left( \sup_{x \in [-\phi, \phi]} \left( \mathbb{W}_{n;(x,z-t)}^{(0,z+t)} - \mathbb{W}_{n;(x,z-t)}^{(0,z)} \right) \geq st^{1/3} \right) \\ &\leq \mathbb{P} \left( \sup_{x \in [-\phi, \phi]} \left| \mathbb{W}_{n;(x,z-t)}^{(0,z+t)} \right| > 2^{-1}st^{1/3} \right) + \mathbb{P} \left( \sup_{x \in [-\phi, \phi]} \left| \mathbb{W}_{n;(x,z-t)}^{(0,z)} \right| > 2^{-1}st^{1/3} \right). \end{aligned} \quad (4.40)$$

The latter two probabilities will each be bounded above by a union bound over several applications of Theorem 4.2.4. Addressing the first of these probabilities to begin with, we set parameters for a given application of the theorem, taking  $\mathbf{I}$  to be a given interval of length at most  $t^{2/3}$  contained in  $[-\phi, \phi]$  and  $\mathbf{J} = \{0\}$ , and also setting  $\mathbf{n} = n$ ,  $\mathbf{t}_1 = z-t$ ,  $\mathbf{t}_2 = z$  and  $\mathbf{s} = 4^{-1}s$ .

The theorem's hypothesis concerning inclusion for the interval  $\mathbf{I}$  (and  $\mathbf{J}$ ) is ensured because

$$|x| \leq D^{-1}s^{1/2}(2t)^{2/3} \leq 2^{2/3}10^{1/4}D^{-1}s^{1/4}n^{1/6}t^{5/6} < C_0^{-1}s^{1/4}n^{1/6}t^{5/6},$$

for  $x \in [-\phi, \phi]$ , where here we use  $s \leq 10(nt)^{2/3}$  and  $D = 2^{2/3}10^{1/2}C_0 > 2^{2/3}10^{1/4}C_0$ .

In these applications of Theorem 4.2.4, the parabolic curvature term inside the supremum,  $t^{-4/3}x^2$ , is at most  $t^{-4/3}\phi^2$ . It is thus also at most  $s/4$ , because  $\phi = D^{-1}s^{1/2}(2t)^{2/3}$  and  $D \geq 2^{5/3}$ .

Thus, dividing  $[-\phi, \phi]$  into  $\lceil 2^{5/3}D^{-1}s^{1/2} \rceil$ -many consecutive intervals of length at most  $t^{2/3}$ , we are indeed able to apply Theorem 4.2.4 and a union bound, finding that, for  $n_0 \in \mathbb{N}$

and  $C, c > 0$  the constants furnished by the theorem, and for  $nt \geq n_0$ ,

$$\begin{aligned} & \mathbb{P} \left( \sup_{x \in [-\phi, \phi]} \left| \mathbf{W}_{n; (x, z-t)}^{(0, z)} \right| > 2^{-1} s t^{1/3} \right) \\ & \leq \mathbb{P} \left( \sup_{x \in [-\phi, \phi]} \left| t^{-1/3} \mathbf{W}_{n; (x, z-t)}^{(0, z)} + t^{-4/3} x^2 \right| > 4^{-1} s \right) \leq \lceil 2^{5/3} D^{-1} s^{1/2} \rceil C e^{-c s^{3/2}} \leq e^{-c' s^{3/2}}, \end{aligned}$$

for  $c' = 2^{-1}c$  and  $s \geq s_0$  where  $s_0$  is chosen in such a way that  $e^{2^{-1}c s_0^{3/2}} \geq C \lceil 2^{5/3} D^{-1} s_0^{1/2} \rceil$ .

The second probability in (4.40) is bounded above by similar means. Several applications of Theorem 4.2.4 will be made. In a given application, the parameters  $\mathbf{I}, \mathbf{J}, \mathbf{n}$  and  $\mathbf{s}$  are chosen as before, but we now set  $\mathbf{t}_1 = z - t$  and  $\mathbf{t}_2 = z + t$ , so that  $\mathbf{t}_{1,2}$  equals  $2t$ , rather than  $t$ . The curvature term  $(2t)^{-4/3} x^2$  is bounded above by  $(2t)^{-4/3} \phi^2$ , a smaller bound than before, so that the preceding bound of  $s/4$  remains valid. The condition for inclusion for the intervals  $\mathbf{I}$  (and  $\mathbf{J}$ ), namely  $\phi \leq C_0^{-1} s^{1/4} n^{1/6} (2t)^{5/6}$ , is weaker than it was previously and is thus satisfied. Hence, using Theorem 4.2.4 and a union bound, we find that, for all  $n \geq 2^{-1} n_0 t^{-1}$ ,

$$\mathbb{P} \left( \sup_{x \in [-\phi, \phi]} \left| \mathbf{W}_{n; (x, z-t)}^{(0, z+t)} \right| > 2^{-1} s t^{1/3} \right) \leq e^{-c' s^{3/2}},$$

for  $s \geq s_0$ .

Combining (4.39) and (4.40) with the two bounds just derived, we obtain Lemma 4.6.5 by taking  $c_0 > 0$  to be less than  $\min\{cD^{-3}, c'\}$ ,  $s_2$  to be suitably greater than  $\max\{s_0, s_1\}$ , and  $r_1 = 2^{-1} n_0$ .  $\square$

**Proof of Lemma 4.6.1.** This follows immediately using (4.11) and from Lemmas 4.6.4 and 4.6.5 and a union bound.  $\square$

## 4.7 Lower bound on transversal fluctuation: Proof of Proposition 4.1.4

In this last section we shall prove the lower bound on the transversal fluctuation of the polymer, the corresponding upper bound of which was proved in [16, Theorem 11.1] (and is stated here, with the optimal exponent in the bound, as Theorem 4.2.6). In fact, Proposition 4.1.4 does slightly more than just providing a corresponding lower bound on the quantity whose upper bound is proved in Theorem 4.2.6. Indeed, in Proposition 4.1.4, one takes the minimum over the transversal fluctuations of all the polymers between two fixed points, and not just the transversal fluctuation of the leftmost one. The proof of Proposition 4.1.4 crucially uses the polymer weight lower tail Theorem 4.2.3. We also fix the constant  $\alpha_0$  in this Proposition 4.1.4 as  $\alpha_0 = C_0^{-2} 3^{-5/3} 10^{-1/2}$ , where  $C_0$  is as in Theorem 4.2.4. This choice of  $\alpha_0$  ensures that the condition in the hypothesis of Theorem 4.2.4 is met whenever it is applied.

**Proof of Proposition 4.1.4.** We prove the proposition for  $t_1 = 0$  and  $t_2 = 1$ . The case for general  $t_1 < t_2$  follows readily using the scaling principle (4.13).

A box is a subset of  $\mathbb{R}^2$  of the form  $[a, b] \times [r_1, r_2]$ , where  $a \leq b$  and  $r_1 \leq r_2$ . Any box has a lower and an upper side, namely  $[a, b] \times \{r_1\}$  and  $[a, b] \times \{r_2\}$ .

The key box for the proof is Strip, now specified to be  $[-s, s] \times [0, 1]$ . Proposition 4.1.4 is, after all, a lower bound on the probability that there exists a polymer between  $(0, 0)$  and  $(0, 1)$  that escapes Strip.

We divide Strip into three further boxes, writing Mid for the box  $[-s, s] \times [1/3, 2/3]$ , and South and North for the boxes obtained from Mid by vertical translations of  $-1/3$  and  $1/3$ . We further set West to be the box obtained from Mid by a horizontal translation of  $-2s$ . See Figure 4.6.

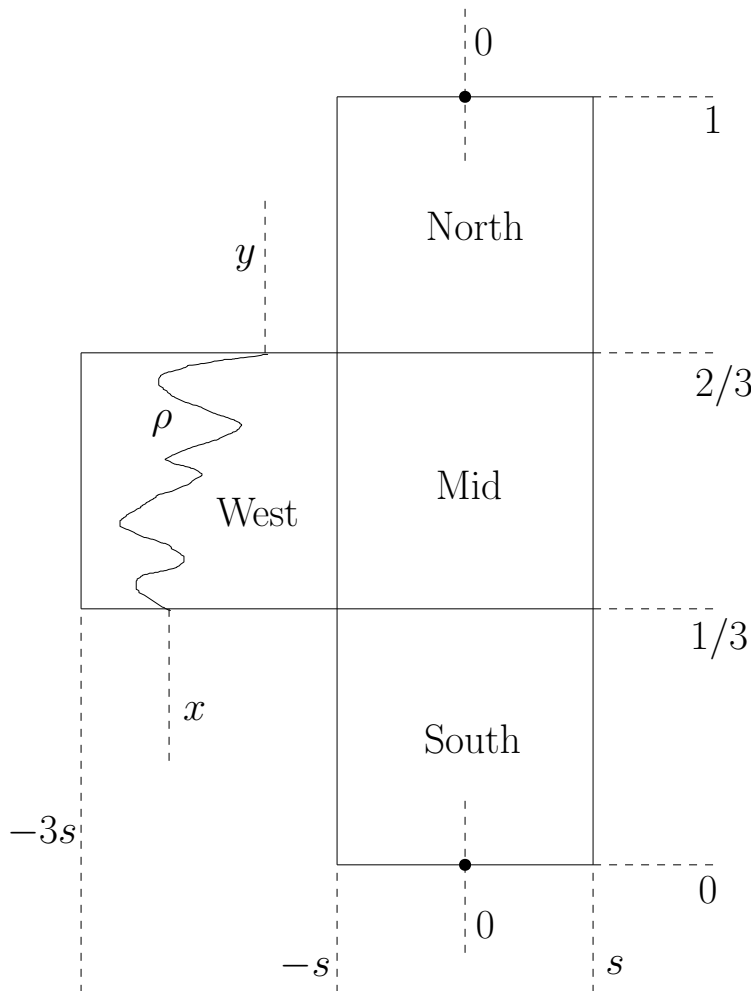


Figure 4.6: In Case High, the high weight path  $\rho$  is extended to form a path from  $(0, 0)$  to  $(0, 1)$  whose weight exceeds that of any path between these points that remains in  $\text{Strip} = \text{North} \cup \text{Mid} \cup \text{South}$ .

Recall that, when  $(x, t_1)$  and  $(y, t_2)$  verify  $n^{1/3}t_{1,2} \geq |y-x|$ , we denote the polymer weight with this pair of endpoints by  $W_{n;(x,t_1)}^{(y,t_2)}$ . We now use a set theoretic notational convention to refer in similar terms to the set of weights of polymers between two collections of endpoint locations. Indeed, let  $I$  and  $J$  be compact real intervals. We will write

$$W_{n;(I,t_1)}^{(J,t_2)} = \left\{ W_{n;(x,t_1)}^{(y,t_2)} : x \in I, y \in J \right\};$$

we will ensure that whenever this notation is used,  $(x, t_1) \stackrel{n}{\preceq} (y, t_2)$  for all  $x \in I$  and  $y \in J$  in the sense of Subsection 4.1.1. When an interval is a singleton,  $I = \{x\}$  say, we write  $(x, t_1)$  instead of  $(\{x\}, t_1)$  when using this notation.

To any box  $B$  and  $s \in \mathbb{R}$ , we define the event  $\text{High}(B, s)$  that the weight of some path that is contained in  $B$  with starting point in the lower side of  $B$  and ending point in the upper side of  $B$  is at least  $s$ .

Our approach to proving Proposition 4.1.4 gives a central role to the event  $\text{High}(\text{Mid}, 300s^2)$ . It may be expected that the order of probability of this event is  $\exp \{ -\Theta(1)s^3 \}$ , but we do not attempt to prove this. Rather, we analyse two cases, called *High* and *Low*, according to the value of the event's probability.

We will quantify the notion of high or low probability for  $\text{High}(\text{Mid}, 300s^2)$  in terms of the decay rate for a very high weight polymer between  $(0, 0)$  and  $(0, 1)$ . Indeed, noting from Theorem 4.2.3 that there exists  $C > 0$  such that, for  $s \geq s_0$ ,

$$\mathbb{P} \left( W_{n;(0,0)}^{(0,1)} \geq 1000s^2 \right) \geq \exp \{ -Cs^3 \}, \quad (4.41)$$

we declare that Case High occurs if

$$\mathbb{P}(\text{High}(\text{Mid}, 300s^2)) \geq \exp \{ -2Cs^3 \};$$

Case Low occurs when Case High does not.

In order to analyse Case High, we introduce a *favourable* event  $F$ . The event is specified as the intersection of the following events:

- $G_1 = \left\{ \inf W_{n;(0,0)}^{([-3s, -s], 1/3)} \geq -50s^2 \right\};$
- $G_2 = \left\{ \inf W_{n;([-3s, -s], 2/3)}^{(0,1)} \geq -50s^2 \right\};$
- $G_3 = \left\{ \sup W_{n;(0,0)}^{([-s, s], 1/3)} \leq 50s^2 \right\};$
- $G_4 = \left\{ \sup W_{n;([-s, s], 2/3)}^{(0,1)} \leq 50s^2 \right\};$
- and  $G_5$  is the event that  $\text{High}(\text{Mid}, 50s^2)$  does not occur.

Thus, the occurrence of  $F$  forces the absence of any high weight path inside  $\text{Mid}$  that crosses this box from its lower to its upper side, while also ensuring that any polymer connecting  $(0, 0)$  (or  $(0, 1)$ ) to the lower (or upper) sides of  $\text{Mid}$  and  $\text{West}$  is not of very low weight. We claim that  $F$  is a high probability event, proving this by applying Theorem 4.2.4. Indeed, for the events  $G_1$  and  $G_3$  entailed by  $F$ , we make several applications of Theorem 4.2.4. For a given application, we consider the parameter settings  $\mathbf{n} = n, \mathbf{t}_1 = 0, \mathbf{t}_2 = 1/3, \mathbf{s} = 10s^2, \mathbf{I} = \{0\}$  and

$$\mathbf{J} = [-3s + (i-1)3^{-2/3}, \max\{-3s + i3^{-2/3}, s\}]$$

for some  $i \in \{1, 2, \dots, \lceil 4 \cdot 3^{2/3}s \rceil\}$ . The condition on inclusion for the intervals  $\mathbf{I}$  and  $\mathbf{J}$  is satisfied since for  $y \in [-3s, s]$ ,

$$|y| \leq 3s \leq \mathbf{s}^{1/2} \leq 10^{1/4} \alpha_0^{1/2} \mathbf{n}^{1/6} \mathbf{s}^{1/4} \leq 3^{5/6} 10^{1/4} \alpha_0^{1/2} \mathbf{n}^{1/6} \mathbf{s}^{1/4} \mathbf{t}_{1,2}^{5/6} = C_0^{-1} \mathbf{n}^{1/6} \mathbf{s}^{1/4} \mathbf{t}_{1,2}^{5/6},$$

where we use that  $s \leq \alpha_0 n^{1/3}$  and our given choice of  $\alpha_0$  has been made so that  $\alpha_0 = C_0^{-2} 3^{-5/3} 10^{-1/2}$ . Also the parabolic curvature inside the supremum is

$$\sup_{y \in [-3s, s]} 3^{4/3} y^2 \leq 3^{4/3} \cdot 3^2 s^2 < 40s^2.$$

Thus, dividing  $[-3s, s]$  into  $\lceil 4 \cdot 3^{2/3}s \rceil$ -many intervals of length at most  $3^{-2/3}$  and using Theorem 4.2.4 and a union bound, it follows that, for  $s$  large enough and  $n \geq 3n_0$ ,

$$\mathbb{P}(G_1^c \cup G_3^c) \leq \mathbb{P}\left(\sup_{y \in [-3s, s]} \left| 3^{1/3} W_{n; (0,0)}^{(y, 1/3)} + 3^{4/3} y^2 \right| > 10s^2\right) \leq \lceil 4 \cdot 3^{2/3}s \rceil C e^{-cs^3} \leq 6^{-1}.$$

Similarly for the events  $G_2$  and  $G_4$ , in a given application of Theorem 4.2.4, we set the parameters  $\mathbf{n} = n, \mathbf{t}_1 = 2/3, \mathbf{t}_2 = 1, \mathbf{s} = 10s^2, \mathbf{I} = [-3s + (i-1)3^{-2/3}, \max\{-3s + i3^{-2/3}, s\}]$  and  $\mathbf{J} = \{0\}$ , for some  $i \in \{1, 2, \dots, \lceil 4 \cdot 3^{2/3}s \rceil\}$ . The condition on the inclusion for the intervals  $\mathbf{I}$  and  $\mathbf{J}$  is ensured exactly in the same way as before, and the parabolic curvature is bounded above by  $40s^2$ . Hence, using Theorem 4.2.4 and a union bound, it follows that, for  $s$  large enough and  $n \geq 3n_0$ ,

$$\mathbb{P}(G_2^c \cup G_4^c) \leq \lceil 4 \cdot 3^{2/3}s \rceil C e^{-cs^3} \leq 6^{-1}.$$

Finally, for  $G_5$ , observe that, since paths between two fixed endpoints constrained to stay in a box have smaller weight than does the polymer between these endpoints, we can again use Theorem 4.2.4. For a given application of Theorem 4.2.4, take  $\mathbf{n} = n, \mathbf{t}_1 = 1/3, \mathbf{t}_2 = 2/3, \mathbf{s} = 40s^2, \mathbf{I} = [-s + (i-1)3^{-2/3}, \max\{-s + i3^{-2/3}, s\}]$  and  $\mathbf{J} = [-s + (j-1)3^{-2/3}, \max\{-s + j3^{-2/3}, s\}]$  for  $i \in \{1, 2, \dots, \lceil 2 \cdot 3^{2/3}s \rceil\}$  and  $j \in \{1, 2, \dots, \lceil 2 \cdot 3^{2/3}s \rceil\}$ . As before, the condition on inclusion for  $\mathbf{I}$  and  $\mathbf{J}$  is satisfied, and the parabolic curvature is at most  $3^{4/3}s^2$ , which is less than  $10s^2$ . Thus, applying Theorem 4.2.4 and a union bound, we find that, for  $n \geq 3n_0$  and  $s$  large,

$$\mathbb{P}(G_5^c) \leq \mathbb{P}\left(\sup W_{n; ([-s, s], 1/3)}^{([-s, s], 2/3)} > 50s^2\right) \leq \lceil 2 \cdot 3^{2/3}s \rceil^2 C e^{-cs^3} \leq 6^{-1}.$$

Thus we have  $\mathbb{P}(F) \geq 1/2$  by a union bound. In Case High, we also have

$$\mathbb{P}(\text{High}(\text{West}, 300s^2)) \geq \exp \{ - 2Cs^3 \},$$

because West is a translate of Mid. Since the interior of West is disjoint from the regions that dictate the occurrence of F, we see that

$$\mathbb{P}(\text{High}(\text{West}, 300s^2) \cap F) \geq 2^{-1} \exp \{ - 2Cs^3 \}. \quad (4.42)$$

When  $\text{High}(\text{West}, 300s^2) \cap F$  occurs, a high weight path connecting  $(0, 0)$  to  $(0, 1)$  may be formed by running it through West. Indeed, and as Figure 4.6 depicts, let  $\rho$  denote a polymer running across, and contained in, West, whose weight is at least  $300s^2$ . If  $x, y \in [-3s, -s]$  are such that  $(x, 1/3)$  and  $(y, 2/3)$  are  $\rho$ 's endpoints, then the path  $\rho_{n;(0,0)}^{\leftarrow;(x,1/3)} \circ \rho \circ \rho_{n;(y,2/3)}^{\leftarrow;(0,1)}$  connects  $(0, 0)$  to  $(0, 1)$  and has weight at least  $-50s^2 + 300s^2 - 50s^2$ , in view of the first two conditions that specify F.

On the other hand, the final three conditions specifying F ensure that, when this event occurs, any path from  $(0, 0)$  to  $(0, 1)$  whose  $x$ -coordinate never exceeds  $s$  in absolute value has weight at most  $50s^2 + 50s^2 + 50s^2$ ; indeed, the weight of any such path may be represented as a sum of the weights of the three subpaths formed by cutting the path at heights one-third and two-thirds.

We thus find that, on  $\text{High}(\text{West}, 300s^2) \cap F$ , any path from  $(0, 0)$  to  $(0, 1)$  that remains in Strip has weight at most  $150s^2$ ; at the same time, a path of weight at least  $200s^2$  connects these two points. Thus, we see that any polymer from  $(0, 0)$  to  $(0, 1)$  has maximum transversal fluctuation at least  $s$  in this event. By (4.42), we find that

$$\mathbb{P} \left( \min \{ \text{TF}(\rho) : \rho \in \Phi_{n;(0,0)}^{(0,1)} \} \geq s \right) \geq 2^{-1} \exp \{ - 2Cs^3 \}. \quad (4.43)$$

Suppose now instead that Case Low holds. We will argue that

$$\mathbb{P} \left( \mathbf{W}_{n;(0,0)}^{(0,1)} \geq 1000s^2, \neg \text{High}([-s, s] \times [0, 1], 900s^2) \right) \geq 2^{-1} \exp \{ - Cs^3 \}, \quad (4.44)$$

where  $\neg A$  denotes the complement of the event  $A$ . Before we do so, we show that the event on this left-hand side entails that any polymer from  $(0, 0)$  to  $(0, 1)$  must leave the strip  $[-s, s] \times [0, 1]$ ; thus, (4.43) holds in Case Low, even when the factor of 2 is omitted from the right-hand exponential. When the last left-hand event occurs, any path from  $(0, 0)$  to  $(0, 1)$  that remains in the strip has weight at most  $900s^2$ . At the same time, the weight of any polymer from  $(0, 0)$  to  $(0, 1)$  is at least  $1000s^2$ . It is thus impossible for any polymer to remain in the strip.

To derive (4.44), note that, because North and South are translates of Mid, Case Low entails that

$$\mathbb{P} \left( \text{High}(\text{South}, 300s^2) \cup \text{High}(\text{Mid}, 300s^2) \cup \text{High}(\text{North}, 300s^2) \right) < 3 \exp \{ - 2Cs^3 \}.$$

The bound (4.41) then yields (4.44), since  $3 \exp \{ - 2Cs^3 \} \leq 2^{-1} \exp \{ - Cs^3 \}$  for all  $s$  large enough.

The bound (4.43) has been derived in both of the cases, so that proof of Proposition 4.1.4 is complete.  $\square$

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